Nilpotent algebras and affinely homogeneous surfaces

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1. Introduction

This paper is devoted to the investigation of finite dimensional commutative nilpotent (associative) algebras $\mathcal{N}$ over an arbitrary base field $\mathbb{F}$ of characteristic zero. Our main attention is focused on those algebras which have 1-dimensional annihilator since these algebras naturally occur in connection with various, geometrically motivated problems. The unital extensions $\mathcal{N}^0 = \mathbb{F} \oplus \mathcal{N}$ of such algebras are exactly the Gorenstein algebras of finite positive vector space dimension over $\mathbb{F}$.

There is only very sparse literature concerning the structure of general nilpotent commutative algebras. For example, every such algebra has a realization as a subalgebra of some $\mathrm{End}(V)$, V a vector space over $\mathbb{F}$, which is maximal with respect to the property that it consists of nilpotent and commuting endomorphisms. This means that abstractly given nilpotent algebras and nilpotent subalgebras of endomorphism algebras are essentially the same. While the structure and classification of maximal commutative algebras consisting of semisimple endomorphisms (Cartan subalgebras) is very well understood, there is only little known about the general structure, not to mention a classification, of its nilpotent counterpart (for an approach in terms of Macaulay's inverse systems compare with [3]). This is a bit surprising, as such nilpotent algebras are quite ubiquitous objects which occur in various areas of mathematics. One reason is certainly that the theory of nilpotent algebras is more involved than the theory of Cartan subalgebras, due to the lack of rigidity properties and other obvious visible structure. Since the standard tools from the Cartan theory such as the root theory cannot be applied in the nilpotent case, the desire arises for appropriate objects which help to understand nilpotent commutative algebras. One of the purposes of this paper is to develop such tools.

From a purely algebraic point of view nilpotent commutative algebras are building blocks for general commutative algebras which, for instance, seems to be very important for quantum physics. As already mentioned, commutative nilpotent algebras naturally arise in the context of several geometrically motivated questions as they often serve as invariants attached to certain geometric objects. Our interest in commutative nilpotent algebras also originates from geometry. To be more specific, we mention two types of geometric problems which provide us with commutative nilpotent algebras which in turn encode some of the geometric structure of the original questions.

In Cauchy-Riemann geometry there is the question under which conditions certain CR-manifolds are (locally) equivalent to tube manifolds $S \times i\mathbb{R}^n \subset \mathbb{R}^n \oplus i\mathbb{R}^n = \mathbb{C}^n$ and how many different tube realizations do exist. In [5] we show that this geometric problem (in the case of non-degenerate hyperquadrics) can be reduced to the classification of real *and* complex commutative nilpotent subalgebras with 1-dimensional annihilator.

Another type of problems arises from the study of isolated hypersurface singularities and their versal deformations: Let h be (the germ of) a holomorphic function, defined in a neighbourhood U of $0 \in \mathbb{C}^n$, i.e., $h \in \mathcal{O}_n := \mathbb{C}\{z_1, \dots, z_n\}$ such that the hypersurface $\{h = 0\}$ has an isolated singularity in 0. This implies that $\operatorname{grad} h : U \to \mathbb{C}^n$ is a finite map, and consequently

$$\mathcal{O}_n / h^*(\mathcal{O}_n) \cdot \mathcal{O}_n = \mathcal{O}_n / J(h)$$

is a finite dimensional local algebra. Here, $J(h) := \left(\frac{\partial h}{\partial z_1}, \dots, \frac{\partial h}{\partial z_n}\right)$ denotes the *Jacobi ideal* of h in $\mathcal{O}_n$. As a consequence of Nakayama's Lemma its maximal ideal is a (commutative) nilpotent

2000 Mathematics Subject Classification: 14J70, 32V40 algebra. The full algebra serves as parameter space for the universal deformation of the isolated singularity of h. Consequently also the maximal ideal of the Tjurina algebra $\mathcal{O}_n/(h, J(h))$ is nilpotent and its algebra structure turns out to determine the original isolated singularity up to a biholomorphic equivalence, see e.g. [12] for further details.

