Algebraic Topology 2

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1 Singular cohomology

1.1 Cohomology of chain complexes

Definition 1.1.1. A **cochain complex** is a sequence of homomorphisms of abelian groups

$$\cdots \to A^{k-1} \xrightarrow{d^{k-1}} A^k \xrightarrow{d^k} A^{k+1} \to \ldots$$

such that $d^{k+1}d^k = 0$ for all $k \in \mathbb{Z}$. The map d^k is called the **coboundary operator** or the **differential** of the complex. A **cochain map** $\phi^\bullet : A^\bullet \to B^\bullet$ is a family of group homomorphisms $\phi^k : A^k \to B^k$ such that the diagram

$$\begin{tikzcd} A^k \arrow[r, "d_A^k"] \arrow[d, "\phi^k"] & A^{k+1} \arrow[d, "\phi^{k+1}"] \\ B^k \arrow[r, "d_B^k"] & B^{k+1} \end{tikzcd}$$

commutes for every $k \in \mathbb{Z}$.

So, a cochain complex is just the same as a chain complex with the numbering reversed. That seems silly, but below we give a connection between chain an cochain complexes which makes the distinction useful.

Definition 1.1.2. Let

$$\cdots \to A^{k-1} \xrightarrow{d^{k-1}} A^k \xrightarrow{d^k} A^{k+1} \to \ldots$$

denote a cochain complex. Then its **cohomology** is defined as

$$H^k(A^\bullet) = \ker(d^k)/\operatorname{im}(d^{k-1}).$$

One writes $Z^k = \ker(d^k))$, $B^k = \operatorname{im}(d^{k-1})$, so that $H^k = Z^k/B^k$. Elements of Z^k are called **cocycles** and elements of B^k are **coboundaries**.

Definition 1.1.3. (From chains to cochains) Let $\cdots \to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to \ldots$ be a chain complex of abelian groups. For a given abelian group R let $C^n(R) = \operatorname{Hom}(C_n, R)$ be the dual group. (The group R will later be a ring, which is why we call it R.) Let $d : C^n \to C^{n+1}$ be the operator dual to ∂, so $d\phi(f) = \phi(\partial f)$ for $\phi \in C^n(R)$. Then $\cdots \to C^{n-1} \xrightarrow{d} C^n \xrightarrow{d} C^{n+1} \to$ is a chain complex. The **cohomology group with coefficients** in R is by definition

$$H^k(C, R) := H^k\Big(\operatorname{Hom}(C_\bullet, R)\Big).$$

Definition 1.1.4. Let R be an abelian group and X a topological space. The set of **singular cochains** with coefficient in R is defined as

$$C^n(X, R) = \operatorname{Hom}(C_n(X), R).$$

The corresponding chomology group is called the **singular cohomology**

$$H^k(X, R)$$

of X.

Definition 1.1.5. Let A be an abelian group. An exact sequence of the form

$$\cdots \to F_2 \to F_1 \to F_0 \to A \to 0$$

with free abelian groups F_j is called a **free resolution** of A. Then

$$F \quad \equiv \quad \dots F_1 \to F_0 \to 0$$

is a chain complex of free abelian groups. The cohomology $H^p(F, R)$ is defined as above. Note that F is the "cut-off" complex. One also writes this as $F_\bullet \to A$. For instance, one has

$$\begin{aligned} H^0(F, R) &= \ker(\operatorname{Hom}(F_0, R) \to \operatorname{Hom}(F_1, R)) \\ &= \operatorname{Hom}(F_0/F_1, R) = \operatorname{Hom}(A, R). \end{aligned}$$

This means that $H^0(F, R)$ does not depend on the choice of the resolution.

Lemma 1.1.6.

(a) *Let F, F' be free resolutions of the abelian groups A, A'. Then every group homomorphism $\alpha : A \to A'$ can be extended to a chain map $F \to F'$, such that there is a commutative diagram*

$$\begin{tikzcd} \dots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"] & A \arrow[r] \arrow[d, "\alpha"] & 0 \\ \dots \arrow[r] & F_2' \arrow[r, "f_2'"] & F_1' \arrow[r, "f_1'"] & F_0' \arrow[r, "f_0'"] & A' \arrow[r] & 0. \end{tikzcd}$$

Any two such extensions of α are chain-homotopic.

(b) *For any two free resolutions F, F' of A there are canonical isomorphisms*

$$H^p(F', R) \cong H^p(F, R) = H^p\Big(\operatorname{Hom}(F_\bullet, R)\Big).$$

Proof. Since F_0 is free, the homomorphism $\alpha \circ f_0 : F_0 \to A$ can be lifted to F_0', this defines α_0. Now assume that α_{n-1} is already defined. The homomorphism $\alpha_{n-1} \circ f_n : F_n \to F_{n-1}'$ satisfies $f_{n-1}' \circ (\alpha_{n-1} \circ f_n) = 0$, which means that the image lies in the kernel of f_{n-1}', which equals the image of f_n'. Since F_n is free, this homomorphism can be lifted to $\alpha_n : F_n \to F_n'$.

In order to show that two given extensions of α are chain-homotopic, it suffices to show that in the case $\alpha = 0$ any extension is nullhomotopic.

So let $\alpha = 0$ and let α_n be any extension. We are looking for group homomorphisms $P_n : F_n \to F_{n+1}'$ such that

$$\alpha_n = f_{n+1}' P_n + P_{n-1} f_n.$$

Set $P_{-1} : A \to F_0'$ to be zero. The relation we need, is $\alpha_0 = f_1' P_0$. Such a P_0 exists, since the image of α_0 lies in the kernel of f_0', i.e., in the image of f_1' and F_0 is free.

Inductively, let P_{n-1} already be defined. We are looking for some P_n such that $\alpha_n - P_{n-1} f_n = f_{n+1}' P_n$. To show that such a map exists, it suffices to show that the image of $\alpha_n - P_{n-1} f_n$ lies in the image of f_{n+1}', i.e., in the kernel of f'_n. Because of $\alpha_{n-1} = f'_n P_{n-1} + P_{n-2} f_{n-1}$ one has

$$\begin{aligned} f'_n(\alpha_n - P_{n-1} f_n) &= f'_n \alpha_n - (f'_n P_{n-1}) f_n \\ &= f'_n \alpha_n - (\alpha_{n-1} - P_{n-2} f_{n-1}) f_n \\ &= f'_n \alpha_n - \alpha_{n-1} f_n = 0. \end{aligned}$$

So P_n exists and therefore part (a) is proven.

Part (b) is now easy. Since chain-homotopic maps give the same map on the homology, every homomorphism $\alpha : A \to A'$ induces an uniquely determined homomorphism on the homology. For two different resolutions F, F' of the same group A we apply this to $\alpha = \mathrm{Id} : A \to A$ and we get uniquely determined homomorphisms on the homology, $\phi : H(F) \to H(F')$ and by the same token, $\psi : H(F') \to H(F)$. By uniqueness, $\phi \circ \psi$ must coincide with the identity and the same for $\psi \circ \phi$, to the two are isomorphisms. □

It follows, that up to canonical isomorphy, the groups $H^k(F, R)$ only depend on A and R, not on the resolution. We call this group

$$\mathrm{Ext}^k(A, R).$$

Lemma 1.1.7. *For a given abelian group A there is an exact sequence*

$$0 \to F_1 \to F_0 \to A \to 0,$$

with free groups F_0, F_1. So any abelian group has a free resolution of length 2.

Proof. Let S be any set of generators of A. Let F_0 be the free abelian group with generator set S' of the same cardinality as S. Any bijection $f : S' \to S$ extends to a surjective group homomorphism $f : F_0 \to A$. Let F_1 be the kernel of f. As F_1 is a subgroup of a free abelian group, it is a free abelian group itself (Lang, Algebra). Therefore the sequence $F_1 \to F_0 \to A$ satisfies the claim. □

The lemma implies that for abelian groups A, B one has $\mathrm{Ext}^k(A, B) = 0$ for $k \geq 2$. Further we have calculated, that

$$\mathrm{Ext}^0(A, B) = \mathrm{Hom}(A, B).$$

So the only interesting group is $\mathrm{Ext}^1(A, B)$.

* * *

1.2 Triviality of Ext

Definition 1.2.1. An abelian group B is called **divisible**, if for every $b \in B$ and every $n \in \mathbb{N}$ there exists $x \in B$ with $nx = b$. For example $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible, $\mathbb{Z}$ is not, nor is and non-trivial finite group.

Lemma 1.2.2. *Lat A, A', B, B' be abelian groups.*

(a) *One has*

$$\mathrm{Ext}^1(A \oplus A', B) \cong \mathrm{Ext}^1(A, B) \oplus \mathrm{Ext}^1(A', B),$$
$$\mathrm{Ext}^1(A, B \oplus B') \cong \mathrm{Ext}^1(A, B) \oplus \mathrm{Ext}^1(A, B')$$

(b) *If A is free or if B is divisible, one has*

$$\mathrm{Ext}^1(A, B) = 0.$$

(c) $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, B) \cong B/nB$.

Proof. (a) The first follows from the fact that the direct sum of free resolutions of A and A' is a free resolution of $A \oplus A'$. The second follows from $\mathrm{Hom}(F, B \oplus B') \cong \mathrm{Hom}(F, B) \oplus \mathrm{Hom}(F, B')$ applied to members F of a free resolution of A.

(b) The case of A being free is clear, as $0 \to A \to A \to 0$ is a free resolution. Now let B be divisible and let $0 \to A_1 \to A_0 \to A \to 0$ be a free resolution. We need to show that the induced homomorphism $\mathrm{Hom}(A_0, B) \to \mathrm{Hom}(A_1, B)$ is surjective. So let $\phi : A_1 \to B$ be a group homomorphism. We need to show that it can be extended to A_0. Let S be the set of all pairs (H, ψ), where $H \subset A_0$ is a subgroup containing A_1 and $\psi : H \to B$ is an extension of ϕ. We order S by

$$(H, \psi) \le (H', \psi') \quad \Leftrightarrow \quad H \subset H',\ \psi'|_H = \psi.$$

Zorn's lemma shows that there exists a maximal element (H, ψ). We claim that $H = A_0$. **Assume** otherwise and let $a_0 \in A_0 \smallsetminus H$. If there exists $n \in \mathbb{N}$ such that $na_0 \in H$, then pick $b_0 \in B$ with $nb_0 = \psi(na_0)$ and set $\tilde{\psi}(a_0) = b_0$. Then $\tilde{\psi}$ extends ψ to the group generated by H and a_0, so ψ is not maximal, a **contradiction**. If no such n exists, set $\tilde{\psi}(a_0) = 0$ and then as well $\tilde{\psi}$ extends ψ.

(c) comes from the resolution $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$. □

Lemma 1.2.3. *Let*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

be an exact sequence of abelian groups. Then the following are equivalent:

(a) *There exists a group homomorphism $s : C \to B$ with $\beta s = \mathrm{Id}_C$.*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] \arrow[l, bend left, "s"] & 0 \end{tikzcd}$$

(b) *There is a group homomorphism $t : B \to A$ with $t\alpha = \mathrm{Id}_A$.*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] \arrow[l, bend left, "t"] & C \arrow[r] & 0 \end{tikzcd}$$

(c) *There is an isomorphism $\psi : B \to A \oplus C$ such that the diagram*

$$\begin{tikzcd} & & B \arrow[dr] \arrow[dd, "\psi"] & & \\ 0 \arrow[r] & A \arrow[ur] \arrow[dr] & & C \arrow[r] & 0 \\ & & A \oplus C \arrow[ur, "p_2"'] & & \end{tikzcd}$$

commutes.

Definition 1.2.4. If these equivalent conditions are satisfied, we say that the sequence $0 \to A \to B \to C \to 0$ is a **split exact sequence**.

Proof. (c) $\Rightarrow$ (a) and (c) $\Rightarrow$ (b) are clear.

(a) $\Rightarrow$ (c): Let $s : C \to B$ with $\beta s = \mathrm{Id}_C$. For $b \in B$ one has $b - s\beta(b) \in \operatorname{im} \alpha$, since $\operatorname{im} \alpha = \ker \beta$ and

$$\beta(b - s\beta(b)) = \beta(b) - \beta s\beta(b) = \beta(b) - \beta(b) = 0.$$

So one can define $\psi : B \to A \oplus C$ by

$$\psi(b) = \alpha^{-1}(b - s\beta(b)) \oplus \beta(b).$$

For $a \in A$ one has $\psi(\alpha(a)) = \alpha^{-1}\alpha(a) \oplus 0 = a \oplus 0$. For $b \in B$ one has $p_2\psi(b) = \beta(b)$. Hence the diagram commutes.

(b) $\Rightarrow$ (c): Let $t : B \to A$ be given with $t\alpha = \mathrm{Id}_A$. Define

$$\psi(b) = t(b) \oplus \beta(b).$$

The commutativity of the diagram is clear. □

Lemma 1.2.5. (a) *Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} F \to 0$ be an exact sequence of abelian groups. If F is free, then the sequence splits.*

(b) *Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of abelian groups. Then the dual sequence*

$$0 \to C^* \xrightarrow{\beta^*} B^* \xrightarrow{\alpha^*} A^*$$

is exact. If the first sequence splits, then α^* is also surjective. In general, α^* is not always surjective.

Proof. (a) Let F be free with generating set X. For each $x \in X$, pick any element $b = s(x) \in B$ with $\beta(b) = x$. The map $s : X \to B$ extends to a group homomorphism $s : F \to B$ with $\beta \circ s = \mathrm{Id}_F$.

(b) Let $f : C \to R$ in C^* with $\beta^*(f) = 0$, i.e., $f \circ \beta = 0$, so f vanishes on the image of β. Since β is surjective, we get $f = 0$ and so β^* is injective.

One has $\lambda^* \beta^* = (\beta\alpha)^* = 0^* = 0$.

Let $f \in B^*$ with $\alpha^*(f) = 0$, i.e., $f \circ \alpha = 0$, so f vanishes on the image of α, which is the kernel of β. That means that f factors through $B/\ker\beta \cong C$, therefore there exists $g : C \to R$ with $f = g \circ \beta = \beta^*(g)$.

If the first sequence splits, say $B \cong B_1 \oplus B_2$, then this sequence splits into two isomorphisms $A \xrightarrow{\cong} B_1$ and $B_2 \xrightarrow{\cong} C$, which dualise to isomorphisms of the dual groups.

For the addendum consider the following counterexample. Let $R = \mathbb{Z}$ and $n \in \mathbb{N}$. Since $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$, the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

dualises to

$$0 \to 0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z}. \qquad \square$$

One can show that $\mathrm{Ext}^1(A, B)$ is in natural bijection with the set of all isomorphism classes of exact sequences

$$0 \to A \to C \to B \to 0.$$

The trivial group element stands for the class of splitting sequences.

* * *

1.3 The Universal Coefficient Theorem

Theorem 1.3.1 (Universal Coefficient Theorem for cohomology)**.** *For a given chain complex C of free abelian groups and an abelian group R there is a canonical split exact sequence*

$$0 \to \operatorname{Ext}^1(H_{n-1}(C), R) \to H^n(C, R) \xrightarrow{h} \operatorname{Hom}(H_n(C), R) \to 0.$$

Here $n \geq 0$, where we formally set $H_{-1}(C) = 0$. This sequence is functorial in C, i.e., any chain map $f : C \to D$ induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}^1(H_{n-1}(C), R) \arrow[r] & H^n(C, R) \arrow[r, "h"] & \operatorname{Hom}(H_n(C), R) \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Ext}^1(H_{n-1}(D), R) \arrow[r] \arrow[u, "f^*"] & H^n(D, R) \arrow[r, "h"] \arrow[u, "f^*"] & \operatorname{Hom}(H_n(D), R) \arrow[r] \arrow[u, "f^*"] & 0
\end{tikzcd}$$

Proof. We define a homomorphism

$$h : H^n(C, R) \to \operatorname{Hom}(H_n(C), R)$$

as follows: Let $[\phi] \in H^n(C, R)$, so $\phi : C_n \to R$ with $\phi \circ \partial = d\phi = 0$. This means that $\phi(B_n) = 0$, where $B_n = \operatorname{im}(\partial_{n+1})$. So the restriction of ϕ to $Z_n = \ker \partial_n$ induces a homomorphism $\overline{\phi} : Z_n/B_n \to R$, i.e., an element of $\operatorname{Hom}(H_n(C), R)$. We set $h([\phi]) = \overline{\phi}$. For the well-definedness let $\psi = \phi + d\alpha = \phi + \alpha \circ \partial$ and let $z \in Z_n$. Then $\alpha \circ \partial(z) = \alpha(\underbrace{\partial(z)}_{0}) = 0$, so we get $\overline{\psi} = \overline{\phi}$.

Lemma 1.3.2. *The group homomorphism h is surjective. The sequence*

$$0 \to \ker h \to H^n(C, R) \xrightarrow{h} \operatorname{Hom}(H_n(C), R) \to 0$$

splits.

Proof. Since B_{n-1} is a subgroup of the free group C_{n-1}, the group B_{n-1} is free abelian. Hence the sequence

$$0 \to Z_n \to C_n \xrightarrow{\partial} B_{n-1} \to 0$$

splits. Therefore there is a projection $p : C_n \to Z_n$ with $p|_{Z_n} = \operatorname{Id}$. Let $\eta \in \operatorname{Hom}(H_n(C), R)$, so $\eta : Z_n/B_n \to R$. Define $\hat{\eta} = \eta \circ p : C_n \to R$. One has $d\hat{\eta} = \hat{\eta} \circ \partial = \eta \circ p \circ \partial = \eta \circ \partial = 0$, since $p \equiv \operatorname{Id}$ on $Z_n \supset B_n$ and η is zero on B_n. This means that $\hat{\eta}$ defines a cohomology class $[\hat{\eta}]$ with $h([\hat{\eta}]) = \eta$ (by construction of h), which implies that h is surjective and the sequence of the lemma is split by the map $\eta \mapsto [\hat{\eta}]$. □

We consider a commutative diagram with split exact rows

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "0"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

As the rows are split, the dual diagram has exact rows as well:

$$\begin{tikzcd}
0 & Z^*_{n+1} \arrow[l] & C^*_{n+1} \arrow[l] & B^*_n \arrow[l, "d"'] & 0 \arrow[l] \\
0 & Z^*_n \arrow[l] \arrow[u, "0"] & C^*_n \arrow[l] \arrow[u, "d"] & B^*_{n-1} \arrow[l] \arrow[u, "0"] & 0 \arrow[l]
\end{tikzcd}$$

Here $A^* = \mathrm{Hom}(A, R)$. This means that we get an exact sequence of cochain complexes $0 \leftarrow Z^* \leftarrow C^* \leftarrow B^* \leftarrow 0$. As the complexes Z and B have zero boundary maps, the corresponding long exact sequence of cohomology groups has the form

$$\cdots \leftarrow B^*_n \leftarrow Z^*_n \leftarrow H^n(C, R) \leftarrow B^*_{n-1} \leftarrow Z^*_{n-1} \leftarrow \ldots$$

The **connection homomorphism** $\delta : Z^*_n \to B^*_n$ in this sequence is the dual map of the inclusion $i_n : B_n \to Z_n$, *since* one gets $\delta(z)$ by picking a pre-image of $z \in Z^*_n$ in C^*_n, then applies d and takes the pre-image in B^*_n. In this first step, the homomorphism $z : Z_n \to R$ is extended to C_n, in the second, it is composed with ∂ and in the third this extension is undone again by restriction to B_n. So in the end, z is only restricted to B_n. Hence one has $\delta = i^*_n$. One gets the exact sequence

$$\cdots \leftarrow B^*_n \xleftarrow{i^*_n} Z^*_n \leftarrow H^n(C, R) \leftarrow B^*_{n-1} \xleftarrow{i^*_{n-1}} Z^*_{n-1} \leftarrow \ldots$$

which gives the short exact sequence

$$0 \leftarrow \ker(i^*_n) \leftarrow H^n(C, R) \leftarrow \mathrm{coker}(i^*_{n-1}) \leftarrow 0.$$

Elements of $\ker(i^*_n)$ are homomorphisms $Z_n \to R$, which vanish on B_n, i.e., the homomorphisms $B_n/Z_n \to R$. In other words: $\ker(i^*_n) = \mathrm{Hom}(H_n(C), R)$. The map $H^n(C, R) \to \ker(i^*_n) = \mathrm{Hom}(H_n(C), R)$ equals the map h. So there is a canonical isomorphism $\ker h \cong \mathrm{coker}\, i^*_{n-1}$. The sequence

$$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(C) \to 0$$

is a free resolution of $H_{n-1}(C)$, so there is a canonical isomorphism $\ker h \cong \mathrm{coker}\, i^*_{n-1} \cong \mathrm{Ext}^1(H_{n-1}(C), R)$. The theorem is proven, except for the functoriality. For this, one looks at the construction of the sequence, i.e., the map h, which comes about by interpreting a cohomology class as a homomorphism on homology. One finds that a chain map f would map this construction for the complex D to the corresponding construction for C and thus induce the same map, i.e., the diagram is commutative. □

Proposition 1.3.3. *Let $\phi : C_\bullet \to D_\bullet$ be a morphism of chain complexes, which induces isomorphisms in the* *homology groups* $\phi_* : H_k(C_\bullet) \xrightarrow{\cong} H_k(D_\bullet)$*. Then the corresponding pullback maps on cohomology*

$$\phi^* : H^k(D, R) \to H^k(C, R)$$

are isomorphisms, too.

In particular, it follows that singular cohomology of a simplicial complex can also be computed using the simplicial chain complex.

Proof. This is clear from the functoriality statement in the Universal Coefficient Theorem and the five lemma. □

Let X, Y be topological spaces. For a continuous map $f : X \to Y$, the chain map $f_\# : C_n(X) \to C_n(Y)$ dualises to a cochain map

$$f^\# : C^n(Y) \to C^n(X),$$

which induces a map $f^* : H^n(Y, R) \to H^n(X, R)$ on the cohomology groups.

Proposition 1.3.4. *Let* $f, g : X \to Y$ *be continuous maps. If* f *and* g *are homotopic, then one has* $f^* = g^*$*.*

Consequently, if A *is a deformation retract of* X*, the inclusion* $A \hookrightarrow X$ *induces isomorphisms* $H^n(X, R) \cong H^n(A, R)$*.*

Proof. If f and g are homotopic, then the induced maps on homology coincide. The universal coefficient theorem yields the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1(H_{n-1}(C), R) \arrow[r] & H^n(C, R) \arrow[r, "h"] & \mathrm{Hom}(H_n(C), R) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}^1(H_{n-1}(D), R) \arrow[r] \arrow[u, "0"] & H^n(D, R) \arrow[r, "h"] \arrow[u, "f^*-g^*"] & \mathrm{Hom}(H_n(D), R) \arrow[r] \arrow[u, "0"] & 0
\end{tikzcd}$$

The zeros to the left and right come from the fact that f and g induce the same map on homology. This implies that $f^* - g^* = 0$ in the middle, too. □

* * *

1.4 The sequence of a pair

Lemma 1.4.1. *Let (X, A) be a pair of spaces. The exact sequence*

$$0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X, A) \to 0$$

splits. Consequently, $C_n(X, A)$ is a free abelian group and the Universal Coefficient Theorem applies to the chain complex $C_\bullet(X, A)$.

Proof. Let F be the free abelian group generated by all singular simplices, whose image does not lie in A. Then $C_n(X) = C_n(A) \oplus F$ and the projection to the first summand yields a splitting. □

Definition 1.4.2. As the group R will be fixed throughout, we occasionally leave it off the notation. We define $C^n(X) = C_n(X)^* = \mathrm{Hom}_R(C_n(X), R)$ as well as $C^n(X, A) = C_n(X, A)^*$ and dualise to get a exact sequence

$$0 \to C^n(X, A) \xrightarrow{j^*} C^n(X) \xrightarrow{i^*} C^n(A) \to 0.$$

The **relative coboundary map** $d : C^n(X, A) \to C^{n+1}(X, A)$ is defined by restriction of $d : C^n(X) \to C^{n+1}(X)$ and gives the **relative cohomology** $H^n(X, A) = Z^n(X, A)/B^n(X, A)$.

Proposition 1.4.3. *There is an exact sequence*

$$0 \to H^0(X, A) \to \dots$$

$$\dots \to H^k(X, A) \xrightarrow{j^*} H^k(X) \xrightarrow{i^*} H^k(A) \xrightarrow{\delta} H^{k+1}(X, A) \to \dots$$

Proof. In the sequence

$$0 \to C^n(X, A) \xrightarrow{j^*} C^n(X) \xrightarrow{i^*} C^n(A) \to 0,$$

the maps j^* and i^* are cochain maps, i.e., $i^*d = di^*$ and $j^*d = dj^*$, since i and j are chain maps. Therefore, the claim follows from Theorem 5.4.3 of AlgTop1. □

Theorem 1.4.4 (Excision in cohomology)**.**

(a) *If $A, Z \subset X$ are subsets with $\overline{Z} \subset \mathring{A}$, then the inclusion $i : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms*

$$H^k(X, A) \xrightarrow{\cong} H^k(X \setminus Z, A \setminus Z).$$

(b) *If X is path-connected and $A \subset X$ regularly closed, then there is an exact sequence*

$$0 \to R \to H^0(A) \xrightarrow{\delta} H^1(X/A) \to \dots$$

$$\dots \to H^k(X/A) \xrightarrow{\pi^*} H^k(X) \xrightarrow{i^*} H^k(A) \xrightarrow{\delta} H^{k+1}(X/A) \to \dots$$

where $i : A \hookrightarrow X$ is the inclusion and $\pi : X \to X/A$ is the projection.

Proof. (a) The inclusion induces a chain map $i_\# : C_n = C_n(X \setminus Z, A \setminus Z) \to C'_n = C_n(X, A)$. By the Universal Coeffient Theorem this yields a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1(H_{n-1}(C), R) \arrow[r] & H^n(C, R) \arrow[r, "h"] & \mathrm{Hom}(H_n(C), R) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}^1(H_{n-1}(C'), R) \arrow[r] \arrow[u, "\alpha"] & H^n(C', R) \arrow[r, "h"] \arrow[u, "i^*"] & \mathrm{Hom}(H_n(C'), R) \arrow[r] \arrow[u, "\beta"] & 0
\end{tikzcd}$$

In this diagram, the maps α and β are induced by maps $i_* : H_n(C) \to H_n(C')$, which by Theorem 5.8.1 from AlgTop1 are isomorphisms. Therefore, α and β are isomorphisms, too. By the five lemma, the map i^* then is an isomorphism.

The proof of (b) is analogous to the proof of the exact pair-sequence in homology. □

Proposition 1.4.5. (a) *Let X be path-connected and $x_0 \in X$ a point. Then one has*

$$\begin{aligned} H^k(X, x_0) &\cong H^k(X), \quad k \geq 1 \\ H^0(X, x_0) &= 0. \end{aligned}$$

(b) *Let $A \subset X$ be a deformation retract of X. For every k one has*

$$H^k(X, A) = 0.$$

(c) *Let $A \subset X$ be regularly closed. Then the quotient map $q : (X, A) \to (X/A, A/A)$ induces isomorphisms*

$$q^* : H^n(X/A, A/A) \xrightarrow{\cong} H^n(X, A).$$

Proof. Analogous to the corresponding proofs in homology. □

* * *

1.5 The Mayer-Vietoris Sequence

Theorem 1.5.1. *Let A, B be subsets with $X = \mathring{A} \cup \mathring{B}$. Then there is an exact sequence*

$$0 \to H^0(X) \xrightarrow{\psi} H^0(A) \oplus H^0(B) \to \dots$$

$$\dots \to H^k(X) \xrightarrow{\psi} H^k(A) \oplus H^k(B) \xrightarrow{\phi} H^k(A \cap B) \xrightarrow{\delta} H^{k+1}(X) \to \dots$$

Proof. We write $C_k(A+B)$ for $C_k(A) + C_k(B) \subset C_p(X)$. As $C_k(A+B)$ is free, Lemma 1.2.5 says that the exact sequence of chain complexes

$$0 \to C_k(A \cap B) \to C_k(A) \oplus C_k(B) \to C_k(A+B) \to 0$$

dualises to an exact sequence of cochain complexes

$$0 \to C^k(A+B, R) \xrightarrow{\psi} C^k(A, R) \oplus C^k(B, R) \xrightarrow{\phi} C^k(A \cap B, R) \to 0.$$

This induces a long exact sequence on cohomology, which coincides with the one in the theorem, except for the term $H^k(X)$, which is replaced by the cohomology of the complex $C^k = C^k(A+B)$. Let D be the complex $D^k = C^k(X)$. The inclusion $i : C_k(A+B) \hookrightarrow C_k(X)$ induces a dual map $i^\# : C^k(X) \to C^k(A+B)$. Let i^* and i_* be the induced maps on the (co-)homology. The map i_* is an isomorphism by Lemma 5.12.1 of AlgTop1. The Universal Coefficient Theorem yields a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1(H_{n-1}(C), R) \arrow[r] & H^n(C, R) \arrow[r, "h"] & \mathrm{Hom}(H_n(C), R) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}^1(H_{n-1}(D), R) \arrow[r] \arrow[u, "\alpha"'] & H^n(D, R) \arrow[r, "h"] \arrow[u, "i^*"'] & \mathrm{Hom}(H_n(D), R) \arrow[r] \arrow[u, "\beta"'] & 0
\end{tikzcd}$$

The maps α and β are isomorphisms, hence so is i^*. □

It is not hard to see that there also is a relative version:

Theorem 1.5.2. *Let A, B be subsets with $X = \mathring{A} \cup \mathring{B}$ and let $K \subset \mathring{A}$ and $L \subset \mathring{B}$ be closed sets. Then there is an exact sequence*

$$0 \to H^0(X, X \smallsetminus (K \cup L)) \xrightarrow{\psi} H^0(A, A \smallsetminus K) \oplus H^0(B, B \smallsetminus L) \to \dots$$

$$\dots \to H^k(X, X \smallsetminus (K \cup L)) \xrightarrow{\psi} H^k(A, A \smallsetminus K) \oplus H^k(B, B \smallsetminus L)$$

$$\xrightarrow{\phi} H^k(A \cap B, (A \cap B) \smallsetminus (K \cap L)) \xrightarrow{\delta} H^{k+1}(X, X \smallsetminus (K \cup L)) \to \dots$$

* * *

1.6 The Cup-product

Definition 1.6.1. From now on, the coefficient group R shall carry an extra structure of a commutative ring with unit, like for instance $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$ oder $\mathbb{Q}$. So maps to R can not only be added, but also multiplied. For two cochains $\alpha \in C^k(X,R)$ and $\beta \in C^l(X,R)$ we define the **cup-product** $\alpha \smile \beta \in C^{k+l}(X,R)$ by

$$(\alpha \smile \beta)(\sigma) = \alpha(\sigma|_{[v_0,\dots,v_k]})\beta(\sigma|_{[v_k,\dots,v_{k+l}]}),$$

where $\sigma : [v_0,\dots,v_{k+l}] \to X$ is a singular simplex and the product on the right takes place in R. The product extends to a bilinear map from $C^k(X,R) \times C^l(X,R)$ to $C^{k+l}(X,R)$.

Lemma 1.6.2. (a) *The cup-product is associative and distributive, i.e., for $\alpha \in C^k(X,R)$, $\beta \in C^l(X,R)$ and $\gamma \in C^r(X,R)$ one has*

$$\begin{aligned}(\alpha \smile \beta) \smile \gamma &= \alpha \smile (\beta \smile \gamma),\\ \alpha \smile (\beta + \gamma) &= \alpha \smile \beta + \alpha \smile \gamma,\\ (\beta + \gamma) \smile \alpha &= \beta \smile \alpha + \gamma \smile \alpha.\end{aligned}$$

The cup product makes $C^\bullet = \bigoplus_k C^k(X,R)$ a generally noncommutative ring.

(b) *The set $C^0(X,R) = \mathrm{Map}(X,R)$ forms a subring. The constant map $\mathbf{1} : x \mapsto 1$ is a unit of the ring $C^\bullet$. It satisfies $d\mathbf{1} = 0$.*

(c) *For $\alpha \in C^k(X,R)$ and $\beta \in C^l(X,R)$ one has*

$$d(\alpha \smile \beta) = d\alpha \smile \beta + (-1)^k \alpha \smile d\beta.$$

Proof. (a) follows from associativity of multiplication in the ring R.

(b) The first assertion is clear by definition. As for the second, let $\sigma : [v_0, v_1] \to X$ be a simplex. Then

$$d\mathbf{1}(\sigma) = \mathbf{1}(\partial\sigma) = \mathbf{1}\big(\sigma(v_1) - \sigma(v_0)\big) = 0.$$

(c) Let $\sigma : [v_0, \dots, v_{k+l+1}] \to X$ be a singular $k + l + 1$ simplex in X. Then one has

$$\begin{aligned}
d(\alpha \smile \beta)(\sigma) &= \sum_{j=0}^{k+l+1} (-1)^j (\alpha \smile \beta)(\sigma|_{[v_0,\dots\hat{v}_j\dots,v_{k+l+1}]}) \\
&= \sum_{j=0}^{k} (-1)^j \alpha(\sigma|_{[v_0,\dots\hat{v}_j\dots,v_{k+1}]})\beta(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]}) \\
&+ \sum_{j=k+1}^{k+l+1} (-1)^j \alpha(\sigma_{[v_0,\dots,v_k]})\beta(\sigma|_{[v_k,\dots\hat{v}_j\dots,v_{k+l+1}]}) \\
&= \sum_{j=0}^{k+1} (-1)^j \alpha(\sigma|_{[v_0,\dots\hat{v}_j\dots,v_{k+1}]})\beta(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]}) \\
&+ \sum_{j=k}^{k+l+1} (-1)^j \alpha(\sigma_{[v_0,\dots,v_k]})\beta(\sigma|_{[v_k,\dots\hat{v}_j\dots,v_{k+l+1}]}) \\
&= d\alpha \smile \beta(\sigma) + (-1)^k \alpha \smile d\beta(\sigma).
\end{aligned}$$

□

Definition 1.6.3. The lemma implies

$$Z^k \smile Z^l \ \subset \ Z^{k+l}$$

and

$$Z^k \smile B^l,\ B^k \smile Z^l \ \subset \ B^{k+l}.$$

Since $H^l = Z^l/B^l$, the cup-product yields an associative and distributive multiplication

$$H^k(X,R) \times H^l(X,R) \ \to \ H^{k+l}(X,R),$$

which turns

$$H^*(X,R) = \bigoplus_{k=0}^{\infty} H^k(X,R)$$

into a ring. This ring is called the **cohomology ring** of X with coefficients in R. This is a (non-commutative) ring with unit. The unit is induced by the unit $\mathbf{1}$ in $C^\bullet(X,R)$.

Proposition 1.6.4. *Let $f : X \to Y$ be continuous, Then the maps*

$$\begin{aligned}
f^\# &: C^\bullet(Y,R) \to C^\bullet(X,R), \\
f^* &: H^\bullet(Y,R) \to H^\bullet(X,R)
\end{aligned}$$

are unital ring homomorphisms.

Proof. This follows from

$$\begin{aligned}
f^\#(\alpha \smile \beta)(\sigma) &= (\alpha \smile \beta)(f \circ \sigma) \\
&= \alpha((f \circ \sigma)|_{[v_0,\dots,v_k]})\beta((f \circ \sigma)|_{[v_k,\dots,v_{k+l}]}) \\
&= f^\#(\alpha)(\sigma|_{[v_0,\dots,v_k]})\, f^\#(\beta)(\sigma|_{[v_k,\dots,v_{k+l}]}) \\
&= f^\#\alpha \smile f^\#\beta(\sigma).
\end{aligned}$$

□

* * *

1.7 Graded algebras

Definition 1.7.1. An R**-algebra** is a (not neccessarily commutative) ring A, which at the same time is an R-module, such that the multiplication map $A \times A \to A$ is R-bilinear. This means that one has

$$r(ab) = (ra)b = a(rb)$$

for $r \in R$ and $a, b \in A$.

Examples 1.7.2. (a) The set of $n \times n$ matrices, $A = \mathrm{M}_n(R)$, is an R-algebra with matrix multiplication.

(b) The polynomial ring $A = R[X]$ is a commutative R-algebra with unit.

(c) Every ring is a $\mathbb{Z}$-algebra.

Definition 1.7.3. An **algebra homomorphism** is a map $\phi : A \to B$ between R-algebras, which is an R-module homomorphism and a ring homomorphism.

Examples 1.7.4. (a) Let $\alpha \in R$, then $\phi_\alpha : R[X] \to R$, given by $f(X) \mapsto f(\alpha)$ is an algebra homomorphism.

(b) If S is an invertible matrix in $\mathrm{M}_n(R)$, then $A \mapsto SAS^{-1}$ is an algebra homomorphism from $\mathrm{M}_n(R)$ to itself.

Definition 1.7.5. An R-algebra A is called a **graded algebra**, if there are R-submodules A_n for $n = 0, 1, \dots,$ such that $A = \bigoplus_{n=0}^{\infty} A_n$ with

$$A_n A_m \subset A_{n+m}.$$

An element $a \in A_n$ is called **homogeneous**. An arbitrary element of A is a sum of homogeneous elements. Let for instance $a = a_0 + \dots + a_n$ with $a_j \in A_j$. If $a_n \neq 0$, then we say that s has **degree** n,

$$\deg(a) = n.$$

.

Examples 1.7.6. (a) The polynomial ring $A = R[x]$ is graded with $A_n = R \cdot x^n$

(b) The cohomology ring $A = H^*(X, R)$ is graded by $A_n = H^n(X, A)$.

Definition 1.7.7. A graded algebra A is called **graded-commutative**, if for $a \in A_k$ and $b \in A_l$ one has

$$ab = (-1)^{kl} ba.$$

Theorem 1.7.8. *The cohomology ring of a space X is graded-commutative. So for $\alpha \in H^k(X, R)$ and $\beta \in H^l(X, R)$. one has*

$$\alpha \smile \beta = (-1)^{kl} \beta \smile \alpha.$$

Proof. For an n-simplex $\sigma : [v_0, \dots, v_n] \to X$ let $\overline{\sigma}$ be the n-simplex $\overline{\sigma} : [v_n, \dots, v_0] \to X$ with the reversed order of vertices. Recall that the notation $[v_0, \dots, v_n]$ stands for the convex hull of the points $v_0, \dots, v_n \in$

$\mathbb{R}^N$, together with the affine isomorphism $[e_0, \ldots, e_n] \to [v_0, \ldots, v_n]$ given by $e_j \mapsto v_j$, where $e_0, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. Therefore, $[v_n, \ldots, v_0]$ is the same convex hull, together with $[e_0, \ldots, v_n] \to [v_n, \ldots, v_0]$ mapping $e_j \mapsto v_{n-j}$.

Therefore one has $\overline{\sigma}(v_i) = \sigma(v_{n-i})$. The reversal of order is a composition of $n + (n-1) + \cdots + 1 = n(n+1)/2$ transpositions of neighboured vertices. Let $\varepsilon_n = (-1)^{n(n+1)/2}$. Define a linear map $\rho : C_n(X) \to C_n(X)$ by $\rho(\sigma) = \varepsilon_n \overline{\sigma}$.

We claim that ρ is a chain map, which is chain-homotopic to the identity. This implies the theorem, since

$$\begin{aligned}(\rho^*\phi \smile \rho^*\psi)(\sigma) &= \phi(\varepsilon_k \sigma|_{v_k,\ldots,v_0]})\psi(\varepsilon_l|_{[v_{k+l},\ldots,v_k]})\\ \rho^*(\psi \smile \phi)(\sigma) &= \varepsilon_{k+l}\psi(\sigma|_{[v_{k+l},\ldots,v_k]})\phi(\sigma|_{[v_k,\ldots,v_0]})\end{aligned}$$

implies $\varepsilon_k\varepsilon_l(\rho^*\phi \smile \rho^*\psi) = \varepsilon_{k+l}\rho^*(\psi \smile \phi)$, as R is commutative. One has

$$\varepsilon_{k+l} = (-1)^{\frac{(k+l)(k+l+1)}{2}} = (-1)^{\frac{k^2+2k+l^2+k+l}{2}} = (-1)^{kl+\frac{k(k+1)+l(l+1)}{2}} = (-1)^{kl}\varepsilon_k\varepsilon_l.$$

Since $\rho^* = \mathrm{Id}$ on the cohomology, we get $\phi \smile \psi = (-1)^{kl}\psi \smile \phi$.

We need to show $\partial\rho = \rho\partial$ holds. For this let σ be an n-simplex. We compute

$$\begin{aligned}\partial\rho(\sigma) &= \varepsilon_n \sum_i (-1)^i \sigma|_{[v_n,\ldots\hat{v}_{n-i}\ldots,v_0]}\\ \rho\partial(\sigma) &= \rho\left(\sum_i (-1)^i \sigma|_{[v_0,\ldots\hat{v}_i\ldots,v_n]}\right)\\ &= \varepsilon_{n-1}\sum_i (-1)^{n-i}\sigma|_{[v_n,\ldots\hat{v}_{n-i}\ldots,v_0]}.\end{aligned}$$

One has $\varepsilon_n = (-1)^{\frac{n(n+1)}{2}} = (-1)^{\frac{n(n-1)}{2}+n} = \varepsilon_{n-1}(-1)^n$. This implies, that ρ is a chain map.

We now construct a chain-homotopy to the identity. Let Δ be an n-simplex. As in the construction of the prism-operator we divide $I \times \Delta \subset \mathbb{R}^{N+1}$ into $(n+1)$ simplices as follows. If

$$\begin{aligned}\left\{0\right\} \times \Delta &= [v_0, \ldots, v_n] \quad \text{and}\\ \left\{1\right\} \times \Delta &= [w_0, \ldots, w_n],\end{aligned}$$

Then $I \times \Delta$ is the union of the simplices

$$[v_0, \ldots, v_i, w_i, \ldots, w_n], \qquad i = 0, \ldots, n.$$

Let $\pi : I \times \Delta \to \Delta$ be the projection. We define $P : C_n(X) \to C_{n+1}(X)$ by

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i \varepsilon_{n-i}(\sigma \circ \pi)|_{[v_0,\ldots,v_i,w_n,\ldots,w_i]}.$$

We want to show $\partial P + P\partial = \rho - \mathrm{Id}$. For this we compute

$$\begin{aligned}\partial P(\sigma) &= \partial\left(\sum_i (-1)^i \varepsilon_{n-i}\sigma \circ \pi|_{[v_0,\ldots,v_i,w_n,\ldots,w_i]}\right)\\ &= \sum_{j\le i}(-1)^{i+j}\varepsilon_{n-i}\sigma\circ\pi|_{[v_0,\ldots\hat{v}_j\ldots,v_i,w_n\ldots,w_i]}\\ &+\sum_{j\ge i}(-1)^{n-j+1}\varepsilon_{n-i}\sigma\pi|_{[v_0,\ldots,v_i,w_n,\ldots\hat{w}_j\ldots,w_i]}.\end{aligned}$$

The terms with $i = j$ yield

$$\underbrace{\varepsilon_n\sigma\circ\pi|_{[w_n,\ldots,w_0]}}_{i=0} + \sum_{i>0}\varepsilon_{n-i}\sigma\circ\pi|_{[v_0,\ldots,v_{i-1},w_n,\ldots,w_i]}$$

$$+\sum_{i<n}(-1)^{n+i+1}\varepsilon_{n-i}\sigma\circ\pi|_{[v_0,\ldots,v_i,w_n,\ldots,w_{i+1}]} - \sigma\circ\pi|_{[v_0,\ldots,v_n]}.$$

Replacing i in the second sum by $i-1$, using $(-1)^{n+i}\varepsilon_{n-i+1} = -\varepsilon_{n-i}$, one sees that these two sums cancel each other. The remaining terms give $\rho(\sigma) - \sigma$. It remains to show that the terms with $i \neq j$ give $-P\partial$. The definition yields

$$\begin{aligned}P\partial(\sigma) &= \sum_{i<j}(-1)^{i+j}\varepsilon_{n-i-1}\sigma\circ\pi|_{[v_o,\ldots,v_i,w_n,\ldots\hat{w}_j\ldots,w_i]}\\ &+\sum_{i>j}(-1)^{i+j-1}\varepsilon_{n-i}\sigma\circ\pi|_{[v_0,\ldots\hat{v}_j\ldots,v_i,w_n,\ldots,w_i]}\end{aligned}$$

By $\varepsilon_{n-i} = (-1)^{n-i}\varepsilon_{n-i-1}$ the claim follows. □

* * *

1.8 The Künneth formula

Definition 1.8.1. The **tensor produkt** of two abelian groups A, B is the group with generators $a \otimes b$ for $a \in A$ and $b \in B$ and relations

$$(a + a') \otimes b = a \otimes b + a' \otimes b,$$
$$a \otimes (b + b') = a \otimes b + a \otimes b'.$$

Examples 1.8.2. (a) For every abelian group one has $\mathbb{Z} \otimes A \cong A$.

(b) $\mathbb{Q} \otimes (\mathbb{Z}/n\mathbb{Z}) = 0$.

Definition 1.8.3. If R is a commutative ring with unit and if M, N are R-modules, then the R-module $M \otimes_R N$ is defined as the quotient $M \otimes N$ modulo the subgroup generated by all elements of the form

$$rm \otimes n - m \otimes rn, \quad r \in R.$$

The group $M \otimes_R N$ becomes an R-module by

$$r(m \otimes n) := rm \otimes n.$$

Examples 1.8.4. (a) For every R-Modul M one has $R \otimes_R M \cong M$.

(b) If $R = \mathbb{Q}(\sqrt{2})$, then $R \otimes_R R \cong R$, but $R \otimes_{\mathbb{Q}} R$ is a 4-dimensional $\mathbb{Q}$-vector space.

(c) If V, W are vector spaces over a field F with bases $v_1, \dots, v_n$ and $w_1, \dots, w_n$, then a basis for the vector space $V \otimes W$ is given by

$$(v_i \otimes w_j)_{\substack{1 \le i \le n \\ 1 \le j \le m}}.$$

In particular, one has

$$\dim\left(V \otimes W\right) = \left(\dim V\right)\left(\dim W\right).$$

Definition 1.8.5. If A, B are algebras over R, then $A \otimes B$ becomes an R-algebra with product

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'.$$

Example 1.8.6. With this product the matrix algebra $\mathrm{M}_m(R) \otimes \mathrm{M}_n(R)$ is isomorphic to $\mathrm{M}_{mn}(R)$. An isomorphism $\mathrm{M}_m(R) \otimes \mathrm{M}_n(R) \to \mathrm{M}_{mn}(R)$ is given as follows: let $A = (a_{ij}) \in M_m(R)$ and $B \in M_n(R)$. Then one maps $A \otimes B$ to the matrix

$$\begin{pmatrix} a_{1,1}B & \dots & a_{1,m}B \\ \vdots & & \vdots \\ a_{m,1}B & \dots & a_{m,m}B \end{pmatrix} \in M_{mn}(R).$$

Definition 1.8.7. If the algebras A and B are graded, then there is a second product, the **graded product**, defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg(b)\deg(a')} aa' \otimes bb',$$

where the elements b and a' are homogeneous. For arbitrary elements this product is extended bilinearly. The ensuing algebra is called the **graded tensor-product algebra**.

The **external cup-product** is defined as a map

$$H^k(X,R) \times H^l(Y,R) \xrightarrow{\times} H^{k+l}(X \times Y, R)$$

by $(a,b) \mapsto a \times b = p_1^*(a) \smile p_2^*(b)$, here p_1 and p_2 are the coordinate projections of $X \times Y$. The same formula defines the relative version

$$H^k(X,A,R) \times H^l(Y,B,R) \xrightarrow{\times} H^{k+l}(X \times Y, A \times B, R).$$

The external product is R-bilinear, hence it defines an R-linear map

$$\psi : H^*(X,R) \otimes_R H^*(Y,R) \ \to \ H^*(X \times Y, R).$$

Theorem 1.8.8. (a) *Let X and Y be topological spaces. Equip $H^*(X,R) \otimes_R H^*(Y,R)$ with the structure of the graded tensor-product algebra, then the external cup-product is an algebra homomorphism.*

(b) *If X is a CW-complex and $H^k(Y,R)$ is a finitely generated free R-module for every $k \geq 0$, then the external cup-product is an isomorphism. For every n one has*

$$H^n(X \times Y, R) \cong \bigoplus_{k+l=n} H^k(X,R) \otimes_R H^l(Y,R).$$

Proof. (a) Let $a, a' \in H^*(X,R)$ and $b, b' \in H^*(Y,R)$ be homogeneous elements. Then one has

$$\begin{aligned}
\psi\Big((a \otimes b)(a' \otimes b')\Big) &= (-1)^{\deg(b)\deg(a')}\psi(aa' \otimes bb') \\
&= (-1)^{\deg(b)\deg(a')}p_1^*(aa') \smile p_2(bb') \\
&= (-1)^{\deg(b)\deg(a')}p_1^*(a) \smile p_1^*(a') \smile p_2^*(b) \smile p_2^*(b') \\
&= p_1^*(a) \smile p_2^*(b) \smile p_1^*(a') \smile p_2^*(b') \\
&= \psi(a \otimes b)\psi(a' \otimes b').
\end{aligned}$$

Therefore, ψ is an algebra homomorphism.

(b) We fix Y and write ψ_X for the algebra-homomorphism given by the external cup product. We write

$$U^k(X) = \bigoplus_{i+j=k} H^i(X,R) \otimes_R H^j(Y,R),$$

and

$$V^k(X) = H^k(X \times Y, R).$$

note that since each $H^j(Y,R)$ is finitely generated and free, we get $U^k(\bigsqcup_\nu Z_\nu) = \prod_\nu U^k(Z_\nu)$ for any family (Z_ν) of spaces. We show by induction on n, that $\psi_{X_n} : U^k(X_n) \to V^k(X_n)$ is an isomorphism. For $n = 0$

this is clear as $\psi_{\mathrm{pt}} : U^k(\mathrm{pt}) \to V^k(\mathrm{pt})$ is an isomorphism. Now let $n \geq 1$ and write

$$X_n/X_{n-1} \cong \bigsqcup_\alpha e_\alpha \Big/ \bigsqcup_\alpha \partial e_\alpha,$$

where $e_\alpha \cong \mathbb{D}^n$, and the interiors are the cells of dimension n. As every e_α is contractible, one gets

$$U^k\left(\bigsqcup_\alpha e_\alpha\right) \cong \prod_\alpha U^k(e_\alpha) \cong \prod_\alpha U^k(\mathrm{pt}) \cong V^k\left(\bigsqcup_\alpha e_\alpha\right).$$

By induction, we can assume that ψ induces isomorphisms

$$U^k\left(\bigsqcup_\alpha \partial e_\alpha\right) \cong V^k\left(\bigsqcup_\alpha \partial e_\alpha\right).$$

The exact sequence of the pair $(\bigsqcup_\alpha e_\alpha, \bigsqcup_\alpha \partial e_\alpha)$ together with the five-lemma implies that ψ gives isomorphisms $U^k(X_n/X_{n-1}) \cong V^k(X_n/X_{n-1})$. The exact sequence of the pair (X_n, X_{n-1}) and the five lemma give that $\psi : U^k(X_n) \cong V^k(X_n)$ for all k, n. The fact that X is the union of the X_n induces $U^k(X) \cong \varprojlim_n U^k(X_n)$ and the same for V^k. The map ψ, being given by the external cup product, is compatible with these projective limits and so ψ is an isomorphism on $V^k(X)$ as it is so on $V^k(X_n)$. □

Example 1.8.9. Let $\mathbb{R}^n/\mathbb{Z}^n \cong \mathbb{T}^n$ be the n=dimensional torus. Let α be a generator of the free R-module $H^1(\mathbb{T}, R)$ and let $\alpha_j = p_j^*(\alpha) \in H^1(\mathbb{T}^n, R)$, where $p_j : T^n \to \mathbb{T}$ is the projection onto the j-th factor. We claim that $H^k(\mathbb{T}^n, R)$ is the free R-module generated by all elements $\alpha_{i_1} \smile \ldots \smile \alpha_{i_k}$, where $1 \leq i_1 < \cdots < i_p \leq n$. In particular it follows $H^k(\mathbb{T}^n, R) \cong R^N$ with $N = \begin{pmatrix} n \\ k \end{pmatrix}$. This indeed follows from the Künneth formula and an induction on n.

* * *

1.9 Orientations

Lemma 1.9.1. *Let $n \in \mathbb{N}$ and let M be an n-dimensional manifold and $x \in M$. Then*

$$H_k\Big(M, M \setminus \{x\}\Big) \cong \begin{cases} \mathbb{Z} & k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The point x has an open neighbourhood $U \cong \mathbb{R}^n$ and by excision we get $H_k(M, M \setminus \{x\}) \cong H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. We first consider the case $n \geq 2$. Note that $\mathbb{R}^n \setminus \{0\}$ is homotopy equivalent to S^{n-1}. The long exact sequence of relative Homology states

$$\cdots \to H_k\Big(\mathbb{R}^n \setminus \{0\}\Big) \to H_k\Big(\mathbb{R}^n\Big) \to H_k\Big(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}\Big) \xrightarrow{\delta} H_{k-1}\Big(\mathbb{R}^n \setminus \{0\}\Big) \to \dots$$

$$\cdots \to \underbrace{H_1\Big(\mathbb{R}^n\Big)}_{0} \to H_1\Big(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}\Big) \to$$
$$\underbrace{H_0\Big(\mathbb{R}^n \setminus \{0\}\Big)}_{\mathbb{Z}} \xrightarrow{\cong} \underbrace{H_0\Big(\mathbb{R}^n\Big)}_{\mathbb{Z}} \to H_0\Big(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}\Big) \to 0,$$

from which the claim can be read off. In the case $n = 1$ we have $H_1\Big(\mathbb{R} \setminus \{0\}\Big) \cong \mathbb{Z}^2$ and the map to $H_0(\mathbb{R})$ is surjective, hence the claim follows in this case, too. ◻

Definition 1.9.2. Let M be an n-dimensional manifold. For a subset $A \subset M$ we write

$$A^c = M \setminus A$$

and call this set the **complement** of A.

Lemma 1.9.3. *Let $B \subset A \subset X$ and suppose that B is a deformation retract of A. Then the inclusion $\alpha : C_k(B) \hookrightarrow C_k(A)$ induces a chain map $\beta : C_k(X, B) \to C_k(X, A)$ and the latter induces an isomorphism*

$$H_k(X, B) \xrightarrow{\cong} H_k(X, A)$$

for every k.

Proof. We get a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & C_k(B) \arrow[r] \arrow[d, "\alpha"] & C_k(X) \arrow[r] \arrow[d, "="] & C_k(X, B) \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & C_k(A) \arrow[r] & C_k(X) \arrow[r] & C_k(X, A) \arrow[r] & 0 \end{tikzcd}$$

The commutativity of this diagram implies that these give commutative diagrams on homology

$$\begin{tikzcd}
\dots \arrow[r] & H_k(B) \arrow[r] \arrow[d, "\alpha_*"] & H_k(X) \arrow[r] \arrow[d, "="] & H_k(X,B) \arrow[r] \arrow[d, "\beta_*"] & H_{k-1}(B) \arrow[r] \arrow[d, "\alpha_*"] & \dots \\
\dots \arrow[r] & H_k(A) \arrow[r] & H_k(X) \arrow[r] & H_k(X,A) \arrow[r] & H_{k-1}(A) \arrow[r] & \dots
\end{tikzcd}$$

As α_* is an isomorphism, the five-lemma implies that β_* is, too. □

Definition 1.9.4. A **local orientation** at the point $x \in M$ is the choice of a generator of the group $H_n(M, M \setminus \{x\}) = H_n(M, \{x\}^c)$.

Given a choice of a local orientation at every point x. For any two x, y in a chart U one chooses a ball B in $U \cong \mathbb{R}^n$ which contains x and y, then B^c is a deformation retract of $\{x\}^c$ as well as $\{y\}^c$ and so there are canonical isomorphisms

$$H_n(M, \{x\}^c) \cong H_n(M, B^c) \cong H_n(M, \{y\}^c).$$

If these isomorphisms map the orientation at x to the one at y, then the choice of local orientations is called **compatible**.

An **orientation** is a compatible choice of local orientations. An orientation does not necessarily exist. If it does, we call the manifold M **orientable**. If M is orientable and connected, there are exactly 2 different orientations.

Proposition 1.9.5. *Let M be a manifold. There exists an orientable covering $\widehat{M} \to M$ of degree 2.*

In particular, if M is connected and the fundamental group Γ has no subgroup of index 2, then M is orientable.

Proof. Let

$$\widehat{M} = \left\{ \tau_x : x \in M \right\}$$

be the set of all local orientations of points $x \in M$. The covering map $p : \widehat{M} \to M$ ist $\tau_x \mapsto x$. For a point $\tau_x \in \widehat{M}$ and a chart (U, ϕ) around the point $x \in M$, let V be the set of all τ_y, $y \in U$ such that τ_x and τ_y induce the same element in $H_n(M, U^c)$. By the uniqueness of τ_y the map $\tau_y \mapsto \phi(y)$ is a chart on V, which makes $\widehat{M}$ a covering manifold of M. This manifold is orientable, since $\tau_x \in \widehat{M}$ can also be viewed as an orientation in $H_n\big(V, V \setminus \{\tau_x\}\big)$.

The addendum follows as a 2-sheeted connected covering is the quotient of the universal covering $\tilde{M}$ by a subgroup of Γ of index 2. □

* * *

1.10 Poincaré duality

Definition 1.10.1. Let R be an abelian group. Let

$$C_k(X,R) = C_k(X) \otimes R.$$

Then $\partial \otimes \mathrm{Id}$ makes this a chain complex and we define its homology to be the **homology with coefficients** in R,

$$H_k(X,R) = H_k\Big(C_\bullet(X,R)\Big).$$

Definition 1.10.2. Let $p : \Omega \to X$ be a continuous map. A **section** to p is a continuous map $s : X \to \Omega$ such that $p(s(x)) = x$ for every $x \in X$.

If a section exists, p must be surjective.

Examples 1.10.3.

- The projection $p : \mathbb{R}^{n+k} \to \mathbb{R}^n$ has a natural section given by the embedding $s : \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k}$.
- The map $p : \mathbb{T} \to \mathbb{T}$, $z \mapsto z^2$ does not have a section, as that would entail the existence of a continuous square root.

Definition 1.10.4. Let M be an n-dimensional manifold and let $M_{\mathbb{Z}}$ denote the disjoint union of all $H_n(M, \{x\}^c)$ as x runs through M. Then, as $\hat{M} \subset M_{\mathbb{Z}}$, which consists of generators only, the set $M_{\mathbb{Z}}$ carries a natural topology, making the map $M_{\mathbb{Z}} \to M$ a covering of infinite degree.

Lemma 1.10.5. *Let M be a manifold of dimension n and let $A \subset M$ be a compact subset.*

(a) *If $x \mapsto \alpha_x$ is a section of the covering space $M_{\mathbb{Z}} \to M$, then there exists a uniquely determined class $\alpha_A \in H_n(M, A^c)$, whose image in $H_n(M, \{x\}^c)$ is α_x for every $x \in A$.*

Note that this does not claim the existence of a section, but only the "globalisation" of a given section.

(b) *$H_k(M, A^c) = 0$ for all $k > n$.*

Proof. (1) We observe that if the lemma holds for compact sets A, B and $A \cap B$, then it holds for $A \cup B$. First we consider the case $k \geq n$. The relative Mayer-Vietoris sequence gives:

$$\begin{aligned} 0 \to H_k\Big(M, (A \cup B)^c\Big) &\xrightarrow{\phi} H_k\Big(M, A^c\Big) \oplus H_k\Big(M, B^c\Big) \\ &\xrightarrow{\psi} H_k\Big(M, (A \cap B)^c\Big). \end{aligned}$$

The zero upfront comes from the assumption that $H_{k+1}\Big(M, (A \cap B)^c\Big) = 0$. One has $\phi(\alpha) = (\alpha, \alpha)$ and $\psi(\alpha, \beta) = \alpha - \beta$. For $k > n$, the middle term is zero by assumption, so then we have $H_k\Big(M, (A \cup B)^c\Big) = 0$. This proves (b).

For (a) let $x \to \alpha_x$ be a section. The hypotheses gives unique classes $\alpha_A \in H_n(M, A^c)$, $\alpha_B \in H_n(M, B^c)$ and $\alpha_{A \cap B} \in H_n\Big(M, (A \cap B)^c\Big)$ having image α_x for all x in $A, B, A \cap B$ respectively. By uniqueness, the images of α_A and α_B in $H_n\Big(M, (A \cap B)^c\Big)$ both equal to $\alpha_{A \cap B}$. The exactness of the sequence implies that

$(\alpha_A, \alpha_B) = \phi(\alpha_{A\cup B})$ for a uniquely determined $\alpha_{A\cup B} \in H_n(M, (A \cup B)^c)$. This then means that $\alpha_{A\cup B}$ has image α_x at every $x \in A \cup B$ as required. This finishes the proof of (1).

(2) We now reduce to the case $M = \mathbb{R}^n$. The compact set A can be written as a union $A_1 \cup \cdots \cup A_m$, where each A_k is contained in a chart $U \cong \mathbb{R}^n$. We repeatedly apply part 1) to reduce the claim to the single A_j or intersections of those. By excision, we then can replace M by $\mathbb{R}^n$.

(3) Let $M = \mathbb{R}^n$ and A a finite union of convex compact sets $A_1, \ldots, A_m$, then, as before, we reduce to the case $m = 1$. When A is convex, then $\{x\}^c$ deformation retracts to A^c for every $x \in A$.

(4) Let now $A \subset \mathbb{R}^n$ be an arbitrary compact set. Let $\alpha \in H_k(\mathbb{R}^n, A^c)$ be represented by a relative cycle z and let $C \subset A^c$ be the union of the images of the singular simplices in ∂z. Since C is compact, it has a positive distance δ from A. We cover A by finitely many closed balls which do not meet C. Let K be the union of these balls, then z defines an element α_K in $H_k(\mathbb{R}^n, K^c)$, mapping to $\alpha \in H_k(\mathbb{R}^n, A^c)$. If $k > n$, then $H_k(\mathbb{R}^n, K^c) = 0$, so $\alpha_K = 0$, hence $\alpha = 0$ and so $H_k(\mathbb{R}^n, A^c) = 0$. If $k = n$ and α_x is zero in $H_n(\mathbb{R}^n, \{x\}^c)$ for all $x \in A$, then the same is true for all $x \in K$, as K is a union of closed balls B meeting A and $H_n(\mathbb{R}^n, B^c) \to H_n(\mathbb{R}^n, \{x\}^c)$ is an isomorphism for $x \in B$. Then by 3), $\alpha_K = 0$, and so is α. This finishes the uniqueness in (a). The existence is clear, as we can choose α_A to be the image of α_B for any ball $B \supset A$. □

Theorem 1.10.6. *Let M be a connected compact connected manifold of dimension n.*

(a) *If M is orientable, the map $H_n(M) \to H_n(M, \{x\}^c) \cong \mathbb{Z}$ is an isomorphism for every $x \in M$.*

(b) *If M is non-orientable, the group $H_n(M)$ is zero.*

(c) $H_k(M) = 0$ *for $k > n$.*

Proof. In Lemma 1.10.5, we can choose $A = M$. Part (c) of the theorem is immediate by part (b) of the lemma. Finally, let $\Gamma(M)$ be the set of sections of $M_{\mathbb{Z}} \to M$. Then $\Gamma(M)$ is a $\mathbb{Z}$-module. There is a map $H_n(M) \to \Gamma(M)$ sending a class α to the section $x \mapsto \alpha_x$ = the image of α under the map $H_n(M) \to H_n(M, \{x\}^c)$. By part (a) of the lemma, this homomorphism is an isomorphism, so $\Gamma(M) \cong H_n(M)$. We claim that for a given $x_0 \in M$ the evaluation map

$$\begin{aligned} \delta_{x_0} : \Gamma(M) &\to H_n(M, \{x_0\}^c) \cong \mathbb{Z} \\ s &\mapsto s(x_0) \end{aligned}$$

is an isomorphism. For injectivity, let $s \in \Gamma(M)$ with $s(x_0) = 0$. Choose an orientation, i.e., a compatible isomorphism $\phi_x : H_n(M, \{x\}^c) \cong \mathbb{Z}$. Let $k \in \mathbb{Z}$ and let $M_k \subset M$ be the set of all $x \in M$ with $\phi_x(s(x)) = k$. Let $x \in M_k$. By compatibility, we have $U \subset M_k$ for any chart U around x. Therefore, M_k is open. Its complement $M_k^c = \bigcup_{l\neq k} M_l$ is open, too. As M is connected and $M_0 \neq \emptyset$, we get $M = M_0$, so $s = 0$ and δ_{x_0} is injective.

For surjectivity, note that orientability of M implies that $\delta_{x_0}(\Gamma(M))$ contains a generator of $H_n(M, \{x_0\}) \cong \mathbb{Z}$. The theorem is proven. □

Definition 1.10.7. If M is compact, connected and orientable, the theorem implies that there is a class $\omega \in H_n(M)$ such that ω induces a generator of $H_n(M, \{x\}^c)$ for one, and hence every, $x \in M$. Such a class ω is called a **fundamental class**. It is uniquely determined up to sign.

Definition 1.10.8. For a ring R, a space X and indices $k \geq l$ we define an R-bilinear pairing

$$\begin{aligned} \frown : C_k(X,R) \times C^l(X,R) &\to C_{k-l}(X,R) \\ (\sigma, \alpha) &\mapsto \alpha\Big(\sigma|_{[v_0,\dots,v_l]}\Big)\, \sigma|_{[v_l,\dots,v_k]} \end{aligned}$$

Lemma 1.10.9. *One has*

$$\partial(\sigma \frown \alpha) = (-1)^l\Big(\partial\sigma \frown \alpha - \sigma \frown d\alpha\Big)$$

Therefore the product of a cycle and a cocycle is a cycle, so the cap product induces a bilinear map

$$H_k(X,R) \times H^l(X,R) \to H_{k-l}(X,R).$$

Proof. By degree reasons, both sides are zero if $k = l$. So we assume $k \geq l+1$. Setting $\sigma_j = \sigma_{[v_0,\dots\widehat{v_j}\dots,v_k]}$ we compute

$$\begin{aligned} \partial\sigma \frown \alpha &= \sum_{j=0}^{k}(-1)^j \sigma_j \frown \alpha \\ &= \sum_{j=0}^{l}(-1)^j \alpha\Big(\sigma|_{[v_0,\dots\widehat{v_j}\dots,v_{l+1}]}\Big)\sigma|_{[v_{l+1},\dots,v_k]} \\ &\quad + \sum_{j=l+1}^{k}(-1)^j \alpha\Big(\sigma|_{[v_0,\dots,v_l]}\Big)\sigma_{[v_l,\dots\widehat{v_j}\dots,v_k]}, \\ \sigma \frown d\alpha &= \sum_{j=0}^{l+1}(-1)^j \alpha\Big(\sigma|_{[v_0,\dots\widehat{v_j}\dots,v_{l+1}]}\Big)\sigma|_{[v_{l+1},\dots,v_k]}, \\ \partial\Big(\sigma \frown \alpha\Big) &= \sum_{j=l}^{k}(-1)^{j-l} \alpha\Big(\sigma|_{v_0,\dots,v_l]}\Big)\, \sigma|_{[v_l,\dots\widehat{v_j}\dots,v_k]}. \end{aligned}$$

This implies the claim. □

Theorem 1.10.10. *Let M be a compact connected orientable manifold and let $\omega \in H_n(M)$ be a fundamental class. Then the map*

$$\begin{aligned} D : H^k(M) &\to H_{n-k}(M), \\ \alpha &\mapsto \omega \frown \alpha \end{aligned}$$

is an isomorphism for every k.