For short, we call from now on a finite dimensional commutative nilpotent algebra over $\mathbb{F}$ with 1-dimensional annihilator simply an *admissible algebra*. In this paper we give a construction of objects naturally associated with an admissible algebra $\mathcal{N}$, encoding sufficient information to recover the original algebra. These objects seem to be easier to deal with than the admissible algebras themselves and also serve as convenient invariants allowing the explicit verification of whether two admissible algebras are isomorphic. These objects are certain classes of smooth subvarieties of $\mathcal{N} \cong \mathbb{F}^{n+1}$ as well as certain classes of polynomials, which we call *nil-polynomials*. These polynomials are closely related to the aforementioned smooth subvarieties. Roughly speaking the nil-polynomials are certain truncated exponential series (here the nilpotency is crucial), concatenated with a linear functional, which essentially is nothing but a linear projection onto the annihilator of $\mathcal{N}$. The constant and linear parts of every nil-polynomial $p \in \mathbb{F}[X_1, \ldots, X_n]$ vanish, but the quadratic part of p is non-degenerate. Up to isomorphism the algebra structure on $\mathcal{N}$ can be recovered from the polynomial p. Even more is true, as the quadratic plus cubic term alone suffice to determine the structure of $\mathcal{N}$, and in turn the entire nil-polynomial p. Unless $n = 0$, its degree coincides with the nil-index of $\mathcal{N}$, i.e., the maximal number ν with $\mathcal{N}^\nu \neq 0$.

Let $\mathcal{A}$ be the annihilator of $\mathcal{N}$ and $\mathcal{K}$ a hyperplane in $\mathcal{N}$ transversal to $\mathcal{A}$, that is, $\mathcal{N} = \mathcal{K} \oplus \mathcal{A}$. The smooth variety associated with $\mathcal{N}$ (and depending on linear isomorphisms $\mathcal{K} \cong \mathbb{F}^n$, $\mathcal{A} \cong \mathbb{F}$) is simply the graph $S \subset \mathbb{F}^{n+1}$ of the corresponding nil-polynomial $p : \mathbb{F}^n \to \mathbb{F}$. We call every such S a *nil-hypersurface*. Among other things we prove that two admissible algebras $\mathcal{N}, \widetilde{\mathcal{N}}$ with nil-hypersurfaces $S, \widetilde{S}$ are isomorphic as algebras if and only if $S, \widetilde{S}$ are affinely equivalent. For an even stronger statement see Theorem 4.2. We also show that affine equivalence for $S, \widetilde{S}$ can be replaced by linear equivalence if and only if the nil-hypersurface S is affinely homogeneous. Linear equivalence gives a stronger and computationally more convenient condition for the isomorphy of the algebras $\mathcal{N}, \widetilde{\mathcal{N}}$. On the polynomial level it means that for the corresponding nil-polynomials p, $\widetilde{p}$ there is a $g \in \mathrm{GL}(n, \mathbb{F})$ such that $\widetilde{p}$ and $p \circ g$ differ by a constant factor from $\mathbb{F}^*$. We further establish a duality between a fixed nil-hypersurface S of $\mathcal{N}$ and the parameter space $\Sigma(\mathcal{N})$ of all such nil-hypersurfaces. Taking this duality a step further, we show that the action of the affine group $\mathrm{Aff}(S)$ on S is equivariantly isomorphic to the action of the algebra automorphism group $\mathrm{Aut}(\mathcal{N})$ on the affine space $\Pi(\mathcal{N})$ of all projections with range the annihilator of $\mathcal{N}$.

As already mentioned, affine homogeneity of an associated nil-hypersurface S of $\mathcal{N}$ makes computations more efficient. However, the question for which admissible algebras $\mathcal{N}$ the nil-hypersurface S is affinely homogeneous is quite involved. Only recently we were able to give a satisfactory answer to this question: While the nil-hypersurface of every admissible algebra of nil-index smaller than 5 is automatically affinely homogeneous, there are non-homogenous counterexamples starting with nil-index 5. In the case however, where $\mathcal{N}$ admits a $\mathbb{Z}^+$-gradation, every corresponding nil-hypersurface S is affinely homogeneous.