Proof. The proof is involved and shall not be given here, as this lecture has other goals. At this point, it is sufficient to understand, how the structure of manifold and the notion of orientation go into the duality theorem. The details of the proof are technical and one learns little from them. $\square$

* * *

1.11 De Rham and group cohomology

Definition 1.11.1. Let M be a smooth manifold and let $\Omega^k(M)$ be the real vector space of smooth k-differential forms. The exterior differential $d^k : \Omega^k \to \Omega^{k+1}$ satisfies $d^{k+1}d^k = 0$, so one can define the **de Rham Cohomology** of M as

$$H^k_{\mathrm{dR}}(M) := \ker(d^k)/\operatorname{im}(d^{k-1}).$$

Let $\sigma : \Delta^k \to M$ be a smooth map. Then a k-differential form ω can be integrated over the image of σ, we denote this as

$$\int_\sigma \omega.$$

We get a bilinear map

$$C_k(M) \times \Omega^k(M) \to \mathbb{R},$$

which a priori is only defined for smooth elements on $C_k(X)$, but can be extended by approximation. Then Stoke's Theorem says

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega.$$

This implies that the above pairing yields a bilinear pairing

$$H_k(M) \times H^k_{\mathrm{dR}}(M) \to \mathbb{R}.$$

One can show that, if M is orientable, this pairing induces an isomorphism

$$H^k_{\mathrm{dR}}(M) \cong H_k(M)^*.$$

Where the right hand side denotes the real vector space of all group homomorphisms from $H_n(M)$ to $(\mathbb{R}, +)$.

The deRham cohomology gives rise to Lie-algebra cohomology, which is connected to group cohomology of Lie groups. For complex manifolds one can decompose the exterior differential into "holomorphic" and "anti-holomorphic" parts, which gives rise to the **Dolbeault cohomology**.

Group cohomology

Definition 1.11.2. For a group Γ, one defines the group cohomology as the cohomology of its classifying space

$$H^k(\Gamma, R) = H^k(B\Gamma, R).$$

This looks a bit roundabout, but there is a host of purely algebraic definitions of group cohomology. I recommend the book by Brown *Cohomology of groups* for a taster. The group cohomology of Galois groups plays an important role in number theory.

Later we shall define group cohomology from a different viewpoint and also with more general coefficients, meaning that we shall replace the group R with a Γ-module.

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2 Categories and functors

2.1 Categories

Definition 2.1.1. A **category** is a triple $(\mathrm{Ob}, \mathrm{Hom}, \circ)$ where Ob is a class, the elements of which are called **objects** of the category. Hom is a family of sets $(\mathrm{Hom}(X,Y))_{X,Y\in \mathrm{Ob}}$. The elements of $\mathrm{Hom}(X,Y)$ are called **morphisms** from X to Y. Finally, $\circ$ is a family of maps: for any three objects X, Y, Z:

$$\begin{aligned} \mathrm{Hom}(X,Y)\times \mathrm{Hom}(Y,Z) &\to \mathrm{Hom}(X,Z)\\ (f,g) &\mapsto g\circ f, \end{aligned}$$

such that

- $g\circ(f\circ h) = (g\circ f)\circ h$ whenever the morphisms are composable.
- For every object X there is a morphism $\mathbf{1}_X \in \mathrm{Hom}(X,X)$ with $f\circ \mathbf{1}_X = f$ and $\mathbf{1}_X\circ g = g$ for all f, g for which the respective composition exists.

Remark 2.1.2. (a) The unit morphism is uniquely determined, for let $\mathbf{1}'_X$ be a second one, then

$$\mathbf{1}_X = \mathbf{1}_X\mathbf{1}'_X = \mathbf{1}'_X.$$

(b) As in the case of maps, the composition changes order, so $g\circ f$ has to be read as "g after f".

Examples 2.1.3. (a) SET is the category of sets and maps with the usual composition.

(b) AB is the category of abelian groups and group homomorphisms.

(c) RING is the category of rings with unit element (not necessarily commutative). Morphisms are unital ring homomorphisms $\phi : R \to S$ with $\phi(1_R) = 1_S$.

(d) TOP is the category of topological spaces and continuous maps.

(e) TOP_* is the category of **pointed spaces**, i.e., objects are pairs (X, x_0) where X is a topological space and $x_0 \in X$ a point. A morphism from (X, x_0) to (Y, y_0) is a continuous map $f : X \to Y$ with $f(x_0) = y_0$.

(f) Let $\mathcal{C}$ be a category. Then $\mathcal{C}^{\mathrm{opp}}$ is the **opposite category** in which all arrows are turned artound. The category $\mathcal{C}^{\mathrm{opp}}$ has the same objects as $\mathcal{C}$, but

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{opp}}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X).$$

(g) A group can be viewed as a category with only one object. This means that for a group G one defines a category $\mathcal{G}$ with only one object X and $\mathrm{Hom}_{\mathcal{G}}(X,X) := G$. The composition in this category is the one given by the group structure.

(h) Let $(A, \geq)$ be a partially ordered set. Then one defines a category with $\mathrm{Ob} = A$, by saying that $\mathrm{Hom}(x,y)$ has exactly one element, if $x \leq y$ and $\mathrm{Hom}(x,y) = \emptyset$ otherwise.

(i) The **homotopy category** [TOP]: The objects are topological spaces and the morphisms are free homotopy classes $[f]$ of continuous maps (See definition below).

(j) Let $\mathcal{A}$ and $\mathcal{B}$ be categories. The **product category** $\mathcal{A}\times\mathcal{B}$ has as object class the class of all pairs (X, Y), where $X \in \mathcal{A}$ and $Y \in \mathcal{B}$. Further one has

$$\mathrm{Hom}_{\mathcal{A}\times\mathcal{B}}\big((A, B), (X, Y)\big) = \mathrm{Hom}(A, X) \times \mathrm{Hom}(B, Y)$$

and the composition is given coordinate-wise.

Definition 2.1.4. Two continuous maps $f, g : X \to Y$ between topological spaces are called (freely) **homotopic**, if there exists a continuous map $h : I \times X \to Y$, where $I = [0, 1]$ is the unit interval, such that

$$h(0, x) = f(x), \quad h(1, x) = g(x)$$

for every $x \in X$. The map h is called a **homotopy** from f to g.

Examples 2.1.5. • Any map $f : X \to \mathbb{R}$ is homotopic to the constant map $g(x) = 0$. A homotopy is given by

$$h(s, x) = (1 - s)f(x).$$

- For $k \in \mathbb{Z}$ let $f_k : S^1 \to S^1$ be defined as $f_k(z) = z^k$. then f_k and f_l are not homotopic, if $k \neq l$.

Definition 2.1.6. We like to visualise morphisms by diagrams like this one:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

We say, that a diagram is **commutative**, if, any two ways to get from one node A to another node B, must coincide. So the above diagram is commutative, if and only if the morphism $h \in \mathrm{Hom}(X, Z)$ is the composition of f and g.

Definition 2.1.7. A morphism $f : X \to Y$ in a category is called an **isomorphism**, if there exists a morphism $g : Y \to X$ with

$$g \circ f = \mathbf{1}_X \quad \text{and} \quad f \circ g = \mathbf{1}_Y.$$

Examples 2.1.8. (a) The isomorphisms in the category of sets are the bijections.

(b) Isomorphisms in the category of groups are the group isomorpisms.

(c) Isomorphisms in the category TOP are the homeomorphisms.

(d) An isomorphism in the homotopy category are called **homotopy equivalence**.

Definition 2.1.9. Let $\mathcal{A}$ be a category. A **subcategory** is a category $\mathcal{B}$ such that $\mathrm{Ob}(\mathcal{B}) \subset \mathrm{Ob}(\mathcal{A})$, one has

$$\mathrm{Hom}_{\mathcal{B}}(X, Y) \subset \mathrm{Hom}_{\mathcal{A}}(X, Y)$$

for all $X, Y \in \mathcal{B}$, and the composition and units in $\mathcal{B}$ are the ones of $\mathcal{A}$. A subcategory $\mathcal{B}$ is called a **full subcategory**, if for any two $X, Y \in \mathcal{B}$ one has $\mathrm{Hom}_{\mathcal{B}}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(X, Y)$. Every subclass of $\mathrm{Ob}(\mathcal{A})$ defines exactly one full subcategory.

Example 2.1.10.

The category of finite groups is a full subcategory of the category GRP of all groups.

Definition 2.1.11. A full subcategory $\mathcal{A}' \subset \mathcal{A}$ is called **dense**, if for every $X \in \mathcal{A}$ there is a $X' \in \mathcal{A}'$, such that X' is isomorphic to X.

Example 2.1.12. Let K be a field and $\mathcal{A}$ the category of all finite-dimensional K-vector spaces and linear maps. Then, as you learn in Linear Algebra, the full subcategory $\mathcal{A}'$, whose objects are $\{0\}, K, K^2, K^3, \ldots$ a dense subcategory of $\mathcal{A}$.

* * *

2.2 Epis, Monos and products

Definition 2.2.1. A morphismus $f : X \to Y$ is called an **epimorphism** or **epi**, if for any two morphisms $\alpha, \beta : Y \to Z$ the following is true: if in the (non-cummutative) diagram

$$X \xrightarrow{f} Y \underset{\beta}{\overset{\alpha}{\rightrightarrows}} Z$$

the upper and lower paths from X to X agree, then one has $\alpha = \beta$. In other words, f is an epi, if the commutativity of a diagram of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & Z \end{tikzcd}$$

implies $\alpha = \beta$. Yet another way to say this is, that f is an epi iff it has the **right-cancellation property**:

$$\alpha \circ f = \beta \circ f \quad \Rightarrow \quad \alpha = \beta$$

holds for all morphisms α and β which are composable with f.

Examples 2.2.2. (a) In SET the epis are exactly the surjective maps.

(b) In the category of Hausdorff spaces and continuous maps the epis are exactly the **dominant** maps, i.e., maps with dense image. (Exercise)

(c) In the category of groups the epis are exactly the surjective group homomorphisms. (Exercise)

(d) In the category RING the inclusion morphism $\mathbb{Z} \to \mathbb{Q}$ is an epi. (Exercise)

Definition 2.2.3. A morphism $f : X \to Y$ is called a **monomorphism** or **mono**, if for any two morphisms $\alpha, \beta : V \to X$ the following is true: if in the (non commutative) diagram

$$V \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X \xrightarrow{f} Y$$

the upper and the lower path from V to Y agree, then one has $\alpha = \beta$. This means, f is a mono, if the commutativity of any given diagram of the form

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[d, "\beta"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

implies $\alpha = \beta$.

Examples 2.2.4. (a) A map in SET is mono if and only if in is injective.

(b) A morphism f is mono in $\mathcal{C}^{\mathrm{opp}}$ iff f is an epi in $\mathcal{C}$.

Products and coproducts

Definition 2.2.5. Let X, Y be objects of a category $\mathcal{C}$. A **product** of X and Y is an object P together with morphisms $p_1 : P \to X$ and $p_2 : P \to Y$, such that the following universal property holds: For every object Z and morphisms $p : Z \to X$ and $q : Z \to Y$ there is exactly one morphism $Z \to P$, such that the diagram

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ X & & Y \\ & Z \arrow[ul] \arrow[ur] \arrow[uu, dotted, "\exists!"] & \end{tikzcd}$$

commutes. This means that the morphisms from Z to X and Y factor over the universal morphisms from P to X and Y.

If it exists, the product is uniquely determined up to isomorphy. It is written as $P = X \times Y$. The maps $X \times Y \to X$ and $X \times Y \to Y$ are called the **projections** of the product.

The universal property yields a bijection

$$\mathrm{Hom}(Z, X \times Y) \cong \mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y).$$

Definition 2.2.6. A **Coprodukt** of X and Y is a product in $\mathcal{C}^{\mathrm{opp}}$.

This means, it is an object K together with morphisms $i_1 : X \to K$ and $i_2 : Y \to K$, such that the following universal property holds: For every object Z and Morphismen $p : X \to Z$ and $q : Y \to Z$ there is exactly one morphism $K \to Z$, such that the diagram

$$\begin{tikzcd} & K \arrow[dd, dotted, "\exists!"] & \\ X \arrow[ur] \arrow[dr] & & Y \arrow[ul] \arrow[dl] \\ & Z & \end{tikzcd}$$

commutes. It if exists, the coproduct is uniquely determined. We write it as $K = X \coprod Y$ or $C = X \oplus Y$. The universal property gives natural bijections:

$$\mathrm{Hom}(X \oplus Y, Z) \cong \mathrm{Hom}(X, Z) \times \mathrm{Hom}(Y, Z).$$

Examples 2.2.7. (a) In the category of sets products and coproducts exist. The product is given by the cartesian product , the coproduct is the disjoint union.

(b) The category of groups allows for products and coproducts. The product is the cartesian product and the coproduct is the free product of groups.

(c) In the category RING of rings with one, the product exists and is given by the cartesian product. The coproduct is the tensor product over $\mathbb{Z}$.

(d) In the category FIELD of fields neither product, nor coproduct exist. (Take two fields with different characteristics.)

(e) Using partially ordered sets, i.e., Example 2.1.3, one easily constructs categories with products, but not with coproducts and vice versa.

* * *

2.3 Pullbacks and pushouts

Definition 2.3.1. A commutative diagramm

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & Z \end{tikzcd}$$

is called **cartesian**, or a **cartesian square**, if for every commutative diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & Z \end{tikzcd}$$

there is exactly one arrow from Q to P such that the diagram

$$\begin{tikzcd} Q \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & P \arrow[r] \arrow[d] & X \arrow[d, "\alpha"] \\ & Y \arrow[r, "\beta"'] & Z \end{tikzcd}$$

commutes. In this case, P is called the **pullback** of X and Y over Z. A pullback is also called a **fiber product**. This property uniquely determines the pullback up to isomorphy. We say that in a category **pullbacks exist**, if every diagram of the form

$$\begin{tikzcd} & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & Z \end{tikzcd}$$

can be extended to a cartesian square. One writes $P = X \times_{\alpha,\beta} Y$ or, if it is clear, which arrows α and β are being used, one writes $P = X \times_Z Y$.

Reversing all arrows, a **Pushout** in C is a pullback in C^{opp}. More precisely, a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] & S \end{tikzcd}$$

is called **co-cartesian**, if for every commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] & Z \end{tikzcd}$$

there is exactly one arrow from S to Z, such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \arrow[ddr] & \\ C \arrow[r] \arrow[drr] & S \arrow[dr, dotted] & \\ & & Z \end{tikzcd}$$

commutes. In this case, S is called the **pushout** of B and C over A. A pushout is also called a **cofiber product**. If a pushout exists, it is uniquely determined up to isomorphy. One writes $S = B \oplus_{f,g} C$ or, if it is clear, which arrows are being used, one writes $S = B \oplus_A C$.

Examples 2.3.2. (a) In the category of sets the pullback is given by

$$X \times_{\alpha,\beta} Y = \Big\{(x, y) \in X \times Y : \alpha(x) = \beta(y)\Big\}.$$

The structure maps to X and Y are given by the projections of the cartesian product.

(b) Let there be given a diagram of sets and maps

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & Y \\ X & \end{tikzcd}$$

Then the co-fiber product C in SET is given by the set

$$C = (X \sqcup Y) / \sim,$$

where $\sim$ is the equivalence relation on the disjoint union generated by

$$f(z) \sim g(z)$$

for all $z \in Z$.

(c) In the category RING of rings with unit, the pullback is just the same as in SET, but the pushout product is the tensor product.

(d) In the category of groups, the pushout exists and equals the amalgam.

Lemma 2.3.3. *If the diagram*

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d] & B \arrow[d] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

is cartesian and f is mono, then so is g. If the diagram

$$\begin{tikzcd} C \arrow[r, "\delta"] \arrow[d] & A \arrow[d] \\ B \arrow[r, "\gamma"] & P \end{tikzcd}$$

is co-cartesian and if δ is epi, then so is γ.

Proof. Let $\alpha, \beta : Z \to F$ be two morphisms, such that $h = g\alpha = g\beta$. We have to show that $\alpha = \beta$ holds. Consider the diagram

$$\begin{tikzcd} Z \arrow[dr, bend left, "\beta"] \arrow[dr, bend right, "\alpha"'] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "h'"'] & & \\ & F \arrow[r, "g"] \arrow[d, "\eta"] & B \arrow[d] \\ & A \arrow[r, "f"] & C \end{tikzcd}$$

The identity $g\alpha = g\beta$ implies $f\eta\alpha = f\eta\beta$ and, since f in injective, we get $h' = \eta\alpha = \eta\beta$, so the diagram commtues. As the diagram we started with, is cartesian, there are, for given h and h' exactly one arrow from Z to F, making the diagram commute, so weg get $\alpha = \beta$.

The claim for co-cartesian diagrams follows by reversing all arrows, i.e., working in C^{opp}. □

* * *

2.4 Functors and natural transformations

Definition 2.4.1. A **functor** from a category A to a category B is a pair $(F, \mathcal{F})$, where $F : \mathrm{Ob}(A) \to \mathrm{Ob}(B)$ is a map and $\mathcal{F}$ is a family of maps $F_{X,Y} : \mathrm{Hom}_A(X, Y) \to \mathrm{Hom}_B(F(X), F(Y))$ such that

- $F_{X,X}(\mathbf{1}_X) = \mathbf{1}_{F(X)}$,
- $F(f \circ g) = F(f) \circ F(g)$,

where in the second point, we have left out the indices with F.

Examples 2.4.2. (a) The **forgetful functor** $F : \mathrm{AB} \to \mathrm{SET}$, which maps a group to its underlying set and group homomorphisms to the maps os sets.

(b) The **homotopy functor** F from the category TOP to the homotopy category [TOP]. It maps every space X to itself and a continuous map f to its homotopy class $[f]$.

(c) Considering groups as categories, functors between them are nothing else but group homomorphisms.

Definition 2.4.3. A functor $F : \mathcal{C} \to \mathcal{D}^{\mathrm{opp}}$ is also called a **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$.

Example 2.4.4. Let K be a field and VECT(K) the category of K-vector spaces and linear maps. The **dualising** $V \mapsto V^* = \mathrm{Hom}(V, K)$ is a contravariant functor from VECT(K) to itself.

Definition 2.4.5. A functor $F : \mathcal{A} \to \mathcal{B}$ is an **isomorphism of categories**, if there is a functor $G : \mathcal{B} \to \mathcal{A}$, such that

$$FG = \mathrm{Id}_{\mathcal{B}} \quad \text{und} \quad GF = \mathrm{Id}_{\mathcal{A}}.$$

Definition 2.4.6. A functor $F : \mathcal{A} \to \mathcal{B}$ is called **faithful**, if for any two $X, Y \in \mathcal{A}$, the map

$$F : \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{B}}(F(X), F(Y))$$

is injective.

The functor F is called **full**, if for any two $X, Y \in \mathcal{A}$ the map

$$F : \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{B}}(F(X), F(Y))$$

is surjective.

The functor is called **fully faithful**, if it is both, full and faithful.

Example 2.4.7. The forgetful functor $\mathrm{AB} \to \mathrm{SET}$ is faithful, but not full.

Lemma 2.4.8. *A functor $F : \mathcal{A} \to \mathcal{B}$ is an isomorphism if and only if F is fully faithful and bijective on the object classes.*

Proof. This is clear. □

Natural transformations

Definition 2.4.9. Let $F, G : \mathcal{A} \to \mathcal{B}$ be functors. A **natural transformation** $t : F \to G$ is a family $(t_X)_{X \in \mathcal{A}}$ of morphisms

$$t_X : F(X) \to G(X),$$

such that for every arrow $f : X \to Y$ in $\mathcal{A}$ the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "t_X"'] & F(Y) \arrow[d, "t_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

commutes. One can compose natural transformations $t : F \to G$ and $s : G \to H$ and gets $st : F \to H$. A natural transformation $t : F \to G$ is called a **natural isomorphism**, if there is a natural transformation $s : G \to F$, such that $st = \mathrm{Id}_F$ and $ts = \mathrm{Id}_G$. If t is a natural isomorphism, then every arrow $t_X : F(X) \to G(X)$ is an isomorphism.

Examples 2.4.10. (a) Every group is naturally isomorphic to its opposite group.

Let G be a group. The opposite group G^{opp} consists of the same set with the composition

$$a \cdot_{\mathrm{opp}} b = ba.$$

Let F be the functor $F : \mathrm{GRP} \to \mathrm{GRP}$ of the category of groups in itself, which maps every group to its opposite.

The "naturally" part in the above assertion means that there is a natural isomorphism $t : \mathrm{Id} \xrightarrow{\cong} F$.

Proof. For every group G the map

$$\begin{aligned} t_G : G &\to G^{\mathrm{opp}}, \\ x &\mapsto x^{-1} \end{aligned}$$

is an isomorphism. If $\phi : G \to H$ is a group homomorphism, then

$$\phi(t_G(x)) = \phi(x^{-1}) = \phi(x)^{-1} = t_H(\phi(x)).$$

Therefore t is a natural transformation from Id to F, but also the other way round, from F to Id and because of

$$t_{G^{\mathrm{opp}}} t_G = \mathrm{Id}_G$$

the transformation t is an isomorphism. □

(b) Let K be a field and let $F : \mathrm{VECT}(K) \to \mathrm{VECT}(K)$ be the functor, which sends each vector space V to its bidual $F(V) = V^{**}$ zuordnet. Then the map

$$\begin{aligned} t_V : V &\to V^{**} \\ v &\mapsto \delta_v, \end{aligned}$$

with $\delta_v(\alpha) = \alpha(v)$ is a natural transformation $t : \mathrm{Id} \to F$.

* * *

2.5 Equivalence of categories

Definition 2.5.1. A functor $F : \mathcal{A} \to \mathcal{B}$ is an **equivalence of categories**, if there exists a functor $G : \mathcal{B} \to \mathcal{A}$, such that

$$FG \cong \mathrm{Id}_{\mathcal{B}} \quad \text{and} \quad GF \cong \mathrm{Id}_{\mathcal{A}}.$$

Every isimorphy of categories is an equivalence of categories.

Example 2.5.2. Let K be a field and let $\mathcal{A}$ be the category of all finite-dimensional K-vector spaces. Then $F : \mathcal{A} \to \mathcal{A}$, $V \mapsto V^{**}$ is an equivalence of categories.

Theorem 2.5.3. (a) *A functor $F : \mathcal{A} \to \mathcal{B}$ is an equivalence of categories if and only if it is fully faithful and has dense image.*

(Recall that a subcategory $\mathcal{A}' \subset \mathcal{A}$ is dense if every $X \in \mathcal{A}$ is isomorphic to some $X' \in \mathcal{A}'$.)

(b) *Two categories $\mathcal{A}, \mathcal{B}$ are equivalent if and only if there are dense subcategories $\mathcal{A}' \subset \mathcal{A}$ and $\mathcal{B}' \subset \mathcal{B}$, which are isomorphic, $\mathcal{A}' \cong \mathcal{B}'$.*

Proof. (a) Let F be an equivalence of categories with quasi-inverse $G : \mathcal{B} \to \mathcal{A}$ and let $t : \mathrm{Id}_{\mathcal{A}} \to GF$ be the natural isomorphy. Then for any two $X, Y \in \mathcal{A}$ the map

$$\mathrm{Hom}(X,Y) \xrightarrow{GF} \mathrm{Hom}(GF(X), GF(Y)) \xrightarrow{t_Y \circ \cdot \circ t_X^{-1}} \mathrm{Hom}(X,Y)$$

is the identity, which implies that F is faithful. Since further t_Y^{-1} and t_X are isomorphisms, it follows that G is full. By symmetry in G and F it follows that F is full as well. Let $s : \mathrm{Id}_{\mathcal{B}} \to FG$ be the natural isomorphy. For $Z \in \mathcal{B}$ the arrow $s_Z : Z \to F(G(Z))$ is an isomorphism, so F has dense image.

Conversely, let $F : \mathcal{A} \to \mathcal{B}$ be fully faithful with dense image. For every $Z \in \mathcal{B}$ choose an $X \in \mathcal{A}$ and an isomorphism $\nu_Z : Z \xrightarrow{\cong} Z' = F(X)$, where we assume that $Z' = Z$ and $\nu_Z = \mathrm{Id}_Z$, if Z lies in the image already. Then set $G(Z) = X$. For $Z, W \in \mathcal{B}$ define $G : \mathrm{Hom}(Z,W) \to \mathrm{Hom}(G(Z), G(W))$ by

$$\mathrm{Hom}(Z,W) \xrightarrow{\nu_W \circ \cdot \circ \nu_Z^{-1}} \mathrm{Hom}(Z' = F(X), W' = F(Y)) \xrightarrow{F^{-1}} \mathrm{Hom}(X = G(Z), Y = G(W)).$$

Then G is a functor, quasi-inverse to F.

(b) Let $F : \mathcal{A} \to \mathcal{B}$ be an equivalence with quasi-inverse $G : \mathcal{B} \to \mathcal{A}$. In every isomorphism class $[X]$ of objects in $\mathcal{A}$ choose an object $X \in \mathrm{im}(G)$, which is possible, as the image is dense by (a). Let $\mathcal{A}'$ be the full subcategory of these chosen objects. Then $\mathcal{A}'$ is dense in $\mathcal{A}$ by construction. Let $\mathcal{B}' = F(\mathcal{A}')$. We claim that $\mathcal{B}'$ is dense $\mathcal{B}$ and that $F|_{\mathcal{A}'}$ is an isomorphism between $\mathcal{A}'$ and $\mathcal{B}'$.

Let $Y \in \mathcal{B}$. Then there exists $X \in \mathcal{A}$ such that $F(X)$ is isomorphic to Y. There exists $X' \in \mathcal{A}'$ such that $X \cong X'$ and thus, as F is fully faithful, we get $F(X') \cong F(X) \cong Y$, hence $\mathcal{B}'$ is dense in $\mathcal{B}$. For the isomorphy, first, as any two objects X, Y in $\mathcal{A}'$ are non-isomorphic, it follows that $F(X) \ncong F(Y)$, because, if $F(\alpha)$ is an isomorphism between $F(X)$ and $F(Y)$, then there exists an inverse $F(\beta)$. Then $F(\alpha\beta) = F(\alpha)F(\beta) = \mathrm{Id}_{F(Y)}$, so $\alpha\beta = \mathrm{Id}_Y$ and the same for $\beta\alpha$, hence α is an isomorpism. This means that F is a bijection from $\mathrm{Ob}(\mathcal{A}')$ to $\mathrm{Ob}(\mathcal{B}')$. It is also bijective on each Hom set, therefore the corresponding inverse maps constitute an inverse functor.

Now for the converse assume that $\mathcal{A}'$ and $\mathcal{B}'$ exist and that $F' : \mathcal{A}' \to \mathcal{B}'$ is an isomorphism. For each $X \in \mathcal{A}$ fix an isomorphism $\alpha_X : X \to X'$ for some object $X' \in \mathcal{A}'$ in a way that if X already lies in $\mathcal{A}'$, then $X' = X$ and $\alpha_X = \mathrm{Id}_X$. Set $F(X) = F'(X')$ for every $X \in \mathcal{A}$ and for any two $X, Y \in \mathcal{A}$ and any $\tau \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ let

$$F(\tau) = F'(\alpha_Y \circ \tau \circ \alpha_X^{-1}).$$

Then F is a functor and the same construction on the other side, using F^{-1} yields a quasi-inverse G. $\square$

* * *

2.6 Additive categories

Definition 2.6.1. An object T of a category C is called **terminal object**, if for every $X \in C$ there is exactly one arrow $X \to T$. If it exists, it is uniquely determined up to isomorphy.

Examples 2.6.2. (a) In AB the trivial group is terminal.

(b) In TOP the one-point space $\{x_0\}$ is terminal.

Definition 2.6.3. An object I of C is called **initial object**, if for every $X \in C$ there is exactly one morphism $I \to X$. If it exists, it is uniquely determined up to isomorphy.

Examples 2.6.4. (a) In AB the trivial group is also initial.

(b) In TOP the empty set is initial. In the category $\mathrm{TOP}_{\neq\emptyset}$ there is no initial object. In TOP_* the one point space is initial.

(c) In RING the ring $\mathbb{Z}$ is initial, whereas the Zero-ring is terminal.

Definition 2.6.5. A **zero object** in a category is an object X_0 which is both, initial and terminal. For any two objects $X, Y \in C$ there then is exactly one morphism 0, which factors through the zero object. This morphism is called the **zero morphism**.

$$\begin{tikzcd} X \arrow[rr, "0"] \arrow[dr] & & Y \\ & X_0 \arrow[ur] & \end{tikzcd}$$

A zero object is written as 0. A category which contains a zero object is called a **pointed category**. Let $f : X \to Y$ be a morphism in a pointed category. A **kernel** for f is a morphism $\alpha : K \to X$ such that

- $f\alpha = 0$ and
- every morphism $g : Z \to X$ with $fg = 0$ factors in a unique way through α, i.e., there is exactly one morphism $\psi : Z \to K$ with $g = \alpha\psi$.

$$\begin{tikzcd} & Z \arrow[dl, dotted, "\exists!\psi"'] \arrow[d, "g"] \arrow[dr, "0"] & \\ K \arrow[r, "\alpha"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Example 2.6.6. In AB for a given morphism $f : A \to B$ the embedding of the subgroup $f^{-1}(0)$ in A is a kernel.

Note that if C is pointed, then C^{opp} is pointed, too.

Definition 2.6.7. Let C be pointed, then **cokernel** for $f : X \to Y$ is a morphism $\gamma : Y \to C$ such that

- $\gamma f = 0$ and

- every norphism $g : Y \to Z$ with $gf = 0$ factors in a unique way through γ, i.e., there is exactly one morphism $\phi : C \to Z$ with $g = \phi\gamma$.

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "0"] \arrow[r, "f"] & Y \arrow[u, "g"] \arrow[r, "\gamma"] & C \arrow[ul, dotted, "\exists!\phi"'] \end{tikzcd}$$

Lemma 2.6.8. *Let C be a pointed category. A kernel is always a mono and a cokernel is always an epimorphism.*

Proof. Let $k : K \to X$ be a kernel for $f : X \to Y$. Let $\alpha, \beta : Z \to K$ morphisms with $k\alpha = k\beta$. We have to show that $\alpha = \beta$. We have the commutative diagram

$$\begin{tikzcd} K \arrow[r, "k"] & X \arrow[r, "f"] & Y \\ Z \arrow[u, shift left, "\beta"'] \arrow[u, shift right, "\alpha"] \arrow[ur, "F"'] & & \end{tikzcd}$$

The arrow $F := k\alpha = k\beta$ has the property that $fF = 0$, hence it factors uniquely through k, which means that $\alpha = \beta$. The second assertion follows by dualizing, since a kernel in C^{opp} is a cokernel in C. □

Examples 2.6.9.

- Let F be a field. The category VECT(F) of F-vector spaces and linear maps is pointed, with zero object being the zero space. Kernels and cokernels do exist and are the usual kernels and cokernels as in Linear Algebra.

- The category of pointed sets. The objects are pairs (X, x_0), where X is a set and $x_0 \in X$ a point. Morphisms from (X, x_0) to (Y, y_0) are maps $f : X \to Y$ with $f(x_0) = y_0$. This category has a zero object: the one-point set $\{x_0\}$. For a map $f : (X, x_0) \to (Y, y_0)$ a kernel exists and is given by the inclusion map $k : f^{-1}(y_0) \hookrightarrow X$. A cokernel also exists and equals the projection onto $C = Y/f(X)$, which means that $f(X)$ is collapsed to a point. More precisely, $Y/f(X)$ equals $Y/\sim$, where $\sim$ is the equivalence relation generated by $f(x) \sim f(x')$ for every two $x, x' \in X$.

Remark 2.6.10. For an object X of a category C let C_X be the class of all arrows $f : Z \to X$ with target X. On C_X we have an equivalence relation, where we say that $f : Z \to X$ and $g : V \to X$ are **equivalent**, if there exists an isomorphism $\alpha : V \to Z$ such that the diagram

$$\begin{tikzcd} Z \arrow[r, "f"] & X \\ V \arrow[u, dotted, "\alpha"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes. The universal property implies that a kernel is uniquely determined up to this equivalence. Therefore, it makes sense to speak of **the kernel** as the equivalence class $[k]$ of one given kernel k. The same goes for **the cokernel**.

Definition 2.6.11. An **additive category** is:

- a pointed category C with

- an abelian group structure + on $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for every pair (X, Y) of, such that the composition

$$\circ : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$$

  is bilinear.

- Further we demand, that for any two objects X, Y a product $X \times Y$ and a coproduct $X \oplus Y$ exists.

In an additive category the zero-morphism $0 \in \mathrm{Hom}(X, Y)$ always equals the zero in the additive group $\mathrm{Hom}(X, Y)$, because the zero-morphism is the only element of the image of the bilinear map $\circ : \mathrm{Hom}(X, 0) \times \mathrm{Hom}(0, Y) \to \mathrm{Hom}(X, Y)$.

Examples 2.6.12.

- For a ring R, the category $\mathrm{MOD}(R)$ is additive.
- For a field F the category $\mathrm{VECT}_{\mathrm{ev}}(F)$ of even dimensional vector spaces of is additive.

* * *

2.7 Abelian categories

Definition 2.7.1. An additive category C is called **abelian category**, if:

(a) For every morphism the kernel and cokernel exist.

(b) A morphism whose kernel and cokernel vanish, is an isomorphism.

Axiom (b) is equivalent to

(b') For every Morphism f, the natural map from the Coimage to the image is an isomorphism.

Here the image an coimage are

$$\mathrm{im}(f) = \ker(\mathrm{coker}(f)), \qquad \mathrm{coim}(f) = \mathrm{coker}(\ker(f)).$$

The definitions of the kernel and of the cokernel both induce the existence of a map $g : \mathrm{coim}(f) \to \mathrm{im}(f)$ and due to the uniqueness, the two definitions of g agree.

Proof of the equivalence of b *and* b'.
(b)⇒(b'): It is easy to see that g has trivial kernel and cokernel. By (b) it is an isomorphism.
(b')⇒(b): Under these conditions $f = g$. □

Remark 2.7.2. Let f be a morphism in an abelian category. If $\ker(f) = 0$, the f is the kernel of its cokernel. If $\mathrm{coker}(f) = 0$, then f is the cokernel of its kernel.

Proof. Let $f : X \to Y$. If $\ker(f) = 0$, then f is the cokernel of its kernel and by (b') f is the kernel of its cokernel. If $\mathrm{coker}(f) = 0$, then f is the kernel of its cokernel and by (b') it is the cokernel of its kernel. □

Examples 2.7.3. (a) Let R be a ring and let $\mathrm{MOD}(R)$ be the category of R-modules and R-linear maps. Then $\mathrm{MOD}(R)$ is an abelian category, where the sum of two homomorphisms is the pointwise sum.

(b) Let $(R_i)_{i\in I}$ be a family of rings and let $\mathcal{A}$ be the category, the objects of which are families $(M_i)_{i\in I}$ where M_i is an R_i-module and a morphism $f : (M_i) \to (N_i)$ is a family $f = (f_i)_{i\in I}$, where each f_i is an R_i-module homomorphism $M_i \to N_i$. Then $\mathcal{A}$ is an abelian category.

(c) An example of an additive category which is not abelian, is given by the category of even-dimensional vector spaces over a given field F. It is not abelian, as a linear map of odd rank has no kernel.

Lemma 2.7.4. *Let $\mathcal{A}$ be an abelian category.*

(a) *An arrow f is mono, iff* $\ker(f) = 0$. *An arrow g is epi, iff* $\mathrm{coker}(g) = 0$.

(b) *The dual category $\mathcal{A}^{\mathrm{opp}}$ is also abelian.*

(c) *For two objects X, Y the product $X \times Y$ is isomorphic to the coproduct $X \oplus Y$.*

(d) *Fiber products and co-fiber products exist.*

(e) *A morphism f which is epi and mono is an isomorphism.*

Proof. (a) If $\ker(f) = 0$, then f is the kernel of its cokernel, so it is mono by Lemma 2.6.8. Conversely, if f is mono and α a kernel, then $f0 = 0 = f\alpha$ and therefore $\alpha = 0$. The epi assertion follows similarly.

(b) is easily verified.

(c) Let $Z \xrightarrow{\alpha} X$ and $Z \xrightarrow{\beta} Y$ be morphisms. Then write $\alpha \times \beta$ for the morphism $Z \to X \times Y$ which is induced by the universal property of the product. The morphisms $X \xrightarrow{1\times 0} X \times Y$ and $Y \xrightarrow{0\times 1} X \times Y$ induce a morphism $\phi : X \oplus Y \to X \times Y$ by the universal property of the sum. This makes the diagram

$$\begin{tikzcd}
X \arrow[rrr, "\mathrm{Id}"] \arrow[dr] & & & X \\
& X \oplus Y \arrow[r, "\phi"] & X \times Y \arrow[ur] \arrow[dr] & \\
Y \arrow[rrr, "\mathrm{Id}"] \arrow[ur] & & & Y
\end{tikzcd}$$

commutative. Let

$$\psi_X : X \times Y \to X \to X \oplus Y$$

and

$$\psi_Y : X \times Y \to Y \to X \oplus Y.$$

Let further

$$\psi : \psi_X + \psi_Y.$$

It is easy to see that ψ is inverse to ϕ and so ϕ is an isomorphism.

(d) Let $f : A \to C$ and $g : B \to C$ be given. Let $\alpha : A \times B \to C$ be the composition $A \times B \to A \xrightarrow{f} C$ and similarly $\beta : A \times B \to B \to C$. Then $K = \ker(\alpha - \beta)$ is a fiber product. Co-fiber products are fiber products in the category $\mathcal{A}^{\mathrm{opp}}$, which is abelian, too.

(e) Let $f : X \to Y$ be a morphism which is epi and mono. Then by (a), f is the kernel of its cokernel, which, again by (a) is zero. The identity morphism $\mathbf{1}_Y : Y \to Y$ also is a kernel of $Y \to 0$. By the uniqueness of a kernel there exist uniquely determined arrows α, β making the diagrams

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \\
Y \arrow[ur, "1_Y"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] & Y \\
Y \arrow[u, "\beta"] \arrow[ur, "1_Y"'] &
\end{tikzcd}$$

It follows $\alpha = f$ and in the usual way it follows that β is an inverse to f. $\square$

Definition 2.7.5. For a morphism f in an abelian category we define

$$\mathrm{im}(f) := \ker(\mathrm{coker}(f)).$$

A sequence of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is called **exact**, if

$$\mathrm{im}(f) = \ker(g)$$

holds, that is to say, if $\mathrm{im}(f)$ is a kernel of g. A sequence

$$\cdots \to A_{i-1} \xrightarrow{d^{i-1}} A_i \xrightarrow{d^i} A_{i+1} \to \ldots$$

is called **exact**, if it is exact at every index i.

Remark 2.7.6. Since a kernel is not uniquely determined, the identity $\mathrm{im}(f) = \ker(g)$ is to be read as saying that any kernel of any cokernel of f is a kernel for g.

Another way to put it is to say that we have $g \circ f = 0$ and hence f factors through $\ker(g)$:

$$\begin{tikzcd} & K \arrow[dr, "\ker(g)"] & & \\ X \arrow[ur, "\alpha"] \arrow[rr, "f"'] & & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

Now the sequence is exact at Y is equivalent to saying that α is an epimorphism.

Definition 2.7.7. A functor $F : \mathcal{C} \to \mathcal{D}$ between additive categories is called an **additive functor**, if for any two objects X, Y the induced map $F : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$ is a group homomorphism.

A functor $F : \mathcal{C} \to \mathcal{D}$ between abelian categories is called an **exact functor**, if it is additive and maps exact sequences to exact sequences.

Diagram chase

Remark 2.7.8 (On diagram chase in abelian categories)**.** The proofs of assertions like the five lemma or the snake lemma depend on diagram chase, where one has to pick elements and chase them along a diagram. So a priori they are not valid in an arbitrary abelian category. There are two ways to fix this.

Firstly, one can give new proofs which only use arguments valid in abelian categories. This can be done, but is very tedious.

Secondly, one makes use of **Mitchell's Embedding Theorem**, which says that for a small abelian category $\mathcal{A}$ there exists a ring R (with 1, not neccessarily commutative) and a full faithful and exact functor $F : \mathcal{A} \to \mathrm{MOD}(R)$.

The functor F yields an equivalence between $\mathcal{A}$ and a full subcategory of $\mathrm{MOD}(R)$ in such a way that kernels and cokernels computed in $\mathcal{A}$ correspond to the ordinary kernels and cokernels computed in $\mathrm{MOD}(R)$. Such an equivalence is necessarily additive. The theorem thus essentially says that the objects of $\mathcal{A}$ can be thought of as R-modules, and the morphisms as R-linear maps, with kernels, cokernels, exact sequences and sums of morphisms being determined as in the case of modules.

* * *

3 Sheaves

3.1 Presheaves

Definition 3.1.1. Let X be a topological space and let $\mathcal{U}(X)$ be the category whose objects are the open sets in X and the morphisms are the inclusion maps $U \to V$ whenever $U \subset V$.

A **presheaf** is a contravariant functor

$$\mathcal{F} : \mathcal{U}(X) \to \mathrm{AB}\,.$$

Remark 3.1.2. Let $\mathcal{F}$ be a presheaf on X. To any open set $U \subset X$, the presheaf attaches an abelian group $\mathcal{F}(U)$ and to any inclusion $V \subset U$, a group homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$, which is called the **restriction** and is written as res^U_V or as $s \mapsto s|_V$. The axioms of a functor imply for $W \subset V \subset U$ and $s \in \mathcal{F}(U)$ that

$$\mathrm{res}^V_W \circ \mathrm{res}^U_V = \mathrm{res}^U_W \quad \text{and} \quad \mathrm{res}^U_U = \mathrm{Id}_{\mathcal{F}(U)}$$

or, in the other notation, for $s \in \mathcal{F}(U)$ one has

$$(s|_V)|_W = s|_W, \quad \text{and} \quad s|_U = s.$$

Definition 3.1.3. The elements of $\mathcal{F}(U)$ are also called **sections** over U of the sheaf $\mathcal{F}$. The reason for these notions will become clear later. An element $s \in \mathcal{F}(X)$ is called a **global section**.

Examples 3.1.4.

Throughout, we fix an abelian group A and a topological space X.

(a) Let $\mathcal{M}_A$ be the **presheaf of all maps**, i.e., for an open set U let $\mathrm{M}_A(U)$ be the set of all maps from $s : U \to A$. Then $\mathcal{M}_A$ is a presheaf with $\mathrm{res}^U_V(s) = s|_V$ being the restriction of the map s.

(b) By $\mathcal{K}_A$ we denote the **constant presheaf** with value group A. By definition, $\mathcal{K}_A(U)$ is the set of all **locally-constant** maps $s : U \to A$. Then $\mathcal{K}_A$ is a presheaf on X, where again the restriction is the usual restriction of maps.

(c) Fix a point $x_0 \in X$ and set

$$\mathcal{S}(U) = \mathcal{S}_{A,x_0}(U) = \begin{cases} A & \text{if } x_0 \in U \\ 0 & \text{otherwise.} \end{cases}$$

With the restriction

$$\mathrm{res}^U_V = \begin{cases} \mathrm{Id}_A & x_0 \in V, \\ 0 & \text{otherwise} \end{cases}$$

the map $\mathcal{S}$ is a presheaf, called the **skyscraper presheaf** at x_0 with value group A.

(d) Finally, let $\mathcal{Z}(U) = A$ for every open U and set

$$\mathrm{res}^U_V = \begin{cases} \mathrm{Id}_A & U = V, \\ 0 & \text{otherwise.} \end{cases}$$

Then these data define a presheaf on X.

Definition 3.1.5. A **morphism of presheaves** $\phi : \mathcal{F} \to \mathcal{G}$ is a family of group homomorphisms $(\phi_U : \mathcal{F}(U) \to \mathcal{G}(U))_{U \subset X \text{ open}}$, such that for every inclusion of open sets $V \subset U$ the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\phi_U"] \arrow[d, "\mathrm{res}^U_V"] & \mathcal{G}(U) \arrow[d, "\mathrm{res}^U_V"] \\ \mathcal{F}(V) \arrow[r, "\phi_V"] & \mathcal{G}(V) \end{tikzcd}$$

commutes.

Examples 3.1.6.

(a) Let $\mathcal{K}_A$ be the constant sheaf on X with value group A. Then any group homomorphism $g : A \to B$ to some abelian group B induces a morphism of presheaves

$$g_* : \mathcal{K}_A \to \mathcal{K}_B$$

by setting

$$g_*(s) = g \circ s.$$

(b) For a fixed point x_0 let $\mathcal{S}_A$ be the skyscraper presheaf at x_0 with value group A. Then, as in the lsat example, a group homomorphism $g : A \to B$ induces a presheaf morphism

$$g_* : \mathcal{S}_A \to \mathcal{S}_B, \qquad g_*(s) = g(s).$$

(c) Notation as before. Let $f_0 : X \to \{0, 1\}$ be a locally-constant map, which means that $f^{-1}(0)$ and $f^{-1}(1)$ both are open. Then f induces a presheaf morphism $\phi_f : \mathcal{K}_A \to \mathcal{K}_A$ given by

$$\phi_f(s)(x) = f(x)s(x).$$

* * *

3.2 Sheaves

Definition 3.2.1. Let $\mathcal{F}$ be a presheaf over X. We call $\mathcal{F}$ a **sheaf**, if two conditions are satisfied.

- **(Uniqueness)** Let $U \subset X$ be open and let $(U_i)_{i\in I}$ an open cover of U, so $U = \bigcup_{i\in I} U_i$. Further let $s \in \mathcal{F}(U)$ such that $s|_{U_i} = 0$ for every $i \in I$. Then $s = 0$.
- **(Existence)** Let $U \subset X$ be open and let $(U_i)_{i\in I}$ be an open cover of U. For every $i \in I$ let there be given some $s_i \in \mathcal{F}(U_i)$, such that for any two $i, j \in I$ one has

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j},$$

  Then there exists an $s \in \mathcal{F}(U)$, such that $s_i = s|_{U_i}$ for every $i \in I$.

One can rephrase this as follows: the Uniqueness axiom says that a section is determined by its local restrictions and the Existence axiom, that compatible local sections can be glued to yield a global section.

Examples 3.2.2.

(a) The presheaf of all maps $\mathcal{M}_A$ is a sheaf.

(b) The constant presheaf $\mathcal{K}_A$ is a sheaf.

(c) The skyscraper presheaf $\mathcal{S}_{A,x_0}$ is a sheaf.

(d) The presheaf $\mathcal{Z}$ with $\mathcal{Z}(U) = A$ and $\mathrm{res}_V^U = 0$ is not a sheaf if $A \neq 0$, since for a sheaf $\mathcal{F}$ we have $\mathcal{F}(\emptyset) = 0$ as we shall see below.

(e) Let A be an abelian group, $X = \mathbb{R}$ and $\mathcal{F}(U) = A$ if $U = X$, but $\mathcal{F}(U) = 0$ otherwise. Then $\mathcal{F}$ is a presheaf, which satisfies Existence, but not Uniqueness.

(f) Let $A \neq 0$ be an abelian group, $X = \mathbb{R}$ and let $\mathcal{F}(U) = 0$ is the diameter of U is bigger than 1. Otherwise, let $\mathcal{F}(U) = A$. The restriction maps are the natural embeddings. Then $\mathcal{F}$ is a presheaf satisfying Uniqueness, but not Existence.

Remark 3.2.3. The sheaf axioms imply, that for every sheaf $\mathcal{F}$ one has $\mathcal{F}(\emptyset) = 0$. To prove this, let $s \in \mathcal{F}(\emptyset)$ and let $(U_i)_{i\in I}$ be the empty cover, i.e., $I = \emptyset$. Then for every $i \in I$ one has $s|_{U_i} = 0$, since I has no elements! By the Uniqueness axiom, we get $s = 0$.

Lemma 3.2.4. *A presheaf $\mathcal{F}$ is a sheaf iff for every open cover $(U_i)_{i\in I}$ of an open set $U \subset X$ the sequence*

$$0 \to \mathcal{F}(U) \xrightarrow{\alpha} \prod_i \mathcal{F}(U_i) \xrightarrow{\beta} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is exact. The products run over I and $I \times I$ and $\alpha(s)_i = s|_{U_i}$, as well as $\beta(s_*)_{i,j} = s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j}$.

Proof. Injectivity of α is equivalent to the Uniqueness axiom. The assertion $\beta \circ \alpha = 0$, so $\ker\beta \supset \operatorname{im}\alpha$ is satisfied for every presheaf. Finally, the assertion $\ker\beta \subset \operatorname{im}\alpha$ is equivalent with the Existence axiom. □

Definition 3.2.5. A **sheaf homomorphism** is the same as a presheaf homomorphism, only between sheaves.

The **direct sum** of two sheaves $\mathcal{F}$ and $\mathcal{G}$ over X is defined as the sheaf

$$U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U).$$

It is easy to see that this indeed is a sheaf.

Definition 3.2.6. A **subsheaf** $\mathcal{H}$ of a given sheaf $\mathcal{F}$ is a sheaf, such that for every open set U the group $\mathcal{H}(U)$ is a subgroup of $\mathcal{F}(U)$ and the restriction homomorphism of $\mathcal{H}$ and $\mathcal{F}$ coincide on these subgroups.

This last condition means that for any two open sets $V \subset U$ the diagram

$$\begin{tikzcd} \mathcal{H}(U) \arrow[r, hook] \arrow[d, "\mathrm{res}_{\mathcal{H}}"] & \mathcal{F}(U) \arrow[d, "\mathrm{res}_{\mathcal{F}}"] \\ \mathcal{H}(V) \arrow[r, hook] & \mathcal{F}(V) \end{tikzcd}$$

commutes.

Examples 3.2.7.

- A presheaf $\mathcal{P}$ on $\mathbb{R}$, which to each open set $U \neq \emptyset$ attaches the group $\mathbb{Z}$, cannot be a sheaf, no matter what the restriction maps look like. **Assume**, it is a sheaf. Let $U = (-\infty, 0)$ and $V = (0, \infty)$, sowie $W = U \cup V$. By Existence, there is $a \in \mathcal{P}(W)$ with $a|_U = 1$ and $a|_V = 0$. Like wise, there is $b \in \mathcal{P}(W)$ with $b|_{U=0}$ and $b|_V = 1$. We get a group homomorphism $\phi : \mathbb{Z}^2 \to \mathbb{Z}$, by $(k, l) \mapsto ka + lb \in \mathcal{P}(W) = \mathbb{Z}$. Let $(k, l) \in \ker(\phi)$, Then $0 = (ka + lb)|_U = k \in \mathcal{P}(U) \cong \mathbb{Z}$ and so $k = 0$. Analogously, it follows $l = 0$. So ϕ injective. But there is no injective group homomorphism $\mathbb{Z}^2 \to \mathbb{Z}$, **Contradiction!**

  (**Assume** there is an injective group homomorphism $\phi : \mathbb{Z}^2 \to \mathbb{Z}$. Then

  $$\phi\left(\phi\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} - \phi\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right) = \phi\begin{pmatrix}1\\0\end{pmatrix}\phi\begin{pmatrix}0\\1\end{pmatrix} - \phi\begin{pmatrix}1\\0\end{pmatrix}\phi\begin{pmatrix}0\\1\end{pmatrix} = 0.$$

  As ϕ is injective, it follows

  $$\phi\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} - \phi\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 0,$$

  therefore $\phi\binom{1}{0} = 0 = \phi\binom{0}{1}$. But that means that ϕ is identically zero. **Contradiction!**)

- The sheaf $\mathcal{O}$ of holomorphic functions on $\mathbb{C}$ is a sheaf of rings.

* * *

3.3 Stalks

Definition 3.3.1. Let $(I, \leq)$ be a partially ordered set. Then $(I, \leq)$ is called a **directed set**, if for any two $a, b \in I$ there exists an upper bound, i.e., an element $c \in I$ with $a \leq c$ and $b \leq c$.

Examples 3.3.2.

- $\mathbb{N}$ is directed.
- Let S be a set and I be the set of all finite subsets $E \subset S$. Then I is directed by inclusion, since for $E, F \in I$ the set $E \cup F$ is finite again, hence it is an upper bound

$$E, F \leq E \cup F.$$

- Let $x \in X$ and X a topological space. Let I be the set of all open neighbourhoods of x with the *reversed inclusion* as ordering, so

$$U \leq V \quad \Leftrightarrow \quad U \supset V.$$

  Then I is directed, as with U and V the set $U \cap V$ is an open neighbourhood, too and hence one has

$$U, V \leq U \cap V.$$

Definition 3.3.3. A **directed system** of abelian groups is a pair $((M_i)_{i \in I}, (\phi_i^j)_{i \leq j})$, where I is a directed set, $(M_i)_{i \in I}$ is a family of abelian groups and for $i \leq j$ the map

$$\phi_i^j : M_i \to M_j$$

is a group homomorphism, such that

$$\phi_i^i = \mathrm{Id}_{M_i}, \qquad \phi_j^k \circ \phi_i^j = \phi_i^k$$

if $i \leq j \leq k$.

Examples 3.3.4.

- Fix a prime number p. Let $I = \mathbb{N}$ and $M_i = \mathbb{Z}$. Further let $\phi_i^j : \mathbb{Z} \to \mathbb{Z}$ be given by $x \mapsto p^{j-i}x$. Then these data establish a directed system.

  The directed set in this case is $\mathbb{N}$. Therefore the directed system is completely determined by the maps ϕ_i^{i+1}, as all others are iterations of these. In this example, the map ϕ_i^{i+1} is the multiplication by p on $\mathbb{Z}$. We write this system as a sequence

$$\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \dots$$

- Let $z_0 \in \mathbb{C}$. Let I be the set of all open neighbourhoods of z_0 in $\mathbb{C}$ with the reversed inclusion as partial order. For $U \in I$ let M_U be the set of all holomorphic functions $f : U \to \mathbb{C}$. For $V \subset U$ let $\phi_U^V : M_U \to M_V$ be given by restriction, so $\phi_U^V(f) = f|_V$. This is called the direct system of all **germs of functions** in z_0.

Definition 3.3.5. The **direct limit** of a directed system (M_i, ϕ_i^j) is defined as

$$\varinjlim_i M_i = \bigsqcup_{i \in I} M_i \Big/ \sim$$

where the equivalence relation $\sim$ on the disjoint union is defined as follows. Two elements $a \in M_i$ and $b \in M_j$ are equivalent, if there exists an index $k \geq i, j$, such that $\phi_i^k(a) = \phi_j^k(b)$. It is easy to see, that this is an equivalence relation. The property of I of being directed is needed for transitivity: Let $a \sim b$ and $b \sim c$ in $\bigsqcup_{i \in I} M_i$. Let's say $a \in M_i$, $b \in M_j$ and $c \in M_k$. Then there exists $l \geq i, j$ such that $\phi_i^l(a) = \phi_j^l(b)$ and there is $m \geq j, k$ such that $\phi_j^m(b) = \phi_j^m(c)$. Let $n \geq l, m$. Then $\phi_i^n(a) = \phi_j^n(b) = \phi_k^n(c)$, so that $a \sim c$.

Lemma 3.3.6. *The rule*

$$[a] + [b] := [a + b] \qquad a, b \in M_k$$

makes $M = \varinjlim_i M_i$ an abelian group with the following universal property: There are group homomorphisms $\phi_i : M_i \to M$, which form the following commutative diagrams:

$$\begin{tikzcd} M_j \arrow[r, "\phi_j"] & M \\ M_i \arrow[u, "\phi_i^j"] \arrow[ur, "\phi_i"'] & \end{tikzcd} \qquad i \leq j$$

such that for every abelian group Z with a family of group homomorphisms $\eta_i : M_i \to Z$, which likewise satisfy $\eta_j \circ \phi_i^j = \eta_i$, there is a uniquely determined group homomorphism $\psi : M \to Z$, such that for every $i \in I$ the diagram

$$\begin{tikzcd} M_i \arrow[r, "\phi_i"] \arrow[dr, "\eta_i"'] & M \arrow[d, dotted, "\exists!\psi"] \\ & Z \end{tikzcd}$$

commutes.

Proof. For the group law, one has to show well-defninedness. So let $a \sim a'$ and $b \sim b'$, say $\phi_k^l(a) = \phi_i^l(a')$ and $\phi_k^l(b) = \phi_i^l(b')$. Then one has $\phi_k^l(a + b) = \phi_k^l(a) + \phi_k^l(b) = \phi_i^l(a') + \phi_i^l(b') = \phi_i^l(a' + b')$, so it follows $(a+b) \sim (a'+b')$ and therefore $[a+b] = [a'+b']$, which establishes well-definedness of addition. The maps ϕ_i are given by composition of the natural maps $M_i \to \bigsqcup_i M_i \to \bigsqcup_i M_i / \sim$. For the universal property one defines $\psi([a]) = \eta_k(a)$, if $a \in M_k$. The well-definedness is straightforward and so is commutativity of the diagrams. Uniqueness of ψ follows from the commutativity of the diagrams, for if ψ' is a second such map and if $[a] \in M$, say $a \in M_k$, then one has $\psi([a]) = \eta_k(a) = \psi'([a])$. □

Examples 3.3.7.

- Assume that every M_i is a subgroup of some given group M, one has $M_i \subset M_j$ for $i \leq j$ and the structure morphisms ϕ_i^j are given by inclusion. Then the union N of all M_i is a subgroup, too and there is a natural isomorphism

$$\varinjlim_i M_i \xrightarrow{\cong} N.$$

- We consider the first example of 3.3.4

$$\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \dots$$

We extend this to a commutative diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "p"] \arrow[d, "1"] & \mathbb{Z} \arrow[r, "p"] \arrow[d, "\frac{1}{p}"] & \mathbb{Z} \arrow[r, "p"] \arrow[d, "\frac{1}{p^2}"] & \mathbb{Z} \arrow[r, "p"] \arrow[d, "\frac{1}{p^3}"] & \dots \\
\mathbb{Q} \arrow[r, "\mathrm{Id}"] & \mathbb{Q} \arrow[r, "\mathrm{Id}"] & \mathbb{Q} \arrow[r, "\mathrm{Id}"] & \mathbb{Q} \arrow[r, "\mathrm{Id}"] & \dots
\end{tikzcd}$$

The union of all images in $\mathbb{Q}$ is the $\mathbb{Z}$-module

$$\mathbb{Z}[1/p] = \left\{ \frac{a}{p^k} \in \mathbb{Q} : a \in \mathbb{Z},\ k \in \mathbb{N} \right\}.$$

According to the last example, this direct limit is isomorphic to $\mathbb{Z}[1/p]$.

Definition 3.3.8. Let $\mathcal{F}$ be a presheaf on the space X and let $x \in X$. Let I be the set of all open neighbourhoods $U \subset X$ of x. The reversed inclusion makes I a directed set and the map $U \mapsto \mathcal{F}(U)$, together with the restriction maps, forms a directed system. The **stalk** at x is the group

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U).$$

Examples 3.3.9.

- Let A be an abelian group and $\mathcal{K}$ the constant sheaf on X with group A. For $x \in X$ the map $f \mapsto f(x)$ is an isomorphism $\mathcal{K}_x \to A$. This means that for a constant sheaf all stalks are the same.

- Let $A \neq \{0\}$ be an abelian group, $x \in X$ and let $\mathcal{F}$ be the skyscraper sheaf with $\mathcal{F}(U) = A \iff x \in U$. Assume that X is a Hausdorff space. For $y \neq x$ in X there is an open neighbourhood V with $\mathcal{F}(V) = 0$, therefore the stalk $\mathcal{F}_y$ at y is 0. The stalk at x is A. This justifies the name skyscraper sheaf.

Definition 3.3.10. Let $U \subset X$ be open and $x \in U$. A section $s \in \mathcal{F}(U)$ induces an element of the stalk $\mathcal{F}_x$, which we denote by $s|_x \in \mathcal{F}_x$. Note that there is no danger of confusing this with the restriction of sections, since, even if $\{x\}$ is an open set, the restriction would be denoted by $s|_{\{x\}}$.

Lemma 3.3.11. *Let $\mathcal{F}$ be a sheaf. If a section vanishes on all stalks, it is zero. More precisely, let $U \subset X$ be open and $s \in \mathcal{F}(U)$. If $s|_x = 0$ holds for every $x \in U$, then $s = 0$.*

Proof. The equation $s|_x = 0$ means that there is an open neighbourhood $U_x \subset U$ with $s|_{U_x} = 0$. These U_x form an open cover of U, on which s vanishes. By the Uniqueness axiom, we get $s = 0$. □

Definition 3.3.12. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a presheaf homomorphism. Then ϕ induces a group homomorphism $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ for every $x \in X$. For composable morphisms one has $(\phi\psi)_x = \phi_x\psi_x$ and $\mathrm{Id}_x = \mathrm{Id}$.

Proposition 3.3.13. *A morphismus of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is an isomorphism iff all induced maps on the stalks $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ are isomorphims.*

Proof. If ϕ is an isomorphism, then there is $\psi : \mathcal{G} \to \mathcal{F}$ such that $\psi\phi = \mathrm{Id}$ and $\phi\psi = \mathrm{Id}$. For every $x \in X$ one has $\mathrm{Id}_x = (\phi\psi)_x = \phi_x\psi_x$ and $\mathrm{Id}_x = \psi_x\phi_x$, so ψ_x is inverse to ϕ_x, hence the latter is an isomorphism.

Conversely, let ϕ_x be an isomorphism for every x. We want so show that ϕ is an isomorphism. It suffices to show that $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism of groups for every open set $U \subset X$, since then one sets $\psi_U = \phi_U^{-1}$ and one sees, that ψ is an inverse to ϕ. So we show that ϕ_U is injective. Let $s \in \mathcal{F}(U)$ with $\phi_U(s) = 0$. Then for every $x \in U$ one gets $0 = \phi_U(s)|_x = \phi_x(s|_x)$, which means that $s|_x = 0$ for every $x \in U$ and by Lemma 3.3.11 it follows that $s = 0$, so ϕ is injective.

For surjectivity let $s \in \mathcal{G}(U)$. For every $x \in U$ the map $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective, so there is $f_x \in \mathcal{F}_x$ with $\phi_x(f_x) = s|_x$. So there exists an open neighbourhood $\tilde{U}_x \subset U$ of x such that $f_x = t_x|_x$ for some section $t_x \in \mathcal{F}(\tilde{U}_x)$. This means that the two sections $\phi_{\tilde{U}_x}(t_x)$ and $s|_{\tilde{U}_x}$ induce the same element in the stalk $\mathcal{G}_x$. Hence there is an open neighbourhood $U_x \subset \tilde{U}_x$, such that $\phi_{U_x}(t_x|_{U_x}) = s|_{U_x}$. The U_x form an open cover of U. We want to show that $t_x = t_y$ holds on $U_x \cap U_y$. Then the Existence axiom guarantees that all t_x come from one section in $\mathcal{F}(U)$, which then is a pre-image of s.

For this purpose let $z \in U_x \cap U_y$. Then one has

$$\phi_z(t_x|_z) = s(z) = \phi_z(t_y|_z)$$

and hence $t_x|_z = t_y|_z$. So there is a neighbourhood V_z of z such that $t_x|_{V_z} = t_y|_{V_z}$. The V_z form an open cover of $U_x \cap U_y$, on which we locally have $t_x - t_y = 0$. By the Uniqueness axiom this also holds on $U_x \cap U_y$.

By the Existence axiom there is a section $t \in \mathcal{F}(U)$ with $t|_{U_x} = t_x$ for every x. The sections s and $\phi_U(t)$ coincide in every stalk, so by Lemma 3.3.11 they are equal and so ϕ_U is surjective. □

* * *

3.4 Sheafification

Proposition 3.4.1. *Let $\mathcal{F}$ be a presheaf. Then there is a sheaf $\mathcal{F}^+$ and a presheaf morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ with the property that every presheaf homomorphism $\phi : \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf, factors in a unique way through θ, so for ϕ there is a uniquely determined presheaf homomorphism ψ such that the diagram*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\theta"] \arrow[dr, "\phi"'] & \mathcal{F}^+ \arrow[d, dotted, "\exists!\,\psi"] \\ & \mathcal{G} \end{tikzcd}$$

commutes. The pair $(\mathcal{F}^+, \theta)$ is uniquely determined up to isomorphy. The sheaf $\mathcal{F}^+$ is called the **sheafification** *of $\mathcal{F}$. For every sheaf $\mathcal{G}$ one has*

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \;\cong\; \mathrm{Hom}(\mathcal{F}^+, \mathcal{G}).$$

Proof. We construct the sheaf $\mathcal{F}^+$ as follows. For an open set $U \subset X$ let $\mathcal{F}^+(U)$ be the set of all maps s from U to the disjoint union $\bigsqcup_{x\in U} \mathcal{F}_x$ such that

- for every $x \in U$ one has $s(x) \in \mathcal{F}_x$ and
- for every $x \in U$ there is an open neighbourhood $V \subset U$ and a $t \in \mathcal{F}(V)$, such that for every $y \in V$ one has $s(y) = t_y$.

We show that $\mathcal{F}^+$ is a sheaf. The restriction res^U_V is defined as the restriction of maps.

For the **Existence Axiom** let $U = \bigcup_{i\in I} U_i$ an open cover and let $s_i \in \mathcal{F}^+(U_i)$, $i \in I$ be given with $s_i = s_j$ on $U_i \cap U_j$, $i, j \in I$. This implies that for every $x \in X$ there is a unique $s(x) \in \mathcal{F}_x$ with $s(x) = s_i(x)$ for every $i \in I$ such that $x \in U_i$. By definition, one then has $s|_{U_i} = s_i$ for every $i \in I$ and thus $s \in \mathcal{F}^+(U)$.

The **Uniqueness Axiom** is clear as the elements of $\mathcal{F}(U)$ are maps on U and the restriction is the restriction of maps.

Finally for the **universal property**, we first define $\theta : \mathcal{F} \to \mathcal{F}^+$ by replacing s in the abstract group $\mathcal{F}(U)$ by the map on U, that sends $x \in U$ to $s(x) \in \mathcal{F}_x$. This is a presheaf homomorphism. Next let $\phi : \mathcal{F} \to \mathcal{G}$ be a presheaf homomorphism to a sheaf $\mathcal{G}$. We construct ψ in the diagram as follows: let $s \in \mathcal{F}^+(U)$ for an open $U \subset X$. By construction, for each $x \in U$ there exists an open neighbourhood $U_x \subset U$ and an element $t_x \in \mathcal{F}(U_x)$ such that $s(u) = t_x|_u$ holds for every $u \in U_x$. For any $x, y \in U$ by construction we have that $(t_x - t_y)|_{U_x\cap U_y}$ lies in the kernel of θ. Hence for given $u \in U_x \cap U_y$ there exists an open neighbourhood $V \subset U_x \cap U_y$ such that $t_x|_V = t_y|_V$, hence

$$\phi(t_x|_{U_x\cap U_y})|_u = \phi(t_x|_V)|_u = \phi(t_y|_V)|_u = \phi(t_y|_{U_x\cap U_y})|_u.$$

By Lemma 3.3.11 it follows that

$$\phi(t_x|_{U_x\cap U_y}) = \phi(t_y|_{U_x\cap U_y})$$

Define $g_x \in \mathcal{G}(U_x)$ by $g_x = \phi(t_x)$. We have an open cover $U = \bigcup_{x\in U} U_x$. For $x, y \in U$ we have just shown that $g_x|_{U_x\cap U_y} = g_y|_{U_x\cap U_y}$. As $\mathcal{G}$ is a sheaf, there exists a unique $g \in \mathcal{G}(U)$ that restricts to the g_x. We set $\psi(s)$

to be this g. All we have done is compatible with restrictions, hence ψ is a sheaf homomorphism. The uniqueness of g above implies the uniqueness of ψ.