Our paper is organized as follows: In Section 2 we fix notation and state some preliminaries. A simple example is given, which indicates why in the rest of the paper we stick to nilpotent algebras that are *commutative*. In Section 3 we introduce the notion of a nil-surface, which is a smooth algebraic variety S_π, associated with a given nilpotent commutative algebra $\mathcal{N}$, however depending also on a projection $\pi \in \mathrm{End}(\mathcal{N})$. Further we discuss various notions of gradations for nilpotent commutative algebras. The main result of the section holds for $\mathcal{N}$ admitting certain types of generalized gradations and compatible projections π: In this case S_π is affinely homogeneous. A special version of this result for base field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ is already contained in [5] and later also was used in [6] for $\mathbb{F} = \mathbb{C}$, compare also with [8]. In Section 4 we restrict our attention to *admissible algebras* algebras $\mathcal{N}$. From that point on we only consider *admissible projections* π on $\mathcal{N}$ which means that the range of π is the (1-dimensional) annihilator of $\mathcal{N}$. The main result of this section is, roughly speaking, that the algebra structure of $\mathcal{N}$ only depends on the hypersurface $S_\pi \subset \mathcal{N}$, and 'essential' properties of S_π do not depend on the admissible projection π. Another statement is that the affine group $\mathrm{Aff}(S_\pi)$ is canonically isomorphic to the algebra automorphism group $\mathrm{Aut}(\mathcal{N})$. The main result of the section, Theorem 4.2, is a generalization and extension of a result in [6] from base field $\mathbb{C}$ to arbitrary $\mathbb{F}$. The proof in [6] is of analytic nature and we get the extension by applying a Lefschetz principle type argument. We also investigate functorial properties of the space $\Sigma(\mathcal{N})$ of all nil-hypersurfaces and of the affine space $\Pi(\mathcal{N})$ of all admissible projections, formulate a duality statement between a member S of $\Sigma(\mathcal{N})$ and the family $\Sigma(\mathcal{N})$ itself and prove the equivariant equivalence of the natural actions of $\mathrm{Aut}(\mathcal{N})$ on $\Pi(\mathcal{N})$ and of $\mathrm{Aff}(S)$ on S. We close the section with an infinitesimal analogon, more precisely, we show for every base field of characteristic 0 that for any admissible algebra $\mathcal{N}$ with associated nil-hypersurface $S \subset \mathcal{N}$ the derivation algebra $\mathfrak{der}(\mathcal{N})$ is isomorphic to the Lie algebra $\mathfrak{aff}(S)$ of all affine transformations $\mathcal{N} \to \mathcal{N}$ that are 'tangent' to S. In Section 5 we associate to every admissible algebra $\mathcal{N} \cong \mathbb{F}^{n+1}$ a class of mutually affinely equivalent polynomials in $\mathbb{F}[x_1, \ldots, x_n]$, called *nil-polynomials.* The graphs of these polynomials are affinely equivalent to S_π. In Section 6 we present for every admissible $\mathcal{N}$ certain canonical decompositions and show: Every $\mathcal{N}$ with nil-index ≤ 3 has a grading, and for every $\mathcal{N}$ with nil-index ≤ 4 every S_π is affinely homogeneous. Both bounds for the nil-index are sharp as will be shown by counterexamples in the last section. We also show for every admissible algebra $\mathcal{N}$ that $\mathfrak{der}(\mathcal{N})$ and $\mathrm{Aut}(\mathcal{N})$ have at least dimension $\dim(\mathcal{N}/\mathcal{N}^4)$. In Section 7 we give large classes of admissible algebras of nil-index 3 and 4. It turns out in particular, that in every dimension ≥ 7 for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ the number of isomorphy classes of admissible algebras is uncountable infinite. We also get a classification of all admissible algebras of nil-index 3. For the special case of algebraically closed base fields this has already been achieved in [3] by completely different methods. In Section 8 we present various counterexamples elaborated with computer aid. Among these we give an admissible algebra $\mathcal{N}$ of dimension 23 and nil-index 5 such that S_π is *not affinely homogeneous.* This disproves the Conjecture at the end of [7], repeated and extended as Conjecture 2.4 in [8a].