Note that for $x \in X$ the stalk $\mathcal{F}_x$ is naturally isomorphic to the stalk $\mathcal{F}_x^+$. If $\mathcal{F}$ is a sheaf already, then θ is an isomorphism, as follows by the universal property. $\square$

Definition 3.4.2. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a presheaf homomorphism. We define the kernel and cokernel as the presheaves

$$U \mapsto \ker \phi_U, \qquad U \mapsto \operatorname{coker} \phi_U,$$

together with the ensuing presheaf morphisms

$$\ker(\phi) \ \to \ \mathcal{F}, \quad \mathcal{G} \ \to \ \operatorname{coker}(\phi).$$

Lemma 3.4.3. *If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then* $\ker \phi$ *is a sheaf, but* $\operatorname{coker} \phi$ *is in general not a sheaf.*

Proof. For the Uniqueness axiom let $U = \bigcup_i U_i$ and let $s \in \ker \phi_U$ with $s|_{U_i} = 0$ for every $i \in I$. Then $s = 0$ since $\ker \phi_U \subset \mathcal{F}(U)$ and $\mathcal{F}$ satisfies the Uniqueness axiom.

For The Existence axiom let $s_i \in \ker \phi_{U_i}$ with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$. Since $\mathcal{F}$ satisfies the Existence axiom, there is an $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$. We have to show that $s \in \ker \phi_U$. We know that $\phi_U(s)|_{U_i} = \phi_{U_i}(s|_{U_i}) = \phi_{U_i}(s_i) = 0$ and so $\phi_U(s) = 0$ because of the Uniqueness axiom for $\mathcal{F}$.

We give an examples, in which $\operatorname{coker} \phi$ is not a sheaf. Let $X = \mathbb{R}/\mathbb{Z}$ and let $0 < \varepsilon < \frac{1}{4}$ and

$$U_1 = \left(-\varepsilon, \frac{1}{2} + \varepsilon\right) + \mathbb{Z}, \qquad U_2 = \left(\frac{1}{2} - \varepsilon, 1 + \varepsilon\right) + \mathbb{Z}.$$

For an open subset U of X let $\mathcal{F}(U)$ be the set of locally-constant functions $U \to \mathbb{R}$ and $\mathcal{G}(U)$ the set of all continuous functions $U \to \mathbb{R}$. Let $\phi_U : \mathcal{F}(U) \hookrightarrow \mathcal{G}(U)$ be the inclusion map. For $-\varepsilon < x < \frac{1}{2} + \varepsilon$ let $s_1(x) = x$ and for $\frac{1}{2} - \varepsilon < x < 1 + \varepsilon$ let $s_2(x) = x$. Then s_1, s_2 are elements of $\mathcal{G}(U_1)$ resp. $\mathcal{G}(U_2)$. The difference $s_1 - s_2$ is locally-constant on $U_1 \cap U_2$, so one has $s_1 \equiv s_2 \mod \phi_{U_1 \cap U_2}$. But there is no section $s \in \mathcal{G}(U_1 \cup U_2) = \mathcal{G}(X)$ with $s|_{U_i} \equiv s_i \mod \phi_{U_i}$, since any section in $\mathcal{G}(X)$ has to take the same value at 0 and 1. $\square$

Definition 3.4.4. We define the **sheaf cokernel** of a sheaf homomorphism ϕ as the sheafification of the presheaf cokernel and we write this sheaf cokernel also as $\operatorname{coker} \phi$. Similarly, we define the **image sheaf** of a sheaf homomorphism $\phi : \mathcal{F} \to \mathcal{G}$ as the sheafification of the presheaf $U \mapsto \operatorname{im} \phi_U$ and we write this sheaf as $\operatorname{im}(\phi)$.

Proposition 3.4.5. *The kernel $k : \mathcal{K} \to \mathcal{F}$ of a sheaf homomorphism $\phi : \mathcal{F} \to \mathcal{G}$ is a categorical kernel in the sense of Definition 2.6.5.*

This means that it has the following universal property: Let $\psi : \mathcal{H} \to \mathcal{F}$ be a sheaf homomorphism with $\phi \circ \psi = 0$. Then there exists a uniquely determined sheaf homomorphism $\theta : \mathcal{H} \to \mathcal{K}$, such that the diagram

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "k"] & \mathcal{F} \arrow[r, "\phi"] & \mathcal{G} \\
& \mathcal{H} \arrow[u, "\psi"'] \arrow[ur, "0"'] \arrow[ul, dotted, "\exists!\,\theta"] &
\end{tikzcd}$$

commutes. The cokernel has the same property with all arrows reversed.

Proof. Let $(\mathcal{H}, \psi)$ as in the proposition. For every open $U \subset X$ the morphism $\phi_U \circ \psi_U : \mathcal{H}(U) \to \mathcal{G}(U)$ the zero morphism, so ψ_U factors through a uniquely determined morphism $\theta_U : \mathcal{H}(U) \to \mathcal{K}(U)$. Since ψ is a sheaf homomorphism, so $\psi_V \circ \mathrm{res}^U_V = \mathrm{res}^U_V \psi_U$ and k has the same property, it follows that

$$k_V \,\mathrm{res}^U_V \,\theta_U = \mathrm{res}^U_V \underbrace{k_U \theta_U}_{=\psi_U} = k_V \theta_V \,\mathrm{res}^U_V \,.$$

As k_V is injective, θ is a sheaf homomorphism. This proves the assertion on the kernel. The proof for the cokernel is left to the reader. □

* * *

3.5 Etale-sheaves

Definition 3.5.1. An **etale-sheaf** over a topological space X is a surjective, continuous map $\pi : F \to X$, together with the structure of an abelian group on each fiber $F_x = \pi^{-1}(x)$, $x \in X$ such that

- π is a local homeomorphism, i.e., for every point $f \in F$ there is an open neighbourhood U, such that $\pi(U)$ is open in X and $\pi|_U$ is a homeomorphism onto its image.
- The structure maps are continuous.

The last property means the following: Let S be the set of all $(f, g) \in F \times F$ with $\pi(f) = \pi(g)$, then the maps

$$\begin{array}{rcl} S & \to & E \\ (x, y) & \mapsto & x + y \end{array} \qquad \begin{array}{rcl} E & \to & E \\ x & \mapsto & -x \end{array}$$

are continuous.

The map π is called the **projection** of the etale-sheaf and for $x \in X$ the set $F_x = \pi^{-1}(x)$ is called the **etale-stalk** over x.

For an open set $U \subset X$ we write $F|_U$ for the etale sheaf $\pi^{-1}(U) \xrightarrow{\pi} U$.

Examples 3.5.2.

- (The constant etale-sheaf) Let A be an abelian group, let $F = X \times A$ and let $\pi : F \to X$ be the projection onto the first coordinate. We equip A with the discrete topology and F with the product topology. Then π is an etale-sheaf, where all etale-stalks are isomorphic to A.
- (The scyscraper etale-sheaf) Let $A \neq 0$ be an abelian group and let $x_0 \in X$ be a **closed point**, i.e., the set $\{x_0\}$ is closed. (In a Hausdorff space every point is closed.) Let $F = (X - \{x_0\}) \sqcup A$ and let $\pi : F \to X$ be defined by $\pi(y) = y$ for $y \in X - \{x_0\}$ and $\pi(a) = x_0$ for $a \in A$. Then there is exactly one topology on F, such that π is a local homeomorphism.

  We describe this topology by giving neighbourhood bases for all points. For $a \in A \subset F$ a neighbourhood base is given by all sets of the form $\{a\} \cup (U - \{x_0\})$, where $U \subset X$ is an open neighbourhood of x_0. If $y \in X - \{x_0\}$, then a neighbourhood basis of y is given by all sets of the form $U \setminus \{x_0\}$, where U is an open neighbourhood of y in X.

Remark 3.5.3. Some etale-sheaves are coverings. But in general they're not , since, for instance, like in the case of a skyscraper sheaf, the fibre varies with the point $x \in X$.

Remark 3.5.4. In the definition, we insisted that π be continuous. This condition is redundant, as it already follows from the local homeomorphy.

Remark 3.5.5. For a given etale sheaf $\pi : F \to X$, the **zero section** is the map $s_0 : X \to F$ with $s_0(x) =$ the zero element of the group F_x. This map is continuous.

For this let $x \in X$. Then there exists an open neighbourhood $U \subset F$ of $s_0(x)$, such that π is a homeomorphism from U to $V = \pi(U)$. Let $\phi : V \to U$ be the inverse map. Then for every $y \in V$ we have

$$s_0(y) = \phi(y) - \phi(y).$$

This means that the map $s_0|_V : V \to F$ is the composition of the continuous maps

$$\begin{aligned} V &\to F \times F, \\ y &\mapsto (\phi(y), \phi(y)), \end{aligned}$$

followed by

$$\begin{aligned} F \times F &\to F \times F, \\ (a,b) &\mapsto (a, -b), \end{aligned}$$

followed by the addition. Hence s_0 is continuous.

Definition 3.5.6. Let an etale-sheaf $F \xrightarrow{\pi} X$ be given. For an open set $U \subset X$ let $\mathcal{F}(U)$ be the set of all local sections of π, i.e., the set of all continuous maps $s : U \to F$ with $\pi \circ s = \mathrm{Id}_U$. Then $\mathcal{F}(U)$ is an abelian group under the pointwise operations.

Proposition 3.5.7. *The map $U \mapsto \mathcal{F}(U)$ is a sheaf: For open sets $V \subset U$ the restriction $\mathrm{res}^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ is a group homomorphism. For $W \subset V \subset U$ one has*

$$\mathrm{res}^U_U = \mathrm{Id}_{\mathcal{F}(U)}, \qquad \mathrm{res}^V_W \circ \mathrm{res}^U_V = \mathrm{res}^U_W .$$

Let $(U_i)_{i \in I}$ be an open cover of the open set $U \subset X$. Then one has

- *(Uniqueness) If $s \in \mathcal{F}(U)$ and one has $s|_{U_i} = 0$ for every $i \in I$, then $s = 0$.*
- *(Existence) For every $i \in I$ let $s_i \in \mathcal{F}(U_i)$ be given, such that for any two $i, j \in I$ one has*

  $$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}.$$

  Then ther exists an $s \in \mathcal{F}(U)$, such that $s_i = s|_{U_i}$ for every $i \in I$.

Proof. Since res^U_V is a restriction of functions, these properties are trivial. □

Examples 3.5.8.

- The constant etale-sheaf induces the constant sheaf. This follows, as the sections $s : X \to X \times A$ are exactly the locally constant functions since we equip A with the discrete topology.
- The scyscraper etale-sheaf induces the corresponding skyscraper sheaf.

Definition 3.5.9. Let $\pi : F \to X$ and $\tau : G \to X$ be etale-sheaves. A **morphism of etale-sheaves** from π to τ is a continuous map $\phi : F \to G$ such that

- the diagram

  $$\begin{tikzcd} F \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & G \arrow[dl, "\tau"] \\ & X & \end{tikzcd}$$

  commutes,

- for every $x \in X$, the map ϕ is a group homomorphism from $F_x \to G_x$.

If $\phi : F \to G$ is an etale-sheaf morphism, then for every open set $U \subset X$ one gets a group homomorphism

$$\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

defined by $\phi_U(s)(x) = \phi(s(x))$. Here $\mathcal{G}$ is the sheaf attached tro the etale-sheaf G.

* * *

3.6 Equivalence of sheaves and etale-sheaves

Definition 3.6.1. Let $\mathcal{F}$ be a sheaf over X. We define the **etale-space** to $\mathcal{F}$ as the disjoint union $F = \mathcal{F}_{\mathrm{et}} = \bigsqcup_{x \in X} \mathcal{F}_x$. We define the projection $\pi : F \to X$ by $\pi(f) = x$ if $f \in \mathcal{F}_x$. We construct a topology, which turns $\pi : F \to X$ into an etale-sheaf. For every open set $U \subset X$, every section $s \in \mathcal{F}(U)$ defines a map $s_{\mathrm{et}} : U \to F$ with $\pi \circ s = \mathrm{Id}_U$, this is the map $x \mapsto s|_x$. We equip F with the topology, generated by the sets $s_{\mathrm{et}}(U)$.

Lemma 3.6.2. *The so defined (π, F) is an etale-sheaf.*

Proof. Let $\mathcal{S}$ be the set of all $s_{\mathrm{et}}(U)$, where $U \subset X$ is open and $s \in \mathcal{F}(U)$. We show that $\mathcal{S}$ is stable under intersections, i.e., that

$$A, B \in \mathcal{S} \quad \Rightarrow \quad A \cap B \in \mathcal{S}.$$

For this let $A = s_{\mathrm{et}}(U)$ and $B = t_{\mathrm{et}}(V)$. Let Z be the set of all $x \in U \cap V$ with $s|_x = s_{\mathrm{et}}(x) = t_{\mathrm{et}}(x) = t|_x \in F_x$. By definition of the stalk $F_x = \mathcal{F}_x$ there is an open set $W \subset U \cap V$ such that $s|_W = t|_W$. This means that $s|_y = t|_y$ for every $y \in W$ and therefore Z is open. By definition, one has $A \cap B = s_{\mathrm{et}}(Z) = t_{\mathrm{et}}(Z)$, so this set lies in $\mathcal{S}$ as desired. It follows that the open sets in F are the unions of sets in $\mathcal{S}$.

We note that, as we have seen, the set, where two sections agree, is open. This implies in particular, that every section $s \in \mathcal{F}(U)$ defines a continuous map s_{et}, since, if $A \subset F$ is an open set, i.e., $A = \bigcup_{i \in I} t_{i,\mathrm{et}}(V_i)$, then

$$s_{\mathrm{et}}^{-1}(A) = \bigcup_{i \in I} s_{\mathrm{et}}^{-1}(t_i(V_i)) = \bigcup_{i \in I} \left\{ x : s|_x = t_i|_x \right\}$$

and this set is open.

The projection π is continuous, since for an open set $U \subset X$, the set $\pi^{-1}(U)$ is the union of all $s_{\mathrm{et}}(W)$ for open sets $W \subset U$ and $s \in \mathcal{F}(W)$. This is a union of open sets, hence open. If $p \in F$ and $x = \pi(p)$, then p lies in the stalk $\mathcal{F}_x$, so there is an open set $W \subset X$ and an $s \in \mathcal{F}(W)$, with $p = s_{\mathrm{et}}(x)$. Then $s_{\mathrm{et}}(W)$ is an open neighbourhood of p and $\pi : s_{\mathrm{et}}(W) \to W$ is bijective and continuous with continuous inverse s_{et}. This means that π is a local homeomorphism.

The continuity of the structure maps is left as an exercise. So F is an etale-sheaf. $\square$

In the last proof we have also shown:

Corollary 3.6.3. *Let $\mathcal{F}$ be a sheaf on X and let s, t two sections. Then the set*

$$U = \left\{ x \in X : s(x) = t(x) \right\}$$

is open in X.

Notation. It is convenient, to identify any $s \in \mathcal{F}(U)$ with its etale map $s_{\mathrm{et}} : U \to F$. So in future we will write

$$s(x) = s_{\mathrm{et}}(x) = s|_x \in \mathcal{F}_x = F_x.$$

This takes a bit getting used to, but in the long run it is quite fruitful, as one can more easily switch between the different descriptions of a sheaf.

Theorem 3.6.4. *Let Ψ be the map, that maps a sheaf $\mathcal{F}$ to its etale-sheaf (F, π) and let Φ be the map, that maps an etale-sheaf to the sheaf of its sections.*

Any etale-sheaf F is naturally isomorphic to $\Psi\Phi F$ and every sheaf $\mathcal{F}$ ist naturally isomorphic to $\Phi\Psi\mathcal{F}$.

For any two etale-sheaves F, G over X the map Φ gives an isomorphism of groups

$$\mathrm{Hom}_X(F, G) \xrightarrow{\cong} \mathrm{Hom}_X(\Phi F, \Phi G).$$

Also, for any two sheaves $\mathcal{F}, \mathcal{G}$, the map Ψ yields a group isomorphism

$$\mathrm{Hom}_X(\mathcal{F}, \mathcal{G}) \xrightarrow{\cong} \mathrm{Hom}_X(\Psi\mathcal{F}, \Psi\mathcal{G}).$$

In particular, this means that Φ is an equivalence of categories:

$$\Big\{\textit{etale-sheaves over } X\Big\} \leftrightarrow \Big\{\textit{sheaves over} X\Big\}.$$

Proof. Let $\mathcal{F} = \Phi F$ be the sheaf of sections of F. Then $\Psi\Phi F = \Psi\mathcal{F}$ is the set of stalks of $\mathcal{F}$. We define a map $u_F : \Psi\mathcal{F} \to F$ as follows. Let $f \in \Psi\mathcal{F}$, then f lies in a stalk $\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U)$. So there is an open subset U of x and a section $s \in \mathcal{F}(U)$ with $f = [U, s]$. We define $u_F(f) = s(x)$. The group homomorphism u_F is injective, since $u_F(f) = 0$ implies that there is an open neighbourhood U of x with $f = [U, 0]$, which implies $f = 0$. It is surjective, because for $f \in F$ there is an open neighbourhood V of f such that $\pi|_V$ is a homeomorphism onto its image, the latter we call U. Let $s : U \to F$ be the inverse map to $\pi|_V$, then s is a continuous section, so it lies in $\mathcal{F}(U)$. It therefore defines an element s of $\mathcal{F}_x$ with $u_F(s) = s(x) = f$.

Conversely, we construct a map $v_{\mathcal{F}} : \Phi\Psi\mathcal{F} \to \mathcal{F}$ as follows. Let $U \subset X$ be open. Then every $s \in \Phi\Psi\mathcal{F}(U)$ is a section of the etale-sheaf $\Psi\mathcal{F}$, i.e., a continuous map $s : U \to \bigsqcup_{x \in U} \mathcal{F}_x$ with $s(x) \in \mathcal{F}_x$ for every $x \in U$. By the definition of the topology on $\Psi\mathcal{F}$, any neighbourhood of $y = s(x)$ contains a neighbourhood of the form $t(V)$ for some open neighbourhood V of x and $t \in \mathcal{F}(V)$. By continuity of S, the set U contains a neighbourhood $V' \subset V$, such that $s(V') \subset t(V)$, but that means $s|_{V'} = t|_{V'}$. In other words, s is locally given by sections of $\mathcal{F}$. By the Existence axiom, s is a section of $\mathcal{F}$, i.e., an element of $\mathcal{F}(U)$. The map $v_{\mathcal{F}}$ sends s to this element. Then $v_{\mathcal{F}}$ is an isomorphism. The rest of the theorem follows easily. □

Definition 3.6.5. A sequence of sheaf homomorphisms

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

is called **exact**, if $g \circ f = 0$ and the induced homomorphism $\mathrm{im}(f) \to \ker(g)$ is an isomorphism of sheaves.

Corollary 3.6.6. *A sequence of sheaf homomorphisms*

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

is exact iff for every $x \in X$ the induced sequence of the stalks

$$\mathcal{F}_x \xrightarrow{f_x} \mathcal{G}_x \xrightarrow{g_x} \mathcal{H}_x$$

is exact.

Proof. Theorem 3.6.4 implies that a sequence of sheaves is exact iff the coresponding sequence of etale sheaves is exact. Consider $\mathcal{F}(U)$ as set of etale-sections. Then it is clear that

$$gf = 0 \quad \Leftrightarrow \quad g_x f_x = 0 \ \forall_{x \in X}.$$

So let $gf = 0$. Let $F \xrightarrow{f_{et}} G \xrightarrow{g_{et}} H$ the corresponding sequence of etale-sheaves. The stalks of $\mathrm{im}(f)$ are

$$\mathrm{im}(f)_x = \lim_{U \ni x} f(\mathcal{F}(U)) = f_x(\mathcal{F}_x).$$

That means that $f_{et}(F)$ is the etale-sheaf of $\mathrm{im}(f)$. Likewise, $\ker(g_{et}) := \left\{x \in G : g_{et}(x) = 0\right\}$ is the etale-sheaf of $\ker(g)$. The induced homomorphism $\mathrm{im}(f) \to \ker(g)$ corresponds to inclusion of the etale-sheaves and the exactness is the equality of $f_{et}(F)$ and $\ker(g_{et})$. The claim follows. □

* * *

3.7 Direct and inverse images

Definition 3.7.1. Let $f : X \to Y$ be a continuous map of topological spaces. For a sheaf $\mathcal{F}$ over X define the **direct image** as the sheaf $f_*\mathcal{F}$ over Y given by

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)).$$

The definitions easily imply that this indeed is a sheaf.

If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves over X, then there is an induced morphism $f_*\phi : f_*\mathcal{F} \to f_*\mathcal{G}$, for an open set $U \subset Y$ given by

$$f_*\phi : f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)) \xrightarrow{\phi_{f^{-1}(U)}} \mathcal{G}(f^{-1}(U)) = f_*\mathcal{G}(U).$$

One has $f_*(\phi \circ \psi) = f_*\phi \circ f_*\psi$, so that f_* is a functor from the category AB(X) of sheaves of abelian groups over X to the category AB(Y).

Example 3.7.2. If $f : X \to Y$ is a constant map with image $y_0 \in Y$ and let $\mathcal{F}$ be a sheaf over X. Then $f_*\mathcal{F}$ is the scyscraper sheaf at the point y_0 with stalk $\mathcal{F}(X)$.

Definition 3.7.3. Let $\mathcal{G}$ be a sheaf over Y. Then one defines the **inverse image**, i.e., the sheaf $f^{-1}\mathcal{G}$, which is the sheafification of the presheaf

$$U \mapsto \varinjlim_{V \supset f(U)} \mathcal{G}(V).$$

If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on Y, then let $f^{-1}\phi$ be the morphism from $f^{-1}\mathcal{F}$ to $f^{-1}\mathcal{G}$ derived from

$$\varinjlim_{V \supset f(U)} \mathcal{F}(V) \xrightarrow{\phi} \varinjlim_{V \supset f(U)} \mathcal{G}(V).$$

Again f^{-1} is a functor from AB(Y) to AB(X).

Example 3.7.4. Let $f(x) = y_0$ be the constant map. Then $f^{-1}\mathcal{G}$ is the constant sheaf with stalk $\mathcal{G}_{y_0}$.

Theorem 3.7.5. *Let $f : X \to Y$ be a continuous map. Let $\mathcal{F}$ be a sheaf over X and $\mathcal{G}$ a sheaf over Y. Then there is a natural bijection*

$$\Phi : \mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \xrightarrow{\cong} \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).$$

We say that the functor f^{-1} is **left-adjoint** to f_* or that f_* is **right-adjoint** zu f^{-1}.

Here "natural" means that Φ is a functor in both arguments, i.e., if $\alpha : \mathcal{F} \to \mathcal{F}'$ is a sheaf homomorphism over X, then the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) & \xrightarrow{\Phi} & \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}) \\
\downarrow{\scriptstyle \alpha\circ} & & \downarrow{\scriptstyle f_*\alpha\circ} \\
\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}') & \xrightarrow{\Phi} & \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}')
\end{array}$$

commutes. Likewise, the corresponding diagram for any sheaf homomorphism $\beta : \mathcal{G} \to \mathcal{G}'$ over Y commutes.

Proof. Since $f^{-1}\mathcal{G}$ is the sheafification of the presheaf $f^{\sim}\mathcal{G} : U \mapsto \lim_{\substack{\to \\ V \supset f(U)}} \mathcal{G}(V)$, there is a natural bijection

$$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_X(f^{\sim}\mathcal{G}, \mathcal{F}).$$

therefore it suffices to give a natural bijection $\Phi : \mathrm{Hom}_X(f^{\sim}\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$.

Let $\alpha : f^{\sim}\mathcal{G} \to \mathcal{F}$ be a presheaf homomorphism. For an open subset $U \subset X$ we have a group homomorphism

$$\alpha_U : \lim_{\substack{\to \\ V \supset f(U)}} \mathcal{G}(V) \to \mathcal{F}(U).$$

If $V \subset Y$ is open, then $U = f^{-1}(V)$ is open in X and we define $\beta_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V)) = f_*\mathcal{F}(V)$ by $\beta_V = \alpha_{f^{-1}(V)}$. Then β is a presheaf homomorphism and we set $\Phi(\alpha) = \beta$.

For the converse direction, let $\beta : \mathcal{G} \to f_*\mathcal{F}$ be a sheaf homomorphism, i.e., for every open $V \subset Y$ the map

$$\beta_V : \mathcal{G}(V) \to f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}V)$$

is a group homomorphism, which is compatible with the restriction maps. For an open $U \subset X$ and $V \supset f(U)$ one has $U \subset f^{-1}(V)$ and so one gets a group homomorphism $\mathcal{G}(V) \to \mathcal{F}(f^{-1}V) \xrightarrow{\mathrm{res}} \mathcal{F}(U)$. By the universal property of the direct limit, these homomorphisms glue to a homomorphism

$$\alpha_U : \lim_{\substack{\to \\ V \supset f(U)}} \mathcal{G}(V) \to \mathcal{F}(U).$$

We get an element $\alpha \in \mathrm{Hom}_X(f^{\sim}\mathcal{G}, \mathcal{F})$. Set $\Psi(\beta) = \alpha$. One has $\Psi \circ \Phi = \mathrm{Id}$ and $\Phi \circ \Psi = \mathrm{Id}$. □

* * *

3.8 Locally-constant sheaves

Definition 3.8.1. A sheaf $\mathcal{F}$ over X is called a **locally-constant sheaf**, if every $x \in X$ has an open neighbourhood U, such that $\mathcal{F}|_U$ is constant.

Example 3.8.2. On the space $X = S^1$ it is possible to give a sheaf of abelian groups with each stalk isomorphic to $\mathbb{Z}$, which is locally-constant, but not constant.

We shall be able to prove this by the end of the section.

Proposition 3.8.3. *Let $\mathcal{F}$ be a locally-constant sheaf over X. If X is connected, then the corresponding etale-sheaf $\pi : F \to X$ is a covering. In particular, paths on X can be lifted to F.*

Proof. Let $x \in X$ and U an open neighbourhood, on which $\mathcal{F}$ is constant, we call U a **trivializing neighbourhood** of x. By Examples 3.5.2 and 3.5.8 we know that the etale space $F|_U$ of $\mathcal{F}|_U$ is homeomorphic to $U \times M$, where $M = \mathcal{F}_x$ is the stalk and we have a commutative diagram

$$\begin{tikzcd} F|_U \arrow[rr, "\cong"] \arrow[dr, "\pi"'] & & U \times M \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

where we have written $F|_U$ for the etale sheaf $\pi^{-1}(U) \xrightarrow{\pi} U$.

Let U_x be the set of all $y \in X$ such that there exists a bijection $\mathcal{F}_x \to \mathcal{F}_y$. If $y \in U_x$ and V is a trivializing neighbourhood of y, then $V \subset U_x$, so U_x is open and for any other point $z \in X$ we either have $U_x = U_z$ or $U_x \cap U_z = \emptyset$. With

$$V = \bigcup_{z \in X \smallsetminus U_x} U_z$$

we have $X = U_x \sqcup V$ and both are open and since X is connected we get $U_x = X$ and therefore π is a covering. $\square$

Definition 3.8.4. Let Γ be a group. A **Γ-module** is an abelian group $(M, +)$ together with an action of Γ on M through group homomorphisms, i.e., for each $\gamma \in \Gamma$ one has

$$\gamma.(m + n) = \gamma.m + \gamma.n$$

holds for all $m, n \in M$.

Examples 3.8.5.

- If R is a ring and Γ a subgroup of the unit group $R^\times$, then every R-module is naturally a Γ module.
- Let A be an abelian group and Γ an arbitrary group. The set A^Γ of all maps $f : \Gamma \to A$ forms a Γ-module with the action
$$\gamma.f(\tau) = f(\gamma^{-1}\tau).$$

Definition 3.8.6. From now on let X be a path-connected space, which is locally simply connected.

Let $\mathcal{F}$ be a locally-constant sheaf over X. Let $x_0 \in X$ be a fixed point and let $\Gamma = \pi_1(X, x_0)$ be the fundamental group. Let $[\gamma] \in \Gamma$ and let $m \in M = \mathcal{F}_{x_0}$. Then the path γ lifts to a uniquely determined path $\gamma_m : [0,1] \to F = \mathcal{F}_{\mathrm{et}}$ with $\gamma_m(0) = m$. Write $\gamma.m = \gamma_m(1)$.

Lemma 3.8.7. *The rule $[\gamma]m = \check{\gamma}.m$ defines an action of Γ on the group M. Every $\gamma \in \Gamma$ acts by a group homomorphism, so M is a Γ-module.*

Proof. We need to show well-definedness. Let γ, τ representatives of the same element of Γ and let $h : I^2 \to X$ be a homotopy with fixed ends. Then h lifts to a homotopy with of γ_m to τ_m. In particular one has $\tilde{h}(0,1) = \gamma.m$ and $\tilde{h}(1,1) = \tau.m$. Also $\tilde{h}(s,1) \in \mathcal{F}_{x_0} = M$ for every $s \in [0,1]$. Therefore, $s \mapsto \tilde{h}(s,1)$ is a path M, which connects $\gamma.m$ to $\tau.m$. Since M is discrete, this path is constant, so $\gamma.m = \tau.m$ and the action is well-defined.

Finally, for $[\gamma], [\tau] \in \Gamma$ we have

$$[\gamma]\Big([\tau]m\Big) = [\gamma]\check{\tau}.m = \check{\gamma}.\Big(\check{\tau}.m\Big) = \Big(\check{\tau}.\check{\gamma}\Big).m = (\gamma.\tau)^{\vee}.m = [\gamma.\tau]m = \Big([\gamma][\tau]\Big)m$$

Since the trivial path acts by the identity, we get an action.

Finally, we need to show that Γ acts by group homomorphisms, i.e.,

$$\gamma.(m+n) = \gamma.m + \gamma.n.$$

Let γ_m be unique the lift with $\gamma_m(0) = m$ and define γ_n and γ_{m+n} in the same way. For each $t \in [0,1]$ the points $\gamma_m(t)$ and $\gamma_n(t)$ sit in the fibre over the point $\gamma(t)$, hence they can be added. Then the path $\eta : t \mapsto \gamma_m(t) + \gamma_n(t)$ is yet another lift of γ. But $\eta(0) = m + n = \gamma_{m+n}(0)$ and hence η and γ_{m+n} agree by the uniqueness of lifts. Then

$$\gamma.(m+n) = \gamma_{m+n}(1) = \eta(1) = \gamma_m(1) + \gamma_n(1) = \gamma.m + \gamma.n. \qquad \square$$

Definition 3.8.8. Let M be a Γ-module. Let $\tilde{X}$ be the universal covering of X and set

$$F = \Gamma \backslash (\tilde{X} \times M),$$

where Γ acts diagonally on $\tilde{X} \times M$, so $g(x,m) = (gx, gm)$. We equip M with the discrete topology, $\tilde{X} \times M$ with the product topology and F with the quotient topology. Define $\pi : F \to X$ by $\pi(\Gamma(x,m)) = \Gamma x \in \Gamma \backslash \tilde{X} = X$.

Lemma 3.8.9. *$\pi : F \to X$ is a locally-constant etale-sheaf.*

Proof. Let $x \in X$ and let U be a neighbourhood, tivialising the universal covering $p : \tilde{X} \to X$. The pre-image $\tilde{U} = p^{-1}(U)$ is a disjoint union of open sets, which all are homeomorphic with U and which are permuted by Γ. Fix one such $\tilde{U}_0$ and let $\phi : U \to \tilde{U}_0$ be the inverse map of the projection. Then ϕ is a homeomorphism and the natural map

$$U \times M \xrightarrow{\phi \times 1} \tilde{U}_0 \times M \hookrightarrow \tilde{U} \times M \to \Gamma \backslash \tilde{U} \times M = F|_U$$

is a homeomorphism, trivialising the sheaf $\mathcal{F}$. $\square$

We now have two constructions. Lemma 3.8.7 gives a functor Φ form the category of all locally-constant sheaves to the category of $\mathbb{Z}[\Gamma]$-modules. Conversely, Lemma 3.8.9 yields a functor Ψ from the category of $\mathbb{Z}[\Gamma]$-modules to the category of locally-constant sheaves.

Theorem 3.8.10. *The functors Φ and Ψ are quasi-inverse to each other. For a path-connected and locally simply connected space X, we have an equivalence of categories:*

$$\Big\{\textit{locally-constant sheaves}\Big\} \leftrightarrow \Big\{\Gamma\textit{-modules}\Big\}$$

where $\Gamma = \pi_1(X, x_0)$ is the fundamental group.

Proof. Let F be a locally-constant etale-sheaf. We construct a natural etale-sheaf isomorphism

$$\tau : F \to \Psi\Phi F = \Gamma\Big\backslash\Big(\tilde{X} \times F_{x_0}\Big).$$

Let $f \in F$ and let $x = \pi(f)$. Choose a path η in X from x_0 to x. Then η has a uniquely determined lift η_f to F with $\eta_f(1) = f$. Let $f_0 = \eta_f(0) \in F_{x_0}$. The homotopy class (with fixed ends) of η defines an element $[\eta]$ of $\tilde{X}$ with $p([\eta]) = x$. We define

$$\tau(f) = \Gamma([\eta], f_0).$$

This construction a priori depends on the choice of the path η, at least modulo homotopy with fixed ends. Another choice yields, modulo homotopy, a path of the form $\gamma.\eta$ for some $[\gamma] \in \Gamma$. In this case, f_0 is replaced by $[\gamma]f_0$, so τ is a well-defined map.
The definition of the inverse map τ^{-1} is obvious: An element of $\Gamma\backslash\tilde{X} \times F_{x_0}$ is of the form $\Gamma([\eta], f_0)$ with $[\eta] \in \tilde{X}$ and $f_0 \in F_{x_0}$. Then η lifts in a unique way to a path $\hat{\eta}_{f_0}$ with $\hat{\eta}_{f_0}(0) = f_0$. Then set $\tau^{-1}\left(\Gamma([\eta], f_0)\right) = \hat{\eta}_{f_0}(1)$. Therefore, τ is bijective. The continuity of τ and τ^{-1} as well as the compatibility with addition and inversion is left as an exercise to the reader.

For the converse direction we start with a Γ-module M and consider $\Phi\Psi(M)$. This is the fiber over x_0 of $\Gamma\backslash\tilde{X} \times M$, which by definition equals $\Gamma(f_0 \times M)$, hence is isomorphic to M, where f_0 is an arbitrary element of the fiber over x_0. □

* * *

3.9 The global sections functor

Definition 3.9.1. Let X be a topological space. We consider the functor

$$H^0 : \left\{\text{sheaves over } X\right\} \to \left\{\text{abelian groups}\right\}$$

given by

$$H^0(\mathcal{F}) = \mathcal{F}(X).$$

This is called the **global sections functor**, or just **sections functor**. If one wants to emphasize the space, one also writes $H^0(X, \mathcal{F})$.

Example 3.9.2. As an example we consider a path-connected, locally simply connected space X and a locally-constant sheaf $\mathcal{F}$. This comes from a module M of the fundamental group $\Gamma = \pi_1(X)$ and the etale-sheaf can be written as $\Gamma\backslash(\tilde{X} \times M)$. A global section $s \in \mathcal{F}(X)$ is a map $s : X \to \Gamma\backslash(\tilde{X} \times M)$ of the form $s(\Gamma\tilde{x}) = \Gamma(\tilde{x}, a_s(\tilde{x}))$, with a uniquely determined continuous map $a_s : \tilde{X} \to M$. Since $\tilde{X}$ is connected and M is discrete, the map a_s is constant. For $\gamma \in \Gamma$ one has

$$\Gamma(\tilde{x}, a_s) = s(\Gamma\tilde{x}) = s(\Gamma\gamma\tilde{x}) = \Gamma(\gamma\tilde{x}, a_s) = \Gamma\gamma^{-1}(\gamma\tilde{x}, a_s) = \Gamma(\tilde{x}, \gamma^{-1}a_s).$$

Comparing the two ends of this equation yields

$$a_s = \gamma.a_s,$$

i.e., a_s lies in the space $\in M^\Gamma$ of Γ-invariants. Conversely, every $a_s \in M^\Gamma$ gives a global section, so

$$H^0(\mathcal{F}) \;\cong\; M^\Gamma = H^0(\Gamma, M).$$

We shall come back to this example later.

Lemma 3.9.3. *Let $0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \to 0$ be an exact sequence of sheaves. Then the sequence*

$$0 \to H^0(\mathcal{F}) \xrightarrow{f_X} H^0(\mathcal{G}) \xrightarrow{g_X} H^0(\mathcal{H})$$

is exact. In general, the map g_X will not be surjective.

Proof. Since f has zero kernel, the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is injective for every open $U \subset X$, so in particular for $U = X$, so f_X is injective.

As $g_x f_x = 0$, for every $s \in \mathcal{F}(X)$ and every $x \in X$ one has $g_X(f_X(s))(x) = g_x f_x(s(x)) = 0$, so $g_X(f_X(s)) = 0$, which means that $g_X f_X = 0$. So we get $\mathrm{im}(f_X) \subset \ker(g_X)$ and we want to show equality. For this let $s \in \ker(g_X)$. Then for a given $x \in X$ the element $s(x)$ lies in $\ker(g_x) = \mathrm{im}(f_x) = \varinjlim_{U \ni x} \mathrm{im}(f_U)$. Therefore, there is an open neighbourhood U_x of x with $s|_{U_x} \in f(\mathcal{F}(U_x))$. For every x fix such a neighbourhood U_x and the (uniquely determined) $t_x \in \mathcal{F}(U_x)$ with $f(t_x) = s|_{U_x}$. These U_x form an open cover of X. For $x, y \in X$ one has $t_x|_{U_x \cap U_y} = t_y|_{U_x \cap U_y}$ since the same is true for s and the t_x are uniquely determined. By the Existence axiom there is $t \in \mathcal{F}(X)$ with $t|_{U_x} = t_x$ and by the Uniqueness axiom we infer $f(t) = s$.

At last we give an example for g_X not being surjective: Let $X = \mathbb{R}/\mathbb{Z}$. Let the fundamental group $\Gamma \cong \mathbb{Z}$

act on $M = \mathbb{Z}^2$ in a way that $1.(x, y) = (y, x)$. Then $M^\Gamma = \{(x, x) : x \in \mathbb{Z}\}$. Let $\mathcal{G}$ be the locally-constant sheaf $\Gamma\backslash(\tilde{X} \times M)$. Further let $\mathcal{H}$ be the constant sheaf with stalk $\mathbb{Z}$, which as a locally-constant sheaf is associated to the trivial action of Γ on $\mathbb{Z}$. Let $g : \mathcal{G} \to \mathcal{H}$ be the sheaf homomorphism associated to the Γ-module homomorphism $M \to \mathbb{Z}$, $(x, y) \mapsto x + y$. This is surjective in every stalk, but on the global sections the group $M^\Gamma \to \mathbb{Z}$ has image $2\mathbb{Z} \neq \mathbb{Z}$. $\square$

Definition 3.9.4. For sheaf homomorphisms $f, g : \mathcal{F} \to \mathcal{G}$ we define $f + g : \mathcal{F} \to \mathcal{G}$ by $(f + g)(U) = f(U) + g(U)$. So $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ becomes an abelian group.

Proposition 3.9.5. *Let X be a space. The category* $\mathrm{AB}(X)$ *of sheaves of abelian groups over X is an abelian category.*

Proof. Composition is bilinear, since this holds for the category of abelian groups. The zero object is the zero sheaf. The product of two sheaves $\mathcal{F}, \mathcal{G}$ is isomorphic to the coproduct and both equal the direct sum $U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$. So $\mathrm{AB}(X)$ is additive. Kernels and cokernels exist by Section 3.4. Finally, the last axiom:

- If $\ker(f) = 0$, then f is the kernel of its cokernel. If $\mathrm{coker}(f) = 0$, then f is the cokernel of its kernel. A morphism f with $\ker(f) = 0 = \mathrm{coker}(f)$ is an isomorphism.

is satisfied, since it holds stalkwise. $\square$

* * *

3.10 Resolutions

Definition 3.10.1. An object P of a category C is called a **projective object**, if for every epi $A \twoheadrightarrow B$ and every arrow $P \to B$ ther is an arrow $P \to A$, such that the diagram

$$\begin{tikzcd} A \arrow[r, two heads] & B \\ & P \arrow[u] \arrow[ul, dotted, "\exists"] \end{tikzcd}$$

commutes. This means that arrows from P can be lifted along epis.

In other words, P is projective iff for every epi $A \twoheadrightarrow B$ the ensuing map given by composition

$$\mathrm{Hom}(P, A) \to \mathrm{Hom}(P, B)$$

is surjective.

Examples 3.10.2.

- In the categorie of sets every obect is projective.
- Let R be a ring. In the category of R-modules, free modules are projective.

Definition 3.10.3. An object I of C is called an **injective object**, if it is projective in C^{opp}, this means if for every mono $A \hookrightarrow B$ and every arrow $A \to I$ there exists an arrow $B \to I$, such that the diagram

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & B \arrow[dl, dotted, "\exists"] \\ I & \end{tikzcd}$$

commutes.

This means that I is injective, if arrows to I can be extended along monos.

In other words, I is injective, if for every mono $A \hookrightarrow B$ the induced map

$$\mathrm{Hom}(B, I) \to \mathrm{Hom}(A, I)$$

is surjective.

Examples 3.10.4.

- In the category of sets and maps, every non-empty set is injective.
- In the category of abelian groups an object, i.e., an abelian group $(A, +)$ is injective iff A is **divisible**, which means that for every $a \in A$ and every $n \in \mathbb{N}$ there is $b \in A$ with $a = nb$ (Exercise).

Definition 3.10.5. We say: a category C has **enough inectives**, if for every object X there is a mono $X \hookrightarrow I$, where I is injective. The category is said to have **enough projectives**, if $\mathcal{A}^{\mathrm{opp}}$ has enough inejctives, which means that for every object X there is an epi $P \twoheadrightarrow X$, where P is projective.

Example 3.10.6. The category MOD(R) of modules of a given ring R has enough projectives, since every module is the image of a free module.

Definition 3.10.7. For an abelian group A let

$$A^* = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$$

be the dual group. As the group $\mathbb{Q}/\mathbb{Z}$ is divisible, it follows that for a free abelian group F the dual F^* is divisible.

Lemma 3.10.8. (a) *The canonical map to the bidual,*

$$A \to A^{**},$$
$$a \mapsto \delta_a$$

with $\delta_a(\alpha) = \alpha(a)$, is injective.

(b) *If M is an R-module for a commutative ring R, then M^* be comes an R-module by setting*

$$r\alpha(m) = \alpha(rm).$$

For an R-module homomorphism $f : M \to N$ the **dual homomorphism** *$f^* : B^* \to A^*$, $f^*(\beta) = \beta \circ f$ is an R-module homomorphism. One has $f^{**} = f$, which means that the diagram*

$$\begin{tikzcd} A^{**} \arrow[r, "f^{**}"] & B^{**} \\ A \arrow[u, hook, "\delta"] \arrow[r, "f"] & B \arrow[u, hook, "\delta"'] \end{tikzcd}$$

commutes.

(c) *The dual of $\delta_A : A \hookrightarrow A^{**}$ is written as $p_A : A^{***} \to A^*$. Then the composition*

$$A^* \xrightarrow{\delta_{A^*}} A^{***} \xrightarrow{p_{A^*}} A^*$$

is the identity map.

(d) *The map δ is an R-module homomorphism. If P is a projective R-module, then P^* is injective.*

Proof. (a) The group $\mathbb{Q}/\mathbb{Z}$ is divisible, hence injective in the category AB of abelian groups. Hence for any subgroup $H \subset A$ ensuing map $A^* \to B^*$ is surjective.

We need to show that for any given $a \neq 0$ in A there exists a homomorphism $\eta : A \to \mathbb{Q}/\mathbb{Z}$ with $\eta(a) \neq 0$. For this let $2 \leq n \leq \infty$ be the order of a. Let

$$\eta(a) = \begin{cases} \frac{1}{n} + \mathbb{Z} & n < \infty, \\ \frac{1}{2} + \mathbb{Z} & n = \infty. \end{cases}$$

Then η extends to a non-trivial group homomorphism from the subgroup $\langle a \rangle$ generated by a to $\mathbb{Q}/\mathbb{Z}$. This eta can be extended to all of A and the claimed η has been found.

(b) For $r \in R$ and $\beta \in B^*$, as well as $a \in A$ we have

$$f^*(r\beta)(a) = (r\beta)(f(a)) = \beta(rf(a)) = \beta(f(ra)) = f^*(\beta(ra)) = (rf^*(\beta)(a),$$

so f^* is an R-module homomorphism. To show commutativity of the diagram, for $a \in A$ and $\beta \in B^*$ we compute

$$f^{**}(\delta_a)(\beta) = \delta_a(f^*(\beta)) = \delta_a(\beta \circ f) = \beta(f(a)) = \delta_{f(a)}(\beta).$$

(c) For $\alpha \in A^*$ and $a \in A$ we compute

$$p_A\Big(\delta_{A^*}(\alpha)\Big)(a) = \delta_{A^*}(\alpha)\Big(\delta_A(a)\Big) = \delta_A(a)(\alpha) = \alpha(a).$$

(d) One has

$$\delta_{ra}(\alpha) = \alpha(ra) = (r\alpha)(a) = \delta_a(r\alpha) = r\delta(\alpha).$$

Next let P be projective and assume given an exact diagram

$$\begin{tikzcd} & P^* & \\ 0 \arrow[r] & A \arrow[u, "\eta"] \arrow[r] & B \end{tikzcd}$$

Dualize it to the solid arrow diagram:

$$\begin{tikzcd} & P \arrow[d, hook] \arrow[dd, dotted, bend left] & \\ & P^{**} \arrow[d] & \\ 0 & A^* \arrow[l] & B^* \arrow[l] \end{tikzcd}$$

Since P is projective, the dotted arrow exists and the outer diagram dualizes to

$$\begin{tikzcd} & P^* & \\ 0 \arrow[r] & A^{**} \arrow[u] \arrow[r] & B^{**} \arrow[ul] \\ 0 \arrow[r] & A \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

We show that the arrow $\phi : A \to P^*$ in this diagram coincides with the original arrow η in the first diagram. For this, recall that by construction we have $\phi = p_{P^*} \circ \delta_{P^*} \circ \eta$. But by part (c) it follows that $p_{P^*} \circ \delta_{P^*} = \mathrm{Id}$, hence $\phi = \eta$. It follows that P^* is an injective object. $\square$

Proposition 3.10.9. *Let R be a commutative ring, then the category* MOD(R) *has enough injectives.*

Proof. Let M be an R-module and let

$$P \to M^* \to 0$$

be an exact sequence with P being a projective module. It dualizes to $0 \to M^{**} \to P^*$ and as M embeds into M^{**}, it embeds into P^*, which is injective by the lemma. □

Definition 3.10.10. An **injective resolution** of an object X of an abelian category is an exact sequence

$$0 \to X \to I^0 \to I^1 \to \dots,$$

in which the objects $I^0, I^1, \dots$ are injective. We write $0 \to X \to I_X$.

Lemma 3.10.11. *If the abelian category $\mathcal{A}$ has enough injectives, then for every object there is an injective resolution.*

Proof. Let X be an object and $X \hookrightarrow I^0$ an injection into an injective object. Let M be the cokernel of $X \to I^0$ and let $M \hookrightarrow I^1$ an injection into an injective I^1, then the sequence $0 \to X \to I^0 \to I^1$ is exact. Now let $n \geq 1$ and $I^0, \dots, I^n$ already constructed. Let M be the cokernel of $I^{n-1} \to I^n$, ten choose an injection $M \hookrightarrow I^{n+1}$ in an injective object. Then the sequence $0 \to X \to I^0 \to \dots \to I^{n+1}$ is exact. This finishes the inductive construction of an injective resolution. □

Remark 3.10.12. A **projective resolution** of an object X is an exact sequence of the form

$$\dots \to P_2 \to P_1 \to P_0 \to X \to 0,$$

where all P_j are projective objects. If the category has enough projectives, then projective resiolutions exist for every object.

In the category MOD(R) one can even take free modules, in which case one speaks of a **free resolution**.

* * *

3.11 Derived functors

Definition 3.11.1. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. An additive functor $F : \mathcal{A} \to \mathcal{B}$ is called an **exact functor** if it translates exact sequences to exact sequences. It is called **left-exact**, if for every exact sequence of the form

$$0 \to A \to B \to C \to 0$$

the sequence

$$0 \to F(A) \to F(B) \to F(C)$$

is exact. Correspondingly, it is called **right exact**, if for every exact sequence $0 \to A \to B \to C \to 0$, the sequence

$$F(A) \to F(B) \to F(C) \to 0$$

is exact.

If F is contravariant, one uses the corresponding notions of $\mathcal{A}^{\mathrm{opp}}$, so F is called left-exact, if for every exact sequence as above the sequence

$$0 \to F(C) \to F(B) \to F(A)$$

is exact.

Example 3.11.2. The global sections functor H^0 from the category of sheaves over a given space X to the category of abelian groups is left-exact.

Lemma 3.11.3. *For every object A in an abelian category $\mathcal{A}$ the functor* $\mathrm{Hom}(A, \bullet)$ *is left-exact and the functor* $\mathrm{Hom}(\bullet, A)$ *is right-excat. Here we consider* $\mathrm{Hom}(\bullet, A)$ *as a covariant functor* $\mathcal{A}^{\mathrm{opp}} \to \mathrm{AB}$.

An object A is projective, iff $\mathrm{Hom}(A, \bullet)$ *is exact. A is injective iff* $\mathrm{Hom}(\bullet, A)$ *is exact.*

Proof. Let $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ be exact. Then α is the kernel of β and β the cokernel of α. Let $f : A \to X$ with $\alpha \circ f = 0$. Since $0 \to X$ is the kernel of α, the morphism f factors through the zero map, hence is zero. Therefore the map $\mathrm{Hom}(A, \alpha)$ is injective. ($\mathrm{Hom}(A, \alpha)$ is the functor $\mathrm{Hom}(A, \cdot)$ applied to α). One has $\mathrm{Hom}(A, \beta) \circ \mathrm{Hom}(A, \alpha) = \mathrm{Hom}(A, \beta \circ \alpha) = 0$, since $\beta \circ \alpha = 0$. Let $f : A \to Y$ be in the kernel of $\mathrm{Hom}(A, \beta)$, i.e., $\beta \circ f = 0$. As α is the kernel of β, the morphism f factors through α, i.e., $f = \alpha \circ h = \mathrm{Hom}(A, \alpha)(h)$ for some h. Together we get that the sequence

$$0 \to \mathrm{Hom}(A, X) \to \mathrm{Hom}(A, Y) \to \mathrm{Hom}(A, Z)$$

is exact. The case of $\mathrm{Hom}(\bullet, A)$ follows by switching to the opposite category.

The assertions on projective and injective objects are now only reformulations of the definitions. □

Lemma 3.11.4. *Given two resolutions: $0 \to M \to I_M$ and $0 \to N \to I_N$, where the second is supposed to be injective, every morphism $\phi : M \to N$ extends to a morphism $\alpha : I_M \to I_N$ of complexes.*

If the first resolution is injective as well, then any two such extensions are homotopic, i.e., for two extensions α and β of $\phi : M \to N$, the difference $\alpha - \beta$ is nullhomotopic.

Proof. We have exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f^0"] \arrow[d, "\phi"] \arrow[dr] & I^0_M \arrow[r, "f^1"] \arrow[d, dotted, "\alpha^0"] & I^1_M \arrow[r] & {,} \\
0 \arrow[r] & N \arrow[r, "g^0"'] & I^0_N \arrow[r, "g^1"'] & I^1_N \arrow[r] &
\end{tikzcd}$$

As I^0_N is mono, one can lift the diagonal morphism $M \xrightarrow{\phi} N \to I^0_N$ to I^0_M, this defines α^0. Write $I^{-1}_M = M$, as well as $I^{-1}_N = M$ and $\alpha^{-1} = \phi$. That means, we have constructed α^{-1} and α^0.

For the induction step assume α^{n-2} and α^{n-1} are constructed.

$$\begin{tikzcd}
 & & I^{n-1}_M / \ker(f^n) \arrow[dr, hook] & \\
I^{n-2}_M \arrow[r, "f^{n-1}"] \arrow[d, "\alpha^{n-2}"] & I^{n-1}_M \arrow[ur] \arrow[rr, "f^n"] \arrow[d, "\alpha^{n-1}"] \arrow[drr, "F"] & & I^n_M \arrow[d, dotted, "\alpha^n"] \\
I^{n-2}_M \arrow[r, "g^{n-1}"'] & I^{n-1}_N \arrow[rr, "g^n"'] & & I^n_N
\end{tikzcd}$$

Consider $F : g^n \circ \alpha^{n-1} : I^{n-1}_M \to I^n_N$. If $f^n(x) = 0$, then one has $x = f^{n-1}(y)$ for some y and one gets $F(x) = F(f^{n-1}(y)) = g^n(g^{n-1}(\alpha^{n-2}(y))) = 0$. This means, that F factors through $I^{n-1}_M / \ker(f^n)$. Since I^n_N is injective, one can lift F to I^n_M and such a lift is named α^n.

This finishes existence. Now for uniqueness modulo homotopy. Now the I^p_M are supposed to be injective, too. The morphism of complexes $\alpha - \beta$ extends $\phi = 0$. So we have to show that any extension of the zero map is nullhomotopic. We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "d_M"] \arrow[d, "0"] & I^0 \arrow[r, "d_I^0"] \arrow[d, "\alpha^0"] & I^1 \arrow[r, "d_I^2"] \arrow[d, "\alpha^1"] & I^2 \arrow[r, "d_I^2"] \arrow[d, "\alpha^2"] & \cdots \\
0 \arrow[r] & N \arrow[r, "d_N"'] & J^0 \arrow[r, "d_J^0"'] & J^1 \arrow[r, "d_J^1"'] & J^2 \arrow[r, "d_J^2"'] & \cdots
\end{tikzcd}$$

consisting of injective objects I^k, J^k. We construct morphisms $P^k : I^k \to J^{k-1}$ such that $\alpha^k = d_J^{k-1} P^k + P^{k+1} d_I^k$. We start with $P^0 : I^0 \to N$, this is the zero map. So let $\dots, P^{k-1}, P^k$ already constructed.

$$\begin{tikzcd}
I^{k-1} \arrow[r, "d_I^{k-1}"] \arrow[d] & I^k \arrow[r, "d_I^k"] \arrow[d, "\alpha^k"] \arrow[dl, "P^k"] & I^{k+1} \arrow[r, "d_I^{k+1}"] \arrow[d, "\alpha^{k+1}"] & \cdots \\
J^{k-1} \arrow[r, "d_J^{k-1}"'] & J^k \arrow[r, "d_J^k"'] & J^{k+1} \arrow[r, "d_J^{k+1}"'] & \cdots
\end{tikzcd}$$

In particular we assume $\alpha^k = d_J^{k-1} P^k$, on in the image of I^{k-1}. That means that $\alpha^k - d_J^{k-1} P^k$ is zero on the kernel of d_I^k, so it factors through the image of d_I^k.

$$\begin{tikzcd}
I^{k-1} \arrow[r, "d_I^{k-1}"] \arrow[dd] & I^k \arrow[rr, "d_I^k"] \arrow[dr] \arrow[dd, "\alpha^k"] \arrow[ddl, "P^k"'] & & I^{k+1} \arrow[r, "d_I^{k+1}"] \arrow[dd, "\alpha^{k+1}"] & \cdots \\
& & \mathrm{im}(d_I^k) \arrow[ur, hook] \arrow[dl, "\alpha^k - d_J^{k-1}P^k"] & & \\
J^{k-1} \arrow[r, "d_J^{k-1}"'] & J^k \arrow[rr, "d_J^k"'] & & J^{k+1} \arrow[r, "d_J^{k+1}"'] & \cdots
\end{tikzcd}$$

As J^k is injective, $\alpha^k - d_J^{k-1}P^k$ extends to an arrow $I^{k+1} \to J^k$, which we call P^{k+1}. Then on the one hand we have $\alpha^k = d_J^{k-1}P^k + P^{k+1}d_I^k$ as announced and on the other hand, $\alpha^{k+1} = d_J^k P^{k+1}$, on the image of I^k, so that the construction can go on. The lemma is proven. $\square$

Definition 3.11.5. Let $\mathcal{A}$ be an abelian category with enough injectives and let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor to the abelian category $\mathcal{B}$. For every object X of $\mathcal{A}$ choose an injective resolution $0 \to X \to I_X$ and define

$$R^k F(X) = H^k(F(I_X)).$$

By Lemma 3.11.4 for every morphism $f : X \to Y$ in $\mathcal{A}$ there is a morphism of complexes $I_X \to I_Y$ and so there exists a morphism $R^k F(f) : R^k F(X) \to R^k F(Y)$. By the lemma these morphisms are uniquely determined. In other words: $R^p F$ is a functor from $\mathcal{A}$ to the category $\mathcal{B}$.

Lemma 3.11.6. (a) *If one applies a left-exact functor F to a split-exact sequence* $0 \to A \to B \to C \to 0$*, then the resulting sequence* $0 \to FA \to FB \to FC \to 0$ *is exact.*

(b) *Let* $0 \to I \to B \to C \to 0$ *be an exact sequence in an abelian category, where I is injective. Then the sequence splits.*

(c) *Let* $0 \to X \to Y \to Z \to 0$ *be an exact sequence in an abelian category with enough injectives. Then there exist injective resolutions* I_X, I_Y, I_Z *and morphisms between them such that the diagram*

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_X^0 \arrow[r] \arrow[d] & I_Y^0 \arrow[r] \arrow[d] & I_Z^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_X^1 \arrow[r] \arrow[d] & I_Y^1 \arrow[r] \arrow[d] & I_Z^1 \arrow[r] \arrow[d] & 0 \\
& \vdots & \vdots & \vdots &
\end{tikzcd}$$

is commutative and exact.

Proof. (a) The splitting of the sequence means that B can be replaced with $A \oplus C$, so the sequence decomposes into two isomorphisms, which are preserved by a left-exact functor.

(b) By injectivity, the identity arrow $I \to I$ can be extended to an arrow $s : B \to I$ making the diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "\mathrm{Id}"'] & B \arrow[dl, "s"] \\ & I & \end{tikzcd}$$

commute, i.e., it is a splitting.

(c) Pick I^0_X and I^0_Z first and then set $I^0_Y = I^0_X \oplus I^0_Z$. Then repeat the same for the sequence of the cokernels. □

Theorem 3.11.7. *Let $\mathcal{A}$ be an abelian category with enough injectives and let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor to an abelian category $\mathcal{B}$.*

(a) *For every $n \geq 0$ the functor R^nF is additive. Up to isomorphism of functors, R^nF is independent of the choices of resolutions.*

(b) *There is a natural isomorphism of functors $F \cong R^0F$.*

(c) *For every exact sequence*

$$0 \to X \to Y \to Z \to 0$$

and every $n \geq 0$ there is a natural morphism

$$\delta^n : R^nF(Z) \to R^{n+1}F(X)$$

such that the sequence

$$\cdots \to R^nF(X) \to R^nF(Y) \to R^nF(Z) \xrightarrow{\delta^n} R^{n+1}F(X) \to \ldots$$

is exact.

(d) *For every morphism of short exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & 0 \end{tikzcd}$$

and every $n \geq 0$ the diagram

$$\begin{tikzcd} R^nF(Z) \arrow[r, "\delta^n"] \arrow[d] & R^{n+1}F(X) \arrow[d] \\ R^nF(Z') \arrow[r, "\delta^n"] & R^{n+1}F(X') \end{tikzcd}$$

commutes.

(e) *If I is an injective object, then one has $R^nF(I) = 0$ for $n \geq 1$.*

Proof. The only non-immediate point is the long exact sequence. For this choose resolutions as in Lemma 3.11.6 part (c). Each sequence $0 \to I_X^k \to I_Y^k \to I_Z^k \to 0$ splits by part (b) of the lemma and so the seuence of complexes $0 \to F(I_X) \to F(I_Y) \to F(I_Z) \to 0$ is exact. From here one proceeds as in Theorem 5.4.3 of AlgTop1, when the long exact sequence for homology was constructed. The connection homomorphisms is constructed using the snake lemma. □

* * *

Throughout, let $F : \mathcal{A} \to \mathcal{B}$ be a left exact additive functor between abelian categories and assume that $\mathcal{A}$ has enough injectives.

Definition 3.11.8. An object A of $\mathcal{A}$ is called **acyclic** with respect to F, if for every $i \geq 1$ the equation $R^iF(A) = 0$ holds. Let $X \in \mathcal{A}$. An exact sequence

$$0 \to X \to A^0 \to A^1 \to \dots$$

is called **acyclic resolution** of X, if all A^j are acyclic.

Example 3.11.9. Injective objects are acyclic, and hence injective resolutions are acyclic resolutions.

Lemma 3.11.10. *Let*

$$0 \to A \to B \to C \to 0$$

be an exact sequence in $\mathcal{A}$ and assume that A is F-acyclic. Then the sequence

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is exact, too.

Proof. This follows from the long exact cohomology sequence in part (c) of Theorem 3.11.7. □

Theorem 3.11.11. *Let $\mathcal{A}$ be an abelian category with enough injectives and let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor to an abelian category $\mathcal{B}$. Let $0 \to X \to A^0 \to \dots$ be an F-acyclic resolution. Then there is a natural isomorphism $R^iF(X) \to H^i(F(A^\bullet))$. That means that derived functors (and thus sheaf cohomology) can be computed with arbitrary acyclic resolutions.*

Proof. We need a lemma.

Lemma 3.11.12. *Let $0 \to Y^0 \to Y^1 \to \cdots$ be an exact sequence of F-acyclic objects. Then the sequence $0 \to F(Y^0) \to F(Y^1) \to \dots$ is exact.*

Proof. As F is left-exact, the sequence

$$0 \to F(Y^0) \to F(Y^1) \to F(Y^2)$$

is exact. Let $Z^j = \operatorname{coker}(Y^{j-1} \to Y^j)$. We get a commutative and exact diagram

$$\begin{tikzcd}
0 \arrow[r] & Y^0 \arrow[r] & Y^1 \arrow[rr] \arrow[dr] & & Y^2 \arrow[rr] \arrow[dr] & & Y^3 \\
& & & Z^1 \arrow[ur] \arrow[dr] & & Z^2 \arrow[ur] \arrow[dr] & \\
& & 0 \arrow[ur] & & 0 \arrow[ur] & & 0.
\end{tikzcd}$$

Applying F we get an exact sequence

$$0 \to F(Y^0) \to F(Y^1) \to F(Z^1) \to R^1F(Y^0) = 0.$$

It follows that $F(Z_1) = \mathrm{coker}(F(Y^0) \to F(Y^1))$. The exact sequence $0 \to Z^1 \to Y^2 \to Y^3$ yields an exact sequence

$$0 \to F(Z^1) \to F(Y^2) \to F(Y^3).$$

Plugging in the previuous, we see that the sequence

$$\mathrm{coker}(F(Y^0) \to F(Y^1)) \to F(Y^2) \to F(Y^3)$$

is exact, too. This amounts to the exactness of

$$F(Y^1) \to F(Y^2) \to F(Y^3)$$

i.e., the claimed exactness at $F(Y^2)$. This argument can be repeated to give the claim. □

To prove the theorem choose an injective resolution

$$0 \to X \to I^0 \to I^1 \to \dots.$$

By Lemma 3.11.4 we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "="'] & A^0 \arrow[r] \arrow[d] & A^1 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & X \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots
\end{tikzcd}$$

where the vertical maps, after enlarging I^k if neccessary, can be assumed to be injective. Let (Y^j) be the sequence of cockernels of the vertical maps. We get an exact and commutative diagram

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X \arrow[r] \arrow[d, "="] & A^0 \arrow[r] \arrow[d] & A^1 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & X \arrow[r] & I^0 \arrow[r] \arrow[d] & I^1 \arrow[r] \arrow[d] & \cdots \\
& 0 \arrow[r] & Y^0 \arrow[r] \arrow[d] & Y^1 \arrow[r] \arrow[d] & \cdots \\
& & 0 & 0 &
\end{tikzcd}$$

Each column yields a long exact sequence as in part (c) of Theorem 3.11.7, which for $j \geq 0$ and $k \geq 1$ contains the exact sequence $0 = R^kF(I^j) \to R^kF(Y^j) \to R^{k+1}F(A^{j+1}) = 0$. Hence we get that Y^j is acyclic, too. Further, Lemma 3.11.10 yields, that each sequence

$$0 \to F(A^j) \to F(I^j) \to F(Y^j) \to 0$$

is exact. Therefore, we get an exact sequence of complexes

$$0 \to F(A) \to F(I) \to F(Y) \to 0.$$

with the corresponding long exact cohomology sequence

$$H^{j-1}F(Y) \to H^jF(A) \to H^jF(I) \to H^jF(Y).$$

By Lemma 3.11.12 both ends are zero, so the arrow in the middle is an isomorphism, hence

$$H^jF(A) \;\cong\; H^jF(I)) \;=\; R^iF(X). \qquad \square$$

* * *

3.12 Delta functors

Definition 3.12.1. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. A **δ-functor** form $\mathcal{A}$ to $\mathcal{B}$ is a sequence of additive functors T^i, $i = 0, 1, 2, \dots$, together with a family of morphisms $\delta^i : T^i(C) \to T^{i+1}(A)$ for every exact sequence $0 \to A \to B \to C \to 0$ such that

- For every short exact sequence as above, the sequence
$$0 \to T^0(A) \to T^0(B) \to T^0(C) \xrightarrow{\delta} T^1(A) \to \dots$$
$$\dots \to T^p(A) \to T^p(B) \to T^p(C) \xrightarrow{\delta} T^{p+1}(A) \to \dots$$
is exact.
- For every morphism of short exact sequences
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0 \end{tikzcd}$$
the δs make a commutative diagram
$$\begin{tikzcd} T^p(C) \arrow[r, "\delta"] \arrow[d] & T^{p+1}(A) \arrow[d] \\ T^p(Z) \arrow[r, "\delta"] & T^{p+1}(X). \end{tikzcd}$$

Definition 3.12.2. A δ-functor T is called a **universal δ-functor**, if for every other δ-functor S and every natural transformation $f^0 : T^0 \to S^0$ there is a uniquely determined sequence of natural transformations $f^p : T^p \to S^p$, such that for each exact sequence $0 \to A \to B \to C \to 0$ the diagram

$$\begin{tikzcd} \cdots \arrow[r] & T^p(A) \arrow[r] \arrow[d, "f_A^p"] & T^p(B) \arrow[r] \arrow[d, "f_B^p"] & T^p(C) \arrow[r, "\delta"] \arrow[d, "f_C^p"] & T^{p+1}(A) \arrow[r] \arrow[d, "f_A^{p+1}"] & \cdots \\ \cdots \arrow[r] & S^p(A) \arrow[r] & S^p(B) \arrow[r] & S^p(C) \arrow[r, "\delta"] & S^{p+1}(A) \arrow[r] & \cdots \end{tikzcd}$$

commutes.