One of the essential parts of the present paper is Theorem 3.2. For the special case of *admissible* algebras it also occurs as Cor. 2.6 in [8a] together with the statement *"We note that Corollary 2.6 was obtained by W. Kaup approximately three months before this paper was written"* on page 3. This Corollary is essentially the same as Theorem 2.5 in [8a]. In later versions such as [8b] any hint to our priority is missing.

2. Preliminaries

Throughout the paper $\mathbb{N}$ is the set of all *non-negative* integers while $\mathbb{Z}^+$ is the semigroup of all *positive* integers. Further, $\mathbb{F}$ is an arbitrary but fixed field of characteristic 0. All algebras in the following are defined over $\mathbb{F}$ and are associative, but may have infinite dimension as $\mathbb{F}$-vector spaces (at least in the first three sections). For every such algebra A, every $x \in A$ and every integer $k \geq 1$ we put

$$x^{(k)} := \frac{1}{k!}x^k \quad \text{and} \quad x^{(0)} := \mathbb{1} \ \text{ if } \ A \ \text{ has a unit } \ \mathbb{1}\,. \tag{2.1}$$

Also we denote for every $j \in \mathbb{Z}^+$ by

$$\exp_j = \sum_{k=j}^{\infty} T^{(k)} \ \in \ \mathbb{F}[[T]]$$

the *j-truncated exponential series.* Then $\exp_1 \circ \log_1 = \log_1 \circ \exp_1 = T$ for

$$\log_1 := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\, T^k\,.$$

2.2 Definition. For every $\mathbb{F}$-algebra A define inductively the characteristic ideals $A^k \subset A$ by $A^1 = A$ and $A^{k+1} = \langle AA^k \rangle_{\mathbb{F}}$. Also put $A_{[0]} = 0$ and $A_{[k]} := \{x \in A : xA^k + A^k x = 0\}$ for $k > 0$. Then A is called *nilpotent* if $A^{k+1} = 0$ for some $k \geq 0$, and the minimal k with this property is called the *nil-index* of A, that we denote by $\nu = \nu(A)$.

For nilpotent A with nil-index ν the inclusion $A^k \subset A_{[\nu+1-k]}$ is obvious for all $k \leq \nu$ as well as $\nu = \inf\{k \geq 0 : A_{[k]} = A\}$. The ideal $\mathrm{Ann}(A) := A_{[1]}$, called the *annihilator* of A, plays a prominent role in the following. The annihilator coincides with the *socle* of $\mathcal{N}$, that is, the sum of all minimal ideals.

Let $\mathcal{N}$ be a nilpotent algebra. From now on we always consider $\exp_1, \log_1 : \mathcal{N} \to \mathcal{N}$ as polynomial mappings that are inverse to each other. Fix an arbitrary projection $\pi = \pi^2 \in \mathrm{End}(\mathcal{N})$. Then for the polynomial mapping

$$f : \mathcal{N} \to \mathcal{N}\,, \qquad f(x) := \pi(\exp_1 x)\,, \tag{2.3}$$

$$S = S_\pi := f^{-1}(0) = \log_1(\ker \pi) \;\subset\; \mathcal{N} \tag{2.4}$$

is a smooth algebraic subvariety of codimension $\mathrm{rank}(\pi)$ containing the origin. We will be mainly interested in the case where the annihilator of $\mathcal{N}$ has dimension 1 and is the range $\pi(\mathcal{N})$ of the projection π. Then S_π is a hypersurface in $\mathcal{N}$.

With $\mathrm{Aff}(\mathcal{N}) \cong \mathrm{GL}(\mathcal{N}) \ltimes \mathcal{N}$ we denote the group of all affine bijections of $\mathcal{N}$ and by $\mathrm{Aff}(S) := \{g \in \mathrm{Aff}(\mathcal{N}) : g(S) = S\}$ the subgroup stabilizing S. Furthermore, $\mathrm{GL}(S) := \{g \in \mathrm{GL}(\mathcal{N}) : g(S) = S\}$ is the isotropy subgroup of $\mathrm{Aff}(S)$ at the origin. We are interested in cases where S is affinely homogeneous, that is, the group $\mathrm{Aff}(S)$ acts transitively on S. This is not always true: As a counterexample consider the matrix algebra $\mathcal{T}$ of all strictly upper triangular $n \times n$-matrices with coordinates x_{jk} for $1 \leq j < k \leq n$ and projection π given by $x \mapsto x_{1n}$ (after identifying $\mathrm{Ann}(\mathcal{T}) \cong \mathbb{F}$ in the obvious way). For instance, for $n = 4$ the corresponding polynomial f is given by

$$f(x) = \frac{1}{6} x_{12} x_{23} x_{34} + \frac{1}{2}(x_{13} x_{34} + x_{12} x_{24}) + x_{14}$$

and it can be seen that S_π is not affinely homogeneous. Notice that the quadratic part of f is a *degenerate* quadratic form on $\ker(\pi)$, in contrast to the commutative case below.

In case $\mathcal{N}$ is commutative, for every projection π on $\mathcal{N}$ with range $\mathcal{N}_{[1]} = \mathrm{Ann}(\mathcal{N})$ it is well known that for the $\mathcal{N}_{[1]}$-valued symmetric 2-form

$$b_\pi : \mathcal{N} \times \mathcal{N} \to \mathcal{N}_{[1]}, \qquad (x, y) \mapsto \pi(xy) \tag{2.5}$$

the radical $\{x \in \mathcal{N} : b_\pi(x, \mathcal{N}) = 0\}$ coincides with $\mathcal{N}_{[1]}$ (for a simple proof compare e.g. Prop. 2.1 in [6]). In particular, the form b_π is non-degenerate on $\pi^{-1}(0) \cong \mathcal{N}/\mathcal{N}_{[1]}$. The $\mathcal{N}_{[1]}$-valued polynomial $f = \pi \circ \exp_1$ has a unique decomposition $f = \sum_{k \geq 1} f^{[k]}$ into homogeneous components $f^{[k]}$ of degree k. Clearly, $f^{[1]} = \pi$, $f^{[2]}(x) = \frac{1}{2} b_\pi(x, x)$ and $f^{[\nu]}(x) = x^{(\nu)}$ for all $x \in \mathcal{N}$ and $\nu = \nu(\mathcal{N})$.

From now on we assume that $\mathcal{N}$ is a *commutative nilpotent* algebra. In this paper we investigate properties of $\mathcal{N}$, the polynomials f and the corresponding nil-surfaces in $\mathcal{N}$ in a fairly general algebraic setting. Our motivation however comes from complex geometry. For instance – as already mentioned in the introduction – every real hyperquadric Y in a complex projective space $\mathbb{P}_n(\mathbb{C})$ gives rise to several (real and complex) commutative nilpotent algebras $\mathcal{N}$. Roughly speaking, the varieties S_π in case $\pi(\mathcal{N}) = \mathcal{N}_{[1]}$ occur as building blocks of bases $F \subset \mathbb{R}^n$ in various tube representations $F \times i\mathbb{R}^n \subset \mathbb{C}^n$ of the CR-manifold Y, compare [5]. Another source of commutative nilpotent algebras arises in the context of isolated hypersurface singularities, compare [6].

With $\mathcal{N}^0 := \mathbb{F}\cdot\mathbb{1}\oplus\mathcal{N}$ we denote the *unital extension* of $\mathcal{N}$, having $\mathbb{1}$ as unit. This notation has been chosen since then the canonical filtration extends as

$$\mathcal{N}^0 \supset \mathcal{N}^1 \supset \cdots \mathcal{N}^k \supset \cdots \quad \text{with} \quad \mathcal{N}^j\mathcal{N}^k \subset \mathcal{N}^{j+k}\,,$$

$\mathcal{N}^1 = \mathcal{N}$ and $\mathcal{N}^0/\mathcal{N}^1 \cong \mathbb{F}$. Clearly, $\mathcal{N}$ is the unique maximal ideal of $\mathcal{N}^0$. In case $\mathcal{N} = \mathcal{N}_1\oplus\mathcal{N}_2\oplus\ldots\oplus\mathcal{N}_d$ for linear subspaces $\mathcal{N}_k$ and a fixed integer $d\ge 1$ we can write every $x\in\mathcal{N}$ as tuple $x=(x_1,x_2,\ldots,x_d)$ with $x_k\in\mathcal{N}_k$ and then

$$\exp(x) = \sum_{\mu\in\mathbb{N}^d} x^{(\mu)} \;\in\; \mathbb{1}+\mathcal{N} \;\subset\; \mathcal{N}^0,$$

where $x^{(\mu)} := x_1^{(\mu_1)}x_2^{(\mu_2)}\cdots x_d^{(\mu_d)}$ for all $\mu=(\mu_1,\ldots,\mu_d)$ and notation (2.1) is in force.