Lemma 3.12.3. *If S and T are universal δ-functors and if $T^0 \cong S^0$, then $T^p \cong S^p$ for every $p \geq 0$.*

Proof. Let $f^0 : T^0 \to S^0$ be an isomorphism with inverse $g^0 : S^0 \to T^0$. Let f^p and g^p the uniquely determined extensions for $p \geq 1$. Then $f^p g^p$ is an extension of $f^0 g^0 = \mathrm{Id}$, commuting with the δs. Since such an extension is uniquely determined, it follows $f^p g^p = \mathrm{Id}$. The other direction works the same, so the f^p are isomorphisms. □

Definition 3.12.4. A functor $F : \mathcal{A} \to \mathcal{B}$ is called **erasable**, if for every object $X \in \mathcal{A}$ there is a mono $u : X \hookrightarrow I$ such that $F(u) = 0$. In the applications one will even have $F(I) = 0$, but the definition is a bit more general.

Example 3.12.5. Let $\mathcal{A}, \mathcal{B}$ be abelian categories, $F : \mathcal{A} \to \mathcal{B}$ be additive and left-exact and assume that $\mathcal{A}$ has enough injectives. Then there are the right derived functors R^pF for $p \geq 1$ and these functors are erasable, since they vanish on injective objects.

Theorem 3.12.6. *Let T be a δ-functor, such that each T^p, $p \geq 1$ is erasable. Then T is universal.*

Proof. Let S be another δ-functor and let $f^0 \colon T^0 \to S^0$ be given. We erase a given object A of $\mathcal{A}$ with an object I and we get an exact sequence

$$0 \to A \xrightarrow{u} I \xrightarrow{v} C \to 0$$

with $T^1(u) = 0$. By the long exact sequence, the solid arrows form a commutative diagram with exact rows:

$$\begin{tikzcd}
T^0(I) \arrow[r, "T^0(v)"] \arrow[d, "f_I^0"] & T^0(C) \arrow[r, "\delta_T"] \arrow[d, "f_C^0"] & T^1(A) \arrow[r] \arrow[d, dotted, "f_A^1"] & 0 \\
S^0(I) \arrow[r, "S^0(v)"] & S^0(C) \arrow[r, "\delta_S"] & S^1(A).
\end{tikzcd}$$

The last zero in the top row comes from $T^1(u) = 0$. It follows $\delta_T = \operatorname{coker}(T^0(v))$. As the second row is exact, we get $\delta_S S^0(v) f_I^0 = 0$ and so $\delta_S f^0(C) T^0(v) = 0$. Therefore there is a uniquely determined arrow $f^1(A)$, such that the entire diagram is commutative.

We have to show that f^1 is a natural transformation of functors, i.e., that for every morphism $\tau : A \to B$ in $\mathcal{A}$ the diagram

$$\begin{tikzcd}
T^1(A) \arrow[r, "T^1(\tau)"] \arrow[d, "f_A^1"'] & T^1(B) \arrow[d, "f_B^1"] \\
S^1(A) \arrow[r, "S^1(\tau)"] & S^1(B)
\end{tikzcd}$$

commutes. For this let $\tau \colon A \to B$ be a morphism in $\mathcal{A}$. Consider the pushout P:

$$\begin{tikzcd}
A \arrow[r, hook, "u"] \arrow[d, "\tau"] & I \arrow[d] \\
B \arrow[r] & P.
\end{tikzcd}$$

Since u is mono, by Lemma 2.7.4 the arrow $B \to P$ is mono as well. Let $P \hookrightarrow N$ be a monomorphism, which erases P. We get a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\tau"] & I \arrow[r] \arrow[d, "\alpha"] & C \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & B \arrow[r] & N \arrow[r] & Y \arrow[r] & 0,
\end{tikzcd}$$

Where $B \to N$ is the composition $B \to P \to N$ and Y is the cokernel. The diagram, the commutativity of which we want to show, is the right side square in the following cube diagram:

$$\begin{tikzcd}
T^0(C) \arrow[rr, "\delta_T"] \arrow[dr, "T^0(\beta)"] \arrow[dd, "f_C^0"'] & & T^1(A) \arrow[dr, "T^1(\tau)"] \arrow[dd] & \\
& T^0(Y) \arrow[rr] \arrow[dd] & & T^1(B) \arrow[dd, "f_B^1"] \\
S^0(C) \arrow[rr] \arrow[dr, "S^0(\beta)"'] & & S^1(A) \arrow[dr, "S^1(\tau)"] & \\
& S^0(Y) \arrow[rr] & & S^1(B).
\end{tikzcd}$$

All side squares of the diagramm commute with the possible exception of the right hand square. But since δ_T is an epi, this square has to commute, as well.

Next we need to show that f_1 commutes with the connection morphism δ. Let

$$0 \to A \to B \to C \to 0$$

be an exact sequence in $\mathcal{A}$. By the same pushout construction construction one gets an erasing monomorphism $A \to I$ and a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{Id}"] & B \arrow[r] \arrow[d, "\alpha"] & C \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & A \arrow[r] & I \arrow[r] & X \arrow[r] & 0.
\end{tikzcd}$$

Consider the diagram:

$$\begin{tikzcd}
& T^0(C) \arrow[d, "f_C^0"] \arrow[ddl, "T^0(\beta)"'] \arrow[ddr, "\delta_T"] & \\
& S^0(C) \arrow[ddl, "S^0(\beta)"'] \arrow[ddr, "\delta_S"] & \\
T^0(X) \arrow[rr, "\delta_T"] \arrow[d, "f_X^0"] & & T^1(A) \arrow[d, "f_A^1"] \\
S^0(X) \arrow[rr, "\delta_S"] & & S^1(A).
\end{tikzcd}$$

We want to show that the right hand square commutes. The triangles above and below are commtuative by the definition of the δ-functor. The left square is commutative, since f^0 is a natural transformation. The front square commutes by the definition of f^1. Hence the last square commutes as well.

An iteration of this argument with the index pair $(n, n+1)$ in stead of $(0, 1)$ gives the theorem. □

* * *

3.13 Sheaf cohomology

Proposition 3.13.1. *Let R be a ring and X a topological space. Then the abelian category* $\mathrm{MOD}_R(X)$ *of all sheaves of R-modules has enough injectives.*

Proof. Let $\mathcal{F}$ be a sheaf of R-modules over X. For every $x \in X$ the stalk $\mathcal{F}_x$ is an R-module, so there exists an injection $\mathcal{F}_x \hookrightarrow J_x$ into an injective R-module. Consider the sheaf $\mathcal{J} : U \mapsto \prod_{x \in U} J_x$. This is the product of the skyscraper sheaves $\mathcal{S}_x(J_x)$ for $x \in X$ in the category $\mathrm{MOD}_R(X)$ of sheaves of R-modules over X. So for every sheaf $\mathcal{G}$ one has

$$\mathrm{Hom}(\mathcal{G}, \mathcal{J}) \cong \prod_{x \in X} \mathrm{Hom}(\mathcal{G}, \mathcal{S}_x(J_x)).$$

On the other hand one has $\mathrm{Hom}(\mathcal{G}, \mathcal{S}_x(J_x)) \cong \mathrm{Hom}(\mathcal{G}_x, J_x)$. So there exists a natural injective homomorphism $\mathcal{F} \to \mathcal{J}$ given by the maps $\mathcal{F}_x \to J_x$. The functor $\mathrm{Hom}(\bullet, \mathcal{J})$ is the direct product over all $x \in X$ of the stalk functor $\mathcal{F} \mapsto \mathcal{F}_x$, which is exact, followed by $\mathrm{Hom}_R(\bullet, J_x)$, which is exact, as J_x is injective. That means that $\mathrm{Hom}(\bullet, \mathcal{J})$ is an exact functor, so $\mathcal{J}$ is an injective object. $\square$

Definition 3.13.2. The **sheaf cohomology** of a sheaf $\mathcal{F}$ is defined to consist of the right derivatives of the global sections functor, i.e.,

$$H^k(\mathcal{F}) = R^k H^0(\mathcal{F}).$$

One also writes this as

$$H^k(X, \mathcal{F}) = R^k H^0(X, \mathcal{F}).$$

Definition 3.13.3. A sheaf $\mathcal{F}$ is called **flabby**, if for any two open sets $V \subset U \subset X$ the restriction $\mathrm{res}^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ is surjective. Skyscraper sheaves are examples of flabby sheaves.

Theorem 3.13.4. (a) *Injective sheaves are flabby.*

(b) *If $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ is an exact sequence of sheaves and if $\mathcal{F}$ is flabby, then for every open set $U \subset X$ the sequence*

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$$

is exact.

(c) *Let $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ be an exact sequence of sheaves and let $\mathcal{F}$ and $\mathcal{G}$ be flabby. Then $\mathcal{H}$ is flabby.*

(d) *Flabby sheaves are acyclic with respect to the global sections functor H^0.*

Proof. (a) Let $\mathbb{Z}$ be the constant sheaf with stalk $\mathbb{Z}$. For an open set $U \subset X$ let $\mathbb{Z}_U = j_!(\mathbb{Z}|_U)$ be the sheafification of the presheaf

$$V \mapsto \begin{cases} \mathbb{Z}(V) & V \subset U, \\ 0 & \text{otherwise.} \end{cases}$$

We show that for every sheaf $\mathcal{F}$ there is a natural isomorphism

$$\mathcal{F}(U) \cong \mathrm{Hom}(\mathbb{Z}_U, \mathcal{F}).$$

For $s \in \mathcal{F}(U)$ let $\phi_s : \mathbb{Z}_U \to \mathcal{F}$ be defined as follows. Let $V \subset X$ be open and $t \in \mathbb{Z}_U(V)$. Then

$$\phi_s(t) = ts|_{U \cap V}.$$

The inverse map to $s \mapsto \phi_s$ is given by $\phi \mapsto s_\phi$ with $s_\phi = \phi(1_U)$.

Let $\mathcal{I}$ be an injective sheaf and let $V \subset U \subset X$ be open sets. These induce a mono $\mathbb{Z}_V \hookrightarrow \mathbb{Z}_U$ and since $\mathcal{I}$ is injective, one gets a surjection $\mathcal{I}(U) = \mathrm{Hom}(\mathbb{Z}_U, \mathcal{I}) \to \mathrm{Hom}(\mathbb{Z}_V, \mathcal{I}) = \mathcal{I}(V)$. So $\mathcal{I}$ is flabby.

(b) We have an exact sequence of sheaves $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ and we want to show that $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$ is exact. Let $s \in \mathcal{H}(U)$. Then there is an open cover $U = \bigcup_{i \in I} U_i$ and $t_i \in \mathcal{G}(U_i)$ such that $s|_{U_i} = \beta(t_i)$. One has that $(t_i - t_j)|_{U_i \cap U_j}$ lies in the kernel of β, i.e., in the image of α, so local pre-images exist. As α is injective, the local pre-images are compatible, so they come from some $f_{i,j} \in \mathcal{F}(U_i \cap U_j)$. As $\mathcal{F}$ is flabby, there is a $\tilde{f}_{i,j} \in \mathcal{F}(U_i)$ with $\tilde{f}_{i,j}|_{U_i \cap U_j} = f_{i,j}$. If one replaces t_i by $t_i - \alpha(\tilde{f}_{i,j})$, one gets $t_i|_{U_i \cap U_j} = t_j|_{U_i \cap U_j}$, so these extend to a section $t \in \mathcal{G}(U)$ with $\beta(t) = s$.

(c) We have an exact sequence of sheaves $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$, where now $\mathcal{F}$ and $\mathcal{G}$ are flabby and we want to show that $\mathcal{H}$ is flabby.

Let $V \subset U \subset X$ be open sets. We get a commutative diagram with exact rows (by part (b))

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) & \longrightarrow & \mathcal{H}(U) & \longrightarrow & 0 \\
 & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}(V) & \longrightarrow & \mathcal{G}(V) & \longrightarrow & \mathcal{H}(V) & \longrightarrow & 0
\end{array}$$

Using diagram chase, one sees that the last vertical arrow is an epi, too.

(d) We now show that flabby sheaf are acyclic, which is the reason why one considers them in the first place. Let $\mathcal{F}$ be a flabby sheaf. Since $\mathcal{F}$ can be embedded in an injective sheaf, we get an exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{H} \to 0,$$

where $\mathcal{I}$ is injective. By (a), $\mathcal{I}$ is flabby and by (c), $\mathcal{H}$ is, too. Since $\mathcal{I}$ is injective, one has $H^p(\mathcal{I}) = 0$ for $p \geq 1$. The long exact cohomology sequence looks like this

$$\begin{aligned}
0 \to H^0(\mathcal{F}) \to H^0(\mathcal{I}) \to H^0(\mathcal{H}) \to& \\
\to H^1(\mathcal{F}) \to 0 \to H^1(\mathcal{H}) \to& \\
\to H^2(\mathcal{F}) \to 0 \to H^2(\mathcal{H}) \to \dots&
\end{aligned}$$

By (b) the first line is exact when one puts a zero at the end, which means that $H^1(\mathcal{F}) = 0$ holds for every flabby sheaf. Further one has $H^k(\mathcal{H}) \cong H^{k+1}(\mathcal{F})$ for $k \geq 1$. But since $\mathcal{H}$ is flabby, one also has $H^1(\mathcal{H}) = 0$ and therefore $H^2(\mathcal{F}) = 0$ and also $H^2(\mathcal{H}) = 0$ and so on. $\square$

* * *

3.14 Fine sheaves

Definition 3.14.1. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a sheaf homomorphism. The **support** of ϕ, written $\mathrm{supp}(\phi)$ is the closure of the set of all $x \in X$ with $\phi_x : \mathcal{F}_x \to \mathcal{G}_x \neq 0$, i.e.,

$$\mathrm{supp}(\phi) = \overline{\left\{x \in X : \phi_x \neq 0\right\}}.$$

Remark 3.14.2. In books on Algebraic Geometry you will find the definition of the support of a sheaf $\mathcal{F}$ and the set of points x, where $\mathcal{F}_x \neq 0$. That means that one doesn't take the closure of this set. This is useful for non-Hausdorff spaces, as the closure can simply be too large. Here we use the notion of support as it is customary in Analysis, as the applications lie in the realm of Analysis.

Definition 3.14.3. A sheaf $\mathcal{F}$ is called a **fine sheaf**, if for every open cover $X = \bigcup_{i\in I} U_i$ there is a family $(\phi_i)_{i\in I}$ of endomorphisms $\phi_i : \mathcal{F} \to \mathcal{F}$ with

(a) $\mathrm{supp}(\phi_i) \subset U_i$ and

(b) the family (ϕ_i) is locally-finite in the sense that for each $x \in X$ there exists an open neighbourhood U, such that $U \cap \mathrm{supp}(\phi_i) = \emptyset$ for all but finitely many $i \in I$,

(c) $\sum_{i\in I} \phi_i = \mathrm{Id}|_{\mathcal{F}}$.

Examples 3.14.4.

(a) Let M be a smooth manifold. Then for every cover (U_i) there is a **partition of unity**, i.e., a locally-finite family $u_i \in C^\infty(M)$ with $\mathrm{supp}(u_i) \subset U_i$ and

$$\sum_{i\in I} u_i = 1.$$

This means that the sheaf C^∞ of germs of smooth functions in fine, and so is the sheaf Ω^p of all smooth p-differential forms, as one can define $\phi_i(\omega) = u_i\omega$.

(b) Skyscraper sheaves are fine.

Lemma 3.14.5. (a) *If $0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \to 0$ is an exact sequence of sheaves and if $\mathcal{F}$ is fine, then the sequence of groups $0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to 0$ is exact.*

(b) *If X is paracompact Hausdorff space, then to every sheaf $\mathcal{F}$ there exists a monomorphism $\mathcal{F} \hookrightarrow \mathcal{J}$, where $\mathcal{J}$ is a product of skyscraper sheaves with injective stalks. The sheaf $\mathcal{J}$ is fine and injective.*

(c) *Let $0 \to \mathcal{F} \to I^0 \to I^1 \to \dots$ be a resolution into sheaves I^p which are products of skyscraper sheaves with injective stalks. If $\mathcal{F}$ is fine, then the sequence $0 \to \mathcal{F}(X) \to I^0(X) \to I^1(X) \to \dots$ is exact.*

Proof. (a) We have to show, that g_X is surjective. So let $t \in \mathcal{H}(X)$. Then there is a cover (U_i) of X and $s_i \in \mathcal{G}(U_i)$ such that $g(s_i) = t|_{U_i}$. The difference

$$s_{ij} = s_i - s_j$$

is a section of $\ker(g) \cong \mathcal{F}$ on $U_i \cap U_j$. Over $U_i \cap U_j \cap U_k$ one has

$$s_{ij} + s_{jk} = s_{ik}.$$

Let ϕ_i be a family of endomorphisms $\mathcal{F} \cong \ker(g)$ associated to (U_i). Since the support of ϕ_j lies in U_j, one can extend $\phi_j(s_{ij})$ to a section in $\ker(g)(U_i) \subset \mathcal{G}(U_i)$. Let

$$s'_i = \sum_j \phi_j(s_{ij}).$$

Then $s'_i \in \ker(g)(U_i)$ and over $U_i \cap U_j$ one has:

$$s'_i - s'_j = \sum_k \phi_k(s_{ik}) - \sum_k \phi_k(s_{jk}) = \sum_k \phi_k(s_{ij}) = s_{ij}.$$

Therefore

$$s_i - s'_i = s_j - s'_j$$

holds on $U_i \cap U_j$. Since $g(s'_i) = 0$ and $g(s_i) = t|_{U_i}$, the prescription $s(x) = (s_i - s'_i)(x)$ for $x \in U_i$ defines a global section s of $\mathcal{G}$ with $g(s) = t$.

(b) Let $\mathcal{J}$ be the product of all skyscaper sheaves $\mathcal{S}_x(J_x)$, as constructed in the proof of the existence of enough injective sheaves, Proposition 3.13.1. Then $\mathcal{J}$ is injective. We need to show that it is fine. Since X is paracompact, it suffices to consider a locally-finite cover $(U_i)_{i \in I}$. As $\mathcal{J}$ is a product of skyscraper sheaves and singletons $\{x\}$, $x \in X$ are closed, there is a family $(\phi_i)_i$ of endomorphisms of $\mathcal{J}$, which in every point only take the value 0 or Id with the property that $\operatorname{supp} \Phi_i \subset U_i$ and $\sum_i \phi_i = \mathrm{Id}$. So $\mathcal{J}$ is fine.

(c) For every object A in an abelian category, on the set $\operatorname{End}(A) = \operatorname{Hom}(A, A)$ addition and composition establish the structure of a ring with unit. The category of cochain complexes for a given category is again an abelian category. So let $C^\bullet = (C^p)_{p \in \mathbb{Z}}$ be a complex of sheaves over a space X and let $R = \operatorname{End}(C^\bullet)$. Let $N \subset R$ be the set of nullhomotopic endomorphisms.

N is a two-sided ideal in R.
To show this, let $\phi \in R$ and let $n \in N$. Let P be a homotopy such that $n = Pd + dP$. Then, as ϕ is a complex morphism, it commutes with the differentials, so $n\phi = p\phi d + dP\phi$, so $p\phi$ is a nullhomotopy for $n\phi$. Likewise, ϕP is a nullhomotopy for ϕn.

Let now $\mathcal{F}$ be a fine sheaf and let $0 \to \mathcal{F} \to I^0 \to I^1 \to \dots$ be a resolution, where each I^p is a product of skyscraper sheaves with injective stalks. As any endomorphism $\phi : \mathcal{F} \to \mathcal{F}$ can be extended to a complex-endomorphism for the resolution by Lemma 3.11.4. This extension can be done stalk-wise, and so the extension ϕ_p will satisfy $\operatorname{supp}(\phi_p) \subset \operatorname{supp} \phi$. The complex-endomorphism $(\phi_\bullet)$ is uniquely determined up to homotopy. Therefore we get a ring homomorphism

$$\operatorname{End}(\mathcal{F}) \to \operatorname{End}(I^\bullet)/N.$$

The sequence $0 \to \mathcal{F}(X) \to I^0(X) \to I^1(X)$ is exact. So let $p \geq 1$ and let $s \in I^p(X)$ with $ds = 0$. Since $I^{p-1} \to I^p \to I^{p+1}$ is exact, there exists a covering $(U_i)_{i \in I}$ such that for every $i \in I$ there is $t_i \in I^{p-1}(U_i)$ with $s|_{U_i} = dt_i$. Let (ϕ_i) be a family of endomorphisms of $\mathcal{F}$ underlying the covering (U_i) with $\operatorname{supp}(\phi_i) \subset U_i$

and $\sum_{i\in I}\phi_i = \mathrm{Id}$. These endomorphisms can be lifted to $I^\bullet$ satisfying $\mathrm{supp}(\phi_{i,p}) \subset U_i$ for each $i \in I$ and one has $\mathrm{Id} = h + \sum_i \phi_{i,\bullet}$, where h is null-homotopic. As $ds = 0$ it follows that $h(s) = dt$ for some $t \in I^{p-1}(X)$. We can extend each $\phi_{i,p}(t_i)$ and each $\phi_{i,p}(dt_i)$ by zero outside U_i to get

$$s = h(s) + \sum_i \phi_{i,p}(s) = dt + \sum_i \underbrace{\phi_{i,p}(dt_i)}_{=d\phi_{i,p}(t_i)} = d\left(t + \sum_i \phi_{i,p}(t_i)\right). \qquad \square$$

Theorem 3.14.6. *Let X be a paracompact Hausdorff space. Then fine sheaves on X are acyclic with respect to the global sections functor H^0.*

Proof. Let $\mathcal{F}$ be a fine sheaf on X. By (b) and (c) of the lemma there is an injective resolution $0 \to \mathcal{F} \to I^0 \to I^1 \to \dots$ such that the sequence $0 \to \mathcal{F}(X) \to I^0(X) \to I^1(X) \to \dots$ is exact. The sheaf cohomology is by definition the cohomology of the complex $0 \to I^0(X) \to I^1(X) \to \dots$, which is exact, except at $I^0(X)$, so $H^k(X, \mathcal{F}) = 0$ for $k \geq 1$. $\square$

* * *

3.15 Cech Cohomology

In this section, X will denote a paracompact Hausdorff space.

Lemma 3.15.1. (a) *Let $U \subset X$ be open, $Z \subset X$ closed with $Z \subset U$. Then there is an open set V with*

$$Z \subset V \subset \overline{V} \subset U.$$

(b) *For every locally-finite cover $(U_i)_{i\in I}$ there is a refinement $(V_i)_{i\in I}$, such that for every $i \in I$ one has $\overline{V}_i \subset U_i$.*

Proof. (a) Let A be the closed set $A = X \smallsetminus U$. We first consider the case $Z = \{z\}$. As X is a Hausdorff space, for every $a \in A$ there is an open neighbourhood B_a with $z \notin \overline{B}_a$. Then $(B_a)_{a\in A} \cup \{U\}$ is an open cover of X. By paracompactness there is a locally-finite refinement $(W_j)_{j\in J} \cup \{U\}$, where we have, that for every $j \in J$ there is an $a \in A$ with $W_j \subset B_a$. Let $\tilde{V}$ be an open neighbourhood of z, that meets only finitely many W_j, say $W_1, \dots, W_n$. Then

$$V = \tilde{V} \smallsetminus (\overline{W}_1 \cup \dots \cup \overline{W}_n)$$

is an open neighbourhood of z, which satisfies the claim, since $z \notin \overline{B_a}$ for all $a \in A$ and $W_j \subset B_a$ for some a.

Now let Z be arbitrary. By the first part there is, to every $z \in Z$, an open neighbourhood V_z with

$$z \in V_z \subset \overline{V}_z \subset U.$$

Therefore $(V_z)_{z\in Z} \cup \{X \smallsetminus Z\}$ is an open cover of X. Let $(V_i)_{i\in I} \cup \{X \smallsetminus Z\}$ be a locally-finite refinement. As this cover is locally-finite, one has

$$\overline{\bigcup_{i\in I} V_i} = \bigcup_{i\in I} \overline{V}_i.$$

Let $V = \bigcup_{i\in I} V_i$. Then V is open and

$$Z \subset V \subset \overline{V} = \bigcup_i \overline{V}_i \subset U.$$

For (b) let $(U_i)_{i\in I}$ be a locally-finite cover. Let S be the set of all families of open sets $(V_i)_{i\in J}$, where $J \subset I$ and $\overline{V}_i \subset U_i$, such that $(V_i)_{i\in J} \cup (U_i)_{i\in I\smallsetminus J}$ is a cover of X. On S we instal the partial order

$$(V_i)_{i\in J} \;\le\; (\tilde{V}_i)_{i\in\tilde{J}} \quad\Leftrightarrow\quad J \subset \tilde{J},\ V_i = \tilde{V}_i\ \forall_{i\in J}.$$

By Zorn's lemma there is a maximal element $(V_i)_{i\in J}$. We claim that $J = I$. **Assume**, this is not the case. Let $i_0 \in I \smallsetminus J$ and let

$$Z = X \smallsetminus \left(\bigcup_{i\in J} V_i \cup \bigcup_{\substack{i\ne i_0 \\ i\in I\smallsetminus J}} U_i \right)$$

Then Z is closed and since the V_i and U_i form a cover, we get $Z \subset U_{i_0}$. By (a) there is an open set $V_{i_0} \subset X$ with $Z \subset V_{i_0} \subset \overline{V}_{i_0} \subset U_{i_0}$. Therefore, J can be enlarged by i_0, the family wasn't maximal, which is a **contradiction** and the claim follows. □

Definition 3.15.2. Let X be a paracompact Hausdorff space. Let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of X. A tuple $(U_0,\dots,U_k)$ of sets of the cover is called a **Cech-k-simplex** or in this section just a k-simplex. For a k-simplex $\sigma = (U_0,\dots,U_k)$ we define its **support** to be the set $|\sigma| = U_0 \cap \dots \cap U_k$. The ***i*-th side** of a k-simplex σ is the $k-1$-Simplex

$$\sigma^i = (U_0,\dots \widehat{U_i} \dots, U_k).$$

Let $\mathcal{F}$ be a sheaf over X and let $C^k(\mathcal{U},\mathcal{F})$ be the set of all maps f, which attach to a k-simplex σ an element of $\mathcal{F}(|\sigma|)$. Note that $\mathcal{F}(\emptyset) = 0$. The elements of $C^k(\mathcal{U},\mathcal{F})$ are called k-cochains. Define

$$d : C^k(\mathcal{U},\mathcal{F}) \ \to\ C^{k+1}(\mathcal{U},\mathcal{F})$$

by

$$df(\sigma) = \sum_{i=0}^{k+1}(-1)^i f(\sigma^i)|_{|\sigma|}.$$

One has $d^2 = 0$, hence one gets a cochain complex, whose cohomology one writes as $\check{\mathrm{H}}^p(\mathcal{U},\mathcal{F})$. If $\phi : \mathcal{F} \to \mathcal{G}$ is a sheaf homomorphism, then one gets a morphism of cochain complexes $C^\bullet(\mathcal{U},\mathcal{F}) \to C^\bullet(\mathcal{U},\mathcal{G})$ and so a morphism $\check{\mathrm{H}}^k(\mathcal{U},\mathcal{F}) \to \check{\mathrm{H}}^k(\mathcal{U},\mathcal{G})$. Therefore $\check{\mathrm{H}}^k(\mathcal{U},\cdot)$ is a functor from the categorie of sheaves over X to the category of abelian groups.

An element f of $C^0(\mathcal{U},\mathcal{F})$ maps an element U_i to a section $s_i \in \mathcal{F}(U_i)$. If $df = 0$, then

$$0 = df(U_i,U_j) = f(U_j)|_{U_i\cap U_j} - f(U_i)|_{U_i\cap U_j},$$

and one then has $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$, which implies that $f(U_i) = s|_{U_i}$ for a unique global section $s \in \mathcal{F}(X)$. Conversely, every global section s yields a map f as above by setting $f(U_i) = s|_{U_i}$. One gets an isomorphism

$$\check{\mathrm{H}}^0(\mathcal{U},\mathcal{F}) \ \cong\ H^0(\mathcal{F},X).$$

Let $\mathcal{V}$ be a refinement of the cover $\mathcal{U}$. Then there is a map $\mu : \mathcal{V} \to \mathcal{U}$ such that $V \subset \mu(V)$ for every $V \in \mathcal{V}$. We call such a map a **refinement map** If $\sigma = (V_0,\dots,V_k)$ is a k-simplex of the cover $\mathcal{V}$, then $\mu(\sigma) = (\mu(V_0),\dots,\mu(V_k))$ is a k-simplex of the cover $\mathcal{U}$. This μ induces a cochain map $\mu_k^\# : C^k(\mathcal{U},\mathcal{F}) \to C^k(\mathcal{V},\mathcal{F})$ given by

$$\mu_k^\#(f)(\sigma) = f(\mu(\sigma))|_{|\sigma|}$$

and yielding a homomorphism

$$\mu_k^* : \check{\mathrm{H}}^k(\mathcal{U},\mathcal{F}) \ \to\ \check{\mathrm{H}}^k(\mathcal{V},\mathcal{F}).$$

We need to show that the latter map on cohomology does not depend on the choice of a refinement map.

Lemma 3.15.3. *Let $\tau : \mathcal{V} \to \mathcal{U}$ be another refinement map. Then one has*

$$\tau_k^* = \mu_k^*.$$

Proof. We construct a homotopy. Let $\sigma = (V_0,\dots,V_{k-1})$ be a $(k-1)$-simplex Then set

$$\tilde{\sigma}_j = (\mu(V_0),\dots,\mu(V_j),\tau(V_j),\dots,\tau(V_{k-1})).$$

Define $P^k : C^k(\mathcal{U}, \mathcal{F}) \to C^{k-1}(\mathcal{V}, \mathcal{F})$ by

$$P^k(f)(\sigma) = \sum_{j=0}^{k-1} (-1)^j f(\tilde{\sigma}_j)|_{|\sigma|}.$$

We want to show:

$$\tau^\# - \mu^\# = dP + Pd.$$

For $f \in C^k(\mathcal{U}, \mathcal{F})$ and $\sigma = (V_0, \dots, V_k)$ we compute:

$$\begin{aligned} dP(f)(\sigma) &= \sum_{i=0}^{k} (-1)^i P(f)(V_0, \dots \widehat{V_i} \dots, V_k) \\ &= \sum_{j<i} (-1)^{i+j} f(\mu(V_0), \dots, \mu(V_j), \tau(V_j), \dots \widehat{\tau(V_i)} \dots, \tau(V_k))|_{|\sigma|} \\ &\quad + \sum_{j>i} (-1)^{i+j+1} f(\mu(V_0), \dots \widehat{\mu(V_i)} \dots, \mu(V_j), \tau(V_j), \dots, \tau(V_k))|_{|\sigma|} \end{aligned}$$

and

$$\begin{aligned} Pd(f)(\sigma) &= \sum_{j=0}^{k} (-1)^j df(\tilde{\sigma}_j)|_{|\sigma|} \\ &= \sum_{j=0}^{k} (-1)^j df(\mu(V_0), \dots, \mu(V_j), \tau(V_j), \dots, \tau(V_k))|_{|\sigma|} \\ &= \sum_{i<j} (-1)^{i+j} f(\mu(V_0), \dots \widehat{\mu(V_i)} \dots, \mu(V_j), \tau(V_j), \dots, \tau(V_k))|_{|\sigma|} \\ &\quad + \sum_{i>j} (-1)^{i+j+1} f(\mu(V_0), \dots, \mu(V_j), \tau(V_j), \dots \widehat{\tau(V_i)} \dots, \tau(V_k))|_{|\sigma|} \\ &\quad + \sum_{j=0}^{k} f(\mu_0(V_0), \dots, \mu(V_{j-1}), \tau(V_j), \dots, \tau(V_k))|_{|\sigma|} \\ &\quad - \sum_{j=0}^{k} f(\mu(V_0), \dots, \mu(V_j), \tau(V_{j+1}), \dots, \tau(V_k))|_{|\sigma|}. \end{aligned}$$

The first two lines in the last expression equal $-dP(f)(\sigma)$ hence cancel in $dP + Pd$. We end up with

$$\begin{aligned} &(dP + Pd)(f)(\sigma) \\ &\quad + \sum_{j=0}^{k} f(\mu_0(V_0), \dots, \mu(V_{j-1}), \tau(V_j), \dots, \tau(V_k))|_{|\sigma|} \\ &\quad - \sum_{j=0}^{k} f(\mu_0(V_0), \dots, \mu(V_j), \tau(V_{j+1}), \dots, \tau(V_k))|_{|\sigma|} \\ &= f(\tau(V_0), \dots, \tau(V_k))|_{|\sigma|} - f(\mu(V_0), \dots, \mu(V_k))|_{|\sigma|} \\ &= (\tau^\# - \mu^\#)(f)(\sigma). \end{aligned}$$

We have shown that the homomorphism of chain complexes $\mu^\# - \tau^\#$ is chain nullhomotopic and therefore μ_* and τ_* agree on cohomology. $\square$

For two open covers $\mathcal{U}$ and $\mathcal{V}$ we write $\mathcal{U} < \mathcal{V}$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$. This makes the set of all open covers is a directed set.

There is a little set-theoretic hickup here, since we have defined a cover as a family and we can change index sets to get new covers. This can be circumvented by allowing only index sets I, which are subsets of, say $\mathcal{P}(X)$.

If $\mathcal{U} < \mathcal{V}$, then we have shown that there is a canonical homomorphism $\check{\mathrm{H}}^k(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}^k(\mathcal{V}, \mathcal{F})$. So the abelian groups $(\check{\mathrm{H}}^k(\mathcal{U}, \mathcal{F}))_{\mathcal{U}}$ form a directed system. We define

$$\check{\mathrm{H}}^k(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} \check{\mathrm{H}}^k(\mathcal{U}, \mathcal{F}).$$

* * *

4 Comparing cohomology theories

4.1 De Rham cohomology

Let X be a smooth manifold and let $k \in \left\{0, \dots, \dim X\right\}$. For an open set $U \subset X$ the set $\Omega^k(U)$ of k-differentialforms is an $\mathbb{R}$-vector space. The map $U \mapsto \Omega^k(U)$ form a sheaf Ω^k. For instance Ω^0 is the sheaf of smooth germs of smooth functions. This contains the constant sheaf $\mathcal{K}_{\mathbb{R}}$ as a subsheaf.

Theorem 4.1.1. *The sequence*

$$0 \to \mathcal{K}_{\mathbb{R}} \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots$$

is a fine resolution of the constant sheaf $\mathcal{K}_{\mathbb{R}}$. Hence we get

$$H^k_{dR}(X) = H^k(X, \mathcal{K}_{\mathbb{R}}).$$

Proof. From the Analysis 4 lecture we take:

Lemma 4.1.2 (Poincaré Lemma)**.** *Let $U \subset \mathbb{R}^n$ open and star-shaped and let ω be a smooth k-form in U with $k \geq 1$. If $d\omega = 0$, then there exists $\eta \in \Omega^{k-1}(U)$ such that $\omega = d\eta$.*

As a smooth manifold is locally diffeomorphic to star-shaped open sets in $\mathbb{R}^n$, the sequence is exact locally, which means it is an exact sequence of sheaves. By example 3.14.4 the sheaves Ω^k are fine. $\square$

* * *

4.2 Singular cohomology

Let X be a topological space and R a $\mathbb{Z}$-module, i.e., an abelian group. For an open subset $U \subset X$ let $C^k(U,R) = \mathrm{Hom}(C_k(U),R)$ be the set of all singular cochains with values in R. For $V \subset U$ let $\mathrm{res}^U_V : C^k(U,R) \to C^k(V,R)$ be the restriction. So $\mathcal{P}^k : U \mapsto C^k(U,R)$ is a presheaf. The coboundary operator $d : C^k(U,R) \to C^{k+1}(U,R)$ commutes with restriction, so it defines a presheaf homomorphism $C^k \to C^{k+1}$. Let $\mathcal{C}^k$ be the sheafification of C^k. Then $\mathcal{C}^0$ is the sheaf of all maps with values in R. It contains the constant sheaf $\mathcal{K}_R$ as subsheaf.

Lemma 4.2.1. *The presheaf $\mathcal{P}^k$ satisfies the axiom of existence, but in general not the axiom of uniqueness.*

In particular, it follows that the canonical map

$$\mathcal{P}^k(X) \to \mathcal{C}^k(X)$$

is surjective.

Proof. Let $U = \bigcup_{i\in I} U_i$ be open sets and let $f_i \in C^k(U_i,R)$ be given such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ holds for all indices $i, j \in I$. Let $E \subset C^k(U,R)$ be the group generated by all $\sigma \in C^k(U,R)$ with the property that there exists an index $i \in I$ with $\sigma \subset U_i$. We then can define $\tilde{f}(\sigma) = f_i(\sigma)$, independent of the index i and this defines a linear map $\tilde{f} : E \to R$. We have $C^k(U) = E \oplus F$, where F is the group generated by all remaining simplices. So for $s = s_E + s_F$ with $s_E \in E$ and $s_F \in F$ we define

$$f(s) = \tilde{f}(s_E).$$

Then $f \in C^k(U,R)$ and we have $f|_{U_i} = f_i$ for every $i \in I$.

Finally we give an example for the failure of the uniqueness axiom. Suppose $X = U \cup V$ with open sets U and V and let E be the abelian group generated by all chains which are contained either in U or in V. Pick a simplex σ, which is not contained in U or in V. Define a linear map $f : \mathbb{Z}\sigma \oplus E \to \mathbb{Z}$ by $f(E) = 0$ and $f(\sigma) = 1$. Extend f to a linear map on $C_k(X)$. Then f is locally zero, but not globally. □

Theorem 4.2.2. *Let X be a paracompact Hausdorff space and locally contractable. The sequence*

$$0 \to \mathcal{K}_R \to \mathcal{C}^0 \xrightarrow{d} \mathcal{C}^1 \xrightarrow{d} \dots$$

is a fine resolution of the constant sheaf $\mathcal{K}_R$. We conclude

$$H^k_{sing}(X,R) = H^k(X,\mathcal{K}_R).$$

Proof. For the exactness at $\mathcal{C}^0$ it suffices to show, that for every $x \in X$ there is an open neighbourhood U, such that the sequence $\mathcal{K}_R(U) \to C^0(U,R) \to C^1(U,R)$ is exact. For this choose U path-connected and let $\alpha \in \ker(d)$, so $\alpha : U \to R$ with $\alpha(\gamma(0)) - \alpha(\gamma(1)) = 0$ for every path γ in U. Since U is path-connected ist, the map α is constant, i.e., in $\mathcal{K}_R(U)$. The exactness at the other places follows from local contractability, since contractible sets have trivial singular cohomology.

Next we show that the sheaves C^k are fine. For this let (U_i) be a locally-finite cover. Choose functions $u_i : X \to \{0,1\}$ with $\operatorname{supp} u_i \subset U_i$ and $\sum_i u_i = 1$. Define an endomorphism ϕ_i of $C^k(U,R)$ by

$$\phi_i(f)(\sigma) = u_i(\sigma(t_0))f(\sigma),$$

where $\sigma : \Delta^k \to U$ is continuous and t_0 is a fixed point. These endomorphisms commute with the restrictions and thus define sheaf endomorphisms of $\mathcal{C}^k$ with $\operatorname{supp} \phi_i \subset U_i$ and $\sum_i \phi_i = \mathrm{Id}$. So the sheaves $\mathcal{C}^k$ are all fine.

For the last assertion we need to show that the canonical map $C^k(X,R) \to \mathcal{C}^k(X)$ is an isomorphism in cohomology. For a covering $\mathcal{U} = (U_i)_{i\in I}$ of X let $C_{k,\mathcal{U}}(X)$ denote the free abelian group generated by $C_k(U_i), i \in I$. Then the canonical map $C_{k,\mathcal{U}} \to C_k(X)$ is an isomorphism in homology, this is shown usinmg barycentric decomposition. Let $C^k_{\mathcal{U}}(X,R) = \mathrm{Hom}(C_{k,\mathcal{U}}(X),R)$. For $\mathcal{U} < \mathcal{V}$, i.e., when $\mathcal{V}$ is a refinement of $\mathcal{U}$, there is a canonical map $C_{k,\mathcal{V}} \to C_{k,\mathcal{U}}$ and thus, dually, $C^k_{\mathcal{U}} \to C^k_{\mathcal{V}}$ and these maps are isomorphisms in cohomology. Set

$$\tilde{C}^k(X,R) = \varinjlim_{\mathcal{U}} C^k_{\mathcal{U}}(X,R),$$

where the direct limit runs over all coverings $\mathcal{U}$. The canonical map $\phi : C^k(X,R) \to \tilde{C}^k(X,R)$ is an isomorphism in cohomology. Finally, by definition of the sheafification, there is a natural isomorphism of complexes $\tau : \mathcal{C}^k(X) \xrightarrow{\cong} \tilde{C}^k(X,R)$. Then the map $\tau^{-1} \circ \phi$ induces the desired isomorphism

$$H^k_{\mathrm{sing}}(X,R) \xrightarrow{\cong} H^k(X\,\mathbb{C}K_R). \qquad \square$$

Remark 4.2.3. The assertion of the theorem becomes false on non-locally contractible spaces. Let for instance X be the Hawaiian earring. Then Eda/Kawamura sowed in *The singular homology of the Hawaiian earring. J. London Math. Soc. (2) 62 (2000), no. 1, 305–310.*, that

$$H^1_{\mathrm{sing}}(X) \cong \left(\prod_{j=1}^{\infty} \mathbb{Z}\right) \oplus \left(\prod_{j=1}^{\infty} \mathbb{Z} \Big/ \bigoplus_{j=1}^{\infty} \mathbb{Z}\right).$$

In particular, this is not a free group. However, the very definition of Cech cohomology implies

$$\check{\mathrm{H}}^1(X) \cong \bigoplus_{j=1}^{\infty} \mathbb{Z}$$

and we shall see in Section 4.4, that the latter also equals the singular cohomology.

* * *

4.3 Group cohomology

Let Γ be a group. Let MOD(Γ) be the abelian category of all Γ-modules. For an Γ-module M let

$$M^\Gamma = \{m \in M : \gamma m = m \ \forall \gamma \in \Gamma\}$$

be the group of all Γ-invariant elements. Then $M \mapsto M^\Gamma$ defines a functor $\mathcal{H}^0(\Gamma, \cdot)$ from MOD(Γ) to the category MOD($\mathbb{Z}$) = AB of $\mathbb{Z}$-modules or abelian groups. It is easy to see that this functor is left-exact. The right-derivatives of this functor are by definition the cohomology groups:

$$H^k(\Gamma, M) = R^k\mathcal{H}^0(\Gamma, M).$$

Let $Y = E\Gamma$ be the universal covering of $B\Gamma = \Gamma\backslash Y$. A Γ-module M induces alocally-constant sheaf $\mathcal{M} = \Gamma\backslash(Y \times M)$ over $X = \Gamma\backslash Y$. Let $H^k(X, \mathcal{M})$ be the corresponding sheaf cohomoloy.

Theorem 4.3.1. *There is a natural isomorphy*

$$H^k(\Gamma, M) \ \cong \ H^k(X, \mathcal{M}).$$

In particular, when $M = R$ is an abelian group with trivial Γ-action, then $\mathcal{M}$ is the constant sheaf with stalk R and so Theorem 4.2.2 implies that

$$H^k(\Gamma, R) = H^k_{\text{sing}}(B\Gamma, R)$$

in accordance with Definition 1.11.2.

Proof. The functors $M \mapsto H^k(\Gamma, M)$ form a universal δ-functor on MOD(Γ). Let $AB(X)$ be the category of sheaves of abelian Groups on X. Then $\mathcal{F} \mapsto H^k(X, \mathcal{F})$ is a universal δ-functor on $AB(X)$. The sheaf-functor MOD(Γ) $\to AB(X)$, that to a module M attaches the locally-constant sheaf $\mathcal{L}_M$, is exact. Therefore $M \mapsto H^k(X, \mathcal{L}_M)$ is a δ-functor on MOD(Γ). It remains to show universality. As usual, we do that by showing erasability of the H^k for $k \geq 1$. Let $M \in$ MOD(Γ) and let

$$I_M = \{\alpha : \Gamma \to M\}$$

be the abelian group of all maps from Γ to M. This becomes a Γ-module by

$$g.\alpha(\tau) = g(\alpha(g^{-1}\tau)).$$

The map, which sends $m \in M$ to the constant map with value m, is an embedding $M \hookrightarrow I_M$ of Γ-modules. So it remains to show that

$$H^k(X, \mathcal{L}_{I_M}) = 0$$

for $k \geq 1$. Let $\pi : Y \to X = \Gamma\backslash Y$ be the projection.

Lemma 4.3.2. *One has*

$$\mathcal{L}_{I_M} \cong \pi_*\mathcal{K}_M,$$

where $\mathcal{K}_M$ is the constant sheaf with stalk M on Y.

Proof. Let $y \in Y$ and let U be an open neighbourhood of y such that $U \cap \gamma U = \emptyset$ for every $\gamma \in \Gamma \smallsetminus \{1\}$. Further, I_M can be identified with $\prod_{\gamma\in\Gamma} M$, where Γ acts by permutations and action on the factors at the same time. We write C^{lc} for the set of all locally-constant maps. Then $\mathcal{L}_{I_M}(\pi(U))$ by definition is the set of all continuous sections of the projection $\Gamma\backslash\big(\Gamma U \times I_M\big) \to \Gamma\backslash\big(\Gamma U\big)$. Such a section s is a map $\Gamma U \to \Gamma\backslash\big(\Gamma U \times I_M\big)$ sending $y \in \Gamma U$ to, say, $\Gamma(y, \alpha(y))$, where the well-definedness implies that for $\gamma \in \Gamma$ we have

$$\Gamma(y, \gamma^{-1}\alpha(\gamma y)) = \Gamma(\gamma y, \gamma\gamma^{-1}\alpha(\gamma y)) = \Gamma(\gamma y, \alpha(\gamma y)) = s(\gamma y) = s(y) = \Gamma(y, \alpha(y)).$$

This means that $\alpha(\gamma y) = \gamma\alpha(y)$, and, as I_M gets the discrete topology here, we have that α is locally-constant. So we get

$$\begin{aligned}
\mathcal{L}_{I_M}(\pi(U)) &\cong \Big\{\alpha \in C^{\mathrm{lc}}(\Gamma U, I_M) : \alpha(\gamma y) = \gamma\alpha(y)\Big\} \\
&\cong \left\{\alpha \in C^{\mathrm{lc}}\left(\Gamma U, \prod_{\gamma\in\Gamma} M\right) : \alpha(\gamma y)_\tau = \gamma\alpha(y)_{\gamma^{-1}\tau}\right\} \\
&\cong \left\{\alpha \in \prod_{\gamma\in\Gamma} C^{\mathrm{lc}}\left(\Gamma U, M\right) : \alpha(\gamma y)_\tau = \gamma\alpha(y)_{\gamma^{-1}\tau}\right\} \\
&\cong \prod_{\gamma\in\Gamma} C^{\mathrm{lc}}\left(\gamma U, M\right) \cong \pi_*\mathcal{K}_M(\pi(U)).
\end{aligned}$$

□

We now show that $\pi_*\mathcal{K}_M$ is acyclic. For this note that the functor $\pi_* : AB(Y) \to AB(\Gamma\backslash Y)$ is exact. This is due to the special properties of the projection $\pi : Y \to \Gamma\backslash Y$, for if $\mathcal{F}$ is a sheaf over Y and if $y_0 \in Y$, then the stalk of $\pi_*\mathcal{F}$ over the image point $\pi(y_0)$ equals

$$\pi_*\mathcal{F}_{\pi(y_0)} = \prod_{y\in Y:\pi(y)=\pi(y_0)} \mathcal{F}_y.$$

Since a sequence of sheaves is exact iff all stalk sequences are exact, we conclude that π_* is exact.

Further, for every sheaf $\mathcal{F}$ on Y one has

$$\mathcal{H}^0(\pi_*\mathcal{F}) = \mathcal{H}^0(\mathcal{F}),$$

where $\mathcal{H}^0$ is the global sections functor (on the left over $\Gamma\backslash Y$, on the right over Y). We choose a special injective resolution of $\mathcal{K}_M$ by products of skyscraper sheaves with injective stalks

$$0 \to \mathcal{K}_M \to I^0 \to I^1 \to \dots$$

The images $\pi_*(I^p)$ are again products of skyscraper sheaves with injective stalks. Since π_* is exact, the sequence

$$0 \to \pi_*\mathcal{K}_M \to \pi_*I^0 \to \pi_*I^1 \to \dots$$

is an injective resolution of $\pi_*\mathcal{K}_M$. It follows that

$$H^k(\Gamma\backslash Y, \pi_*\mathcal{K}_M) = H^k(\mathcal{H}^0(\pi_* I^\bullet)) = H^k(\mathcal{H}^0(I^\bullet)) = H^k(Y, \mathcal{K}_M)$$

The right hand side is zero for $k \geq 1$, by Theorem 4.2.2 and the contractability of Y. □

* * *

4.4 Cech-cohomology

In this section let X be a paracompact Hausdorff space.

Theorem 4.4.1. $(\check{H}^k)_k$ *is a universal δ-functor on the category* AB(X)*. It follows*

$$\check{H}^k(X,\mathcal{F}) \cong H^k(X,\mathcal{F}).$$

Proof. For the δ-functor let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be an exact sequence of sheaves and let $\mathcal{U}$ be an open cover of X. then the sequence

$$0 \to C^k(\mathcal{U},\mathcal{F}) \to C^k(\mathcal{U},\mathcal{G}) \to C^k(\mathcal{U},\mathcal{H})$$

is exact. Let $\overline{C}^k(\mathcal{U},\mathcal{H})$ be the image of $C^k(\mathcal{U},\mathcal{G})$ in $C^k(\mathcal{U},\mathcal{H})$. Then the sequence

$$0 \to C^k(\mathcal{U},\mathcal{F}) \to C^k(\mathcal{U},\mathcal{G}) \to \overline{C}^k(\mathcal{U},\mathcal{H}) \to 0$$

is exact. One gets a short exact sequence of cochain complexes

$$0 \to C^\bullet(\mathcal{U},\mathcal{F}) \to C^\bullet(\mathcal{U},\mathcal{G}) \to \overline{C}^\bullet(\mathcal{U},\mathcal{H}) \to 0.$$

Let $\mathcal{V}$ be a refinement of $\mathcal{U}$. A given refinement map $\mu : \mathcal{V} \to \mathcal{U}$ yields a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C^\bullet(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d, "\mu"] & C^\bullet(\mathcal{U},\mathcal{G}) \arrow[r] \arrow[d, "\mu"] & \overline{C}^\bullet(\mathcal{U},\mathcal{H}) \arrow[r] \arrow[d, "\mu"] & 0 \\ 0 \arrow[r] & C^\bullet(\mathcal{V},\mathcal{F}) \arrow[r] & C^\bullet(\mathcal{V},\mathcal{G}) \arrow[r] & \overline{C}^\bullet(\mathcal{V},\mathcal{H}) \arrow[r] & 0. \end{tikzcd}$$

This in turn gives a commutative diagram of cohomology groups

$$\begin{tikzcd} \dots \arrow[r] & \overline{H}^{k-1}(\mathcal{U},\mathcal{H}) \arrow[r, "\delta"] \arrow[d] & \check{H}^k(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & \check{H}^k(\mathcal{U},\mathcal{G}) \arrow[r] \arrow[d] & \overline{H}^k(\mathcal{U},\mathcal{H}) \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & \overline{H}^{k-1}(\mathcal{V},\mathcal{H}) \arrow[r, "\delta"] & \check{H}^k(\mathcal{V},\mathcal{F}) \arrow[r] & \check{H}^k(\mathcal{V},\mathcal{G}) \arrow[r] & \overline{H}^k(\mathcal{V},\mathcal{H}) \arrow[r] & \dots \end{tikzcd}$$

Taking the direct limit, one gets a lang exact sequence

$$\dots \to \overline{H}^{k-1}(X,\mathcal{H}) \xrightarrow{\delta} \check{H}^k(X,\mathcal{F}) \to \check{H}^k(X,\mathcal{G}) \to \overline{H}^k(X,\mathcal{H}) \to \dots$$

Lemma 4.4.2. *There is a natural isomorphism $\overline{H}^k(X,\mathcal{H}) \cong \check{H}^k(X,\mathcal{H})$, which commutes with the connection homomorphisms.*

Proof. It suffices to show that to every given locally-finite cover $\mathcal{U}$ and to every given $f \in C^k(\mathcal{U},\mathcal{H})$ there is a refinement $\mathcal{O}$ and a refinement map $\mu : \mathcal{O} \to \mathcal{U}$, such that $\mu(f) \in \overline{C}^k(\mathcal{O}, \mathcal{H})$. So let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover and let $f \in C^k(\mathcal{U}, \mathcal{H})$. By Lemma 3.15.1 there exists an open cover $\mathcal{V} = (V_i)_{i\in I}$ with $\overline{V}_i \subset U_i$ For every $i \in I$.

As $(U_\cdot)$ is locally-finite, for every $x \in X$ there is an open neighbourhood O_x with

- $O_x \subset V_i$ for some $i \in I$,
- if $O_x \cap V_i \neq \emptyset$, then $O_x \subset U_i$.
- O_x lies in the intersection of all U_i containing x,
- if σ is a k-simplex of the cover $\mathcal{U}$ and if $x \in |\sigma|$, (so $O_x \subset |\sigma|$), then the restriction $f(\sigma)|_{O_x}$ is the image of a section of $\mathcal{G}$ over O_x.

The last condition can be fulfilled, since there are only finitely many k-simplices for the cover $\mathcal{U}$, which contain x. The cover $(\mathcal{O}_x)_{x\in X}$ is our candidate. For every $x \in X$ choose some $V_x \in \mathcal{V}$ and $U_x \in \mathcal{U}$ with $O_x \subset V_x \subset \overline{V}_x \subset U_x$. We get a refinement map $\mu : \mathcal{O} \to \mathcal{U}$. Now let $\sigma = (\mathcal{O}_{x_0}, \dots, \mathcal{O}_{x_k})$ be a k-simplex to the cover $\mathcal{O}$. Then one has $O_{x_0} \cap V_{x_i} \neq \emptyset$ for $0 \le i \le k$, so it follows $O_{x_0} \subset U_{x_i}$. Hence $O_{x_0} \subset U_{x_0} \cap \dots U_{x_k} = |\mu(\sigma)|$. Therefore,

$$\begin{aligned} \mu(f)(\sigma) &= f(U_{x_0}, \dots, U_{x_k})|_{|\sigma|} \\ &= \underbrace{\underbrace{f(U_{x_0}, \dots, U_{x_k})|_{x_0}}_{\in \mathcal{G}(O_{x_0})}\Big|_{|\sigma|}}_{\in \mathcal{G}(|\sigma|)}. \end{aligned}$$

Finally one gets $\mu(f) \in \overline{C}^k(\mathcal{O}, \mathcal{H})$. □

So the long exact sequence is the one demanded in the definition of a δ-functor. The functoriality of the δ-morphism is clear on the level of $C^k(\mathcal{U}, \mathcal{F})$ and therefore follows for the direct limit.

It follows, that $\check{H}$ is a δ-functor. For universality, we show that $\check{H}^k$ is erasable for $k \ge 1$. By Lemma 3.14.5 and Theorem 3.14.6 it suffices to show the following lemma.

Lemma 4.4.3. *If $\mathcal{F}$ is fine, one has $\check{H}^k(X, \mathcal{F}) = 0$ For every $k \ge 1$.*

Proof. Let $k \ge 1$. It suffices to show $\check{H}^k(\mathcal{U}, \mathcal{F}) = 0$ for every locally-finite cover $\mathcal{U} = (U_i)_{i\in I}$. Let (ϕ_i) be an associate family of Endomorphisms of $\mathcal{F}$ with supp $\phi_i \subset U_i$ and $\sum_i \phi_i = 1$. We show that the Identity on $C^\bullet(\mathcal{U}, \mathcal{F})$ is nullhomotopic. For this we construct maps $h_p : C^p(\mathcal{U}, \mathcal{F}) \to C^{p-1}(\mathcal{U}, \mathcal{F})$ for every $p \ge 1$. Let $f \in C^p(\mathcal{U}, \mathcal{F})$ and let $\sigma = (U_0, \dots, U_{p-1})$ be a $(p-1)$-simplex to the cover $\mathcal{U}$. Then $\phi_i \circ f(U_i, U_0, \dots, U_{p-1})$ has support in $U_i \cap U_0 \cap \dots \cap U_{p-1}$. Extending by zero, we can view $\phi_i \circ f(U_i, U_0, \dots, U_{p-1})$ as a section on $U_0 \cap \dots, \cap U_{p-1}$. Define

$$h_p(f)(\sigma) = \sum_i \phi_i \circ f(U_i, U_0, \dots, U_{p-1}).$$

Then one gets
$$d \circ h_p + h_{p+1} \circ d = \mathrm{Id}$$
for $p \geq 1$. The Lemma follows. □

By the lemma $\check{H}^k$ is erasable for $k \leq 1$ and therefore they form a universal δ-functor. □

* * *

4.5 Leray covers

In this section X continues to be a paracompact Hausdorff space.

Lemma 4.5.1. *Let $\mathcal{P} = \prod_{x\in X} \mathcal{S}_{x,A_x}$ be a product of skyscraper sheaves. Then $\check{H}^k(\mathcal{U},\mathcal{P}) = 0$ for every $k \geq 1$ and every cover $\mathcal{U}$.*

Proof. For every open set $U \subset X$ we have $\mathcal{P}(U) = \prod_{x\in U} A_x$ and this isomorphism commutes with the Cech-differential, hence it suffices to show the claim for a single skyscraper sheaf $\mathcal{S} = \mathcal{S}_{x_0,A}$. In that case, let $I' = \{i \in I : x_0 \in U_i\}$ and $X' = \bigcup_{i\in I'} U_i$. Then the embedding $X' \hookrightarrow X$ induces isomorphisms on $\check{H}^k(\mathcal{U},\mathcal{S})$ for all k as the sheaf vanishes outside X'. Therefore, we can replace X with X' and likewise replace I by I' and henceforth assume that $x_0 \in U_i$ for every $i \in I$. Then for a simplex σ, every section $s \in \mathcal{S}(|\sigma|)$ is uniquely determined by the value $s(x_0)$, which can be every element of A. We therefore can identify $C^k(\mathcal{U},\mathcal{S})$ with the set of maps $f : I^{k+1} \to A$ and the differential is

$$df(i_0,\dots,i_{k+1}) = \sum_{j=0}^{k+1} (-1)^i f(i_0,\dots \widehat{i_j} \dots, i_{k+1}).$$

This is the coboundary operator to the following chain complex C_k is the free abelian group generated by I^{k+1} and the boundary operator is

$$\partial(i_0,\dots,i_k) = \sum_{j=0}^{k} (-1)^k\,(i_0,\dots,\widehat{i_j}\dots,i_k).$$

Let Y be the full simplicial complex on the vertex set I. Mapping $(i_0,\dots,i_k)$ to the singular simplex spanned by $i_0,\dots,i_k$ defines a map from $H_k(C_\bullet)$ to $H_{k,\mathrm{sing}}(Y)$. In the same way as the equivalence of singular and simplicial homology, one shows that this map is an isomorphism. By Proposition 1.3.3 one concludes $\check{H}^k(\mathcal{U},\mathcal{S}) \cong H^k_{\mathrm{sing}}(Y,A)$, but the latter vanishes for $k \geq 1$ since the full simplicial complex Y is contractible. ◻

Definition 4.5.2. A cover $\mathcal{U}$ of X is called **Leray-cover** for the sheaf $\mathcal{F}$, if for every k-simplex $\sigma = (U_0,\dots,U_k)$ the sheaf $\mathcal{F}|_{|\sigma|}$ is acyclic.

Example 4.5.3. Let X be a smooth manifold. By the Poincaré Lemma the constant sheaf $\mathbb{R}$ is acyclic on every open set $U \subset X$, which is diffeomorphic with $\mathbb{R}^n$. Therefore X possesses locally-finite Leray-covers. If X is compact, there is a finite Leray cover.

Theorem 4.5.4. *If $\mathcal{U}$ is a Leray cover for the sheaf $\mathcal{F}$, then the natural map*

$$\check{H}^p(\mathcal{U},\mathcal{F}) \to \check{H}^p(X,\mathcal{F})$$

is an isomorphism.

So it suffices to compute Cech cohomology with one Leray cover.

Proof. Embed $\mathcal{F}$ into a sheaf $\mathcal{J}$ which is a product of skyscraper sheaves with injective stalks. Then $\mathcal{J}|_U$ is injective for every open set $U \subset X$. In particular, $\mathcal{U}$ is a Leray cover for $\mathcal{J}$ as well. Let $\mathcal{G}$ be the cokernel, so we have an exact sequence

$$0 \to \mathcal{F} \to \mathcal{J} \to \mathcal{G} \to 0.$$

As $\mathcal{U}$ is a Leray cover, for every k-simplex the sequence

$$0 \to \mathcal{F}(|\sigma|) \to \mathcal{J}(|\sigma|) \to \mathcal{G}(|\sigma|) \to 0$$

is exact. We get an exact sequence of Cech complexes

$$0 \to C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{U}, \mathcal{J}) \to C^\bullet(\mathcal{U}, \mathcal{G}) \to 0.$$

This yields a long exact sequence on the cohomology. The same holds for the restrictions to $|\sigma|$ for any given cochain σ. As $\mathcal{F}|_{|\sigma|}$ and $\mathcal{J}|_{|\sigma|}$ are acyclic, the long exact sequence implies that $\mathcal{G}|_{|\sigma|}$ is acyclic, too. So $\mathcal{U}$ is a Leray cover for $\mathcal{G}$, too.

By Lemma 4.5.1 we have $\check{H}^k(\mathcal{U}, \mathcal{J}) = 0$ for $k \geq 1$. Together with the homomorphism into the Cech cohomology, we get the following commutative diagrams with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \check{\mathrm{H}}^0(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d, "\cong"] & \check{\mathrm{H}}^0(\mathcal{U},\mathcal{J}) \arrow[r] \arrow[d, "\cong"] & \check{\mathrm{H}}^0(\mathcal{U},\mathcal{G}) \arrow[r] \arrow[d, "\cong"] & \check{\mathrm{H}}^1(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & \check{\mathrm{H}}^0(X,\mathcal{F}) \arrow[r] & \check{\mathrm{H}}^0(X,\mathcal{J}) \arrow[r] & \check{\mathrm{H}}^0(X,\mathcal{G}) \arrow[r] & \check{\mathrm{H}}^1(X,\mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

Since the first three vertical arrows are isomorphisms, the five lemma implies that the arrow α is an isomorphism, too.

For $k \geq 1$ the long exact sequence gives

$$\begin{tikzcd}
0 \arrow[r] & \check{\mathrm{H}}^k(\mathcal{U},\mathcal{G}) \arrow[r] \arrow[d] & \check{\mathrm{H}}^{k+1}(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \check{\mathrm{H}}^k(X,\mathcal{G}) \arrow[r] & \check{\mathrm{H}}^{k+1}(X,\mathcal{F}) \arrow[r] & 0.
\end{tikzcd}$$

In the first diagram the first three vertical arrows are isomorphisms and so is the third. In the second diagram we can apply a seesaw principle since know that $\mathcal{U}$ is a Leray cover for the sheaf $\mathcal{G}$, too. □

Application for Cech-cohomology

Cech-cohomology was invented in complex analysis and has many applications there. Here we only mention the **Cousin-problem**: Let $X \subset \mathbb{C}^n$ be open and let (U_i) be an open cover. A function $f : X \to \mathbb{C}$ is called **holomorphic**, if for every $z \in X$ and every $1 \leq j \leq n$ the map $w \mapsto f(z_1, \ldots, z_{j-1}, w, z_{j+1}, \ldots z_n)$ is holomorphic in a neighbourhood of z_j.

A **meromorphic function** on X is a map $f : X \to \mathbb{C} \cup \{\infty\}$, such that for every $z \in X$ there is an open neighbourhood U and holomorphic functions h_1, h_2 on U and

$$f|_U = \frac{h_1}{h_2}.$$

Let $\mathcal{O}(X)$ be the set of holomorphic functions on X and $\mathcal{M}(X)$ the set of all meromorphic functions. Further let $\mathcal{O}^\times(X)$ be the set of holomorphic functions without zero.

Let there be given an open cover $(U_i)_{i\in I}$ of X. Let $f_i \in \mathcal{M}(U_i)$ be meromorphic functions with $\frac{f_i}{f_j} \in \mathcal{O}^\times(U_i \cap U_j)$ for all $i, j \in I$.

Question: is there $f \in \mathcal{M}(X)$ such that $\frac{f}{f_i} \in \mathcal{O}^\times(U_i)$ for every i?

(This means that the global function f has the same zeros and poles as the f_i.)

Theorem 4.5.5. *If $\check{H}^1(\mathcal{U}, \mathcal{O}) = 0$ and each U_i is simply connected, then each Cousin problem to the covering $\mathcal{U}$ has a solution.*

Proof. It is easy to see that there exists a so called **weak solution**, i.e., a continuous function $\psi \in C(X \smallsetminus P)$, defined outside the set P poles, such that ψ/f_i extends continuously to a continuous, zero-free function.

Then $\psi = \psi_i f_i$ on U_i, where the function ψ_i has no zeros. Note that the exponential function $\exp : \mathbb{C} \to \mathbb{C}^\times$ is the universal covering of $\mathbb{C}^\times$. Therefore the continuous function $\psi : U_i \to \mathbb{C}^\times$, factors throup exp, i.e., there exists a function $\phi_i \in C(U_i)$ with $\psi_i = e^{\phi_i}$, i.e., $\psi = e^{\phi_i} f_i$ on U_i. On $U_i \cap U_j$ one therefore has

$$e^{\phi_j - \phi_i} = \frac{f_i}{f_j} \in \mathcal{O}^\times(U_i \cap U_j). \tag{$*$}$$

This implies $\phi_{i,j} = \phi_i - \phi_j \in \mathcal{O}(U_i \cap U_j)$. The family $s = (\phi_{i,j})_{i,j\in I}$ is a cocycle, i.e., lies in $Z^1(\mathcal{U}, \mathcal{O})$, as the following computation shows. Let $\sigma = (U_0, U_1, U_2)$ be a 2-simplex, then on $U_0 \cap U_1 \cap U_2$ we have

$$\begin{aligned} ds(\sigma) &= s(U_1 \cap U_2) - s(U_0 \cap U_2) + s(U_0 \cap U_1) \\ &= \phi_1 - \phi_2 - (\phi_0 - \phi_2) + \phi_0 - \phi_1 = 0. \end{aligned}$$

As $\check{H}^1(\mathcal{U}, \mathcal{O}) = 0$, this cocycle is a coboundary. Thus there exist holomorphic functions $g_i \in \mathcal{O}(U_i)$ with

$$\phi_{i,j} = \phi_i - \phi_j = g_i - g_j$$

on $U_i \cap U_j$. By $(*)$ we get $e^{g_j - g_i} = f_i/f_j$, so

$$e^{g_i} f_i = e^{g_j} f_j$$

holds on $U_i \cap U_j$. Hence there exists a global meromorphic function $f \in \mathcal{M}(S)$ with $f = e^{g_i} f_i$ on U_i, whence the claim. □