For every nilpotent algebra $\mathcal{N}$ the *Hilbert function* $H=H_{\mathcal{N}}:\mathbb{N}\to\mathbb{N}$ of $\mathcal{N}^0$ is defined by $H(k)=\dim(\mathcal{N}^k/\mathcal{N}^{k+1})$. Clearly, $H(0)=1\le H(\nu)$ for $\nu:=\nu(\mathcal{N})$, and $H(k)=0$ for $k>\nu$. As usual, we write H also as finite sequence $\{H(0),H(1),\ldots,H(\nu)\}$ and call H *symmetric* if

$$H(k)=H(\nu-k)\;\text{ for all }\; 0\le k\le\nu\,. \tag{2.6}$$

The Hilbert function is a rough invariant for nilpotent algebras that in general does not suffice to distinguish two given algebras.

The complement of $\mathcal{N}$ in $\mathcal{N}^0$ is the maximal subgroup of $\mathcal{N}^0$. A special subgroup is the unipotent group $\mathcal{U}:=\mathbb{1}+\mathcal{N}$. The exponential mapping $\exp:\mathcal{N}\to\mathcal{U}$ is a group isomorphism with inverse $\log$, where $\log(\mathbb{1}+x)=\log_1(x)$ for all $x\in\mathcal{N}$.

With $\mathrm{Aut}(\mathcal{N})$ we denote the algebra automorphism group of $\mathcal{N}$. The endomorphism algebra $\mathrm{End}(\mathcal{N})$ endowed with the commutator product $[\lambda,\sigma]=\lambda\sigma-\sigma\lambda$ is a Lie algebra that we also denote by $\mathfrak{gl}(\mathcal{N})$. With $\mathfrak{Der}(\mathcal{N})\subset\mathfrak{gl}(\mathcal{N})$ we denote the Lie subalgebra of all derivations. This is an $\mathcal{N}^0$-leftmodule in an obvious way. For every nilpotent $\lambda\in\mathfrak{Der}(\mathcal{N})$ the operator $\exp(\lambda)\in\mathrm{Aut}(\mathcal{N})$ is unipotent and conversely, every unipotent $u\in\mathrm{Aut}(\mathcal{N})$ is of this form.

Derivations of $\mathcal{N}$ can be obtained in the following way: Let π be a projection on $\mathcal{N}$ with range $\mathcal{N}_{[1]}$ and suppose that $\lambda\in\mathrm{End}(\mathcal{N})$ satisfies $\lambda(\mathcal{N})\subset\mathcal{N}_{[1]}$ and $\lambda(\mathcal{N}^2)=0$. Then for every $a\in\mathcal{N}_{[3]}$ the operator $x\mapsto ax+\pi(ax)+\lambda(x)$ is in $\mathfrak{Der}(\mathcal{N})$. This implies

$$\dim\mathfrak{Der}(\mathcal{N}) \;\ge\; (\dim\mathcal{N}/\mathcal{N}^2)\dim\mathcal{N}_{[1]} \;+\; \dim(\mathcal{N}_{[3]}/\mathcal{N}_{[2]})\,. \tag{2.7}$$

This estimate improves Theorem 5.4 in [10], which gives in case of algebraically closed base field (and finite dimension) the lower bound (2.7) without the summand $\dim(\mathcal{N}_{[3]}/\mathcal{N}_{[2]})$. Also

$$\dim\mathfrak{Der}(\mathcal{N}) \;\le\; (\dim\mathcal{N}/\mathcal{N}^2)\dim\mathcal{N}$$

always holds since every derivation of $\mathcal{N}$ is uniquely determined by its values on a linear subspace L with $\mathcal{N}=L+\mathcal{N}^2$. The latter inequality is an equality e.g. if $\mathcal{N}=\mathfrak{m}/\mathfrak{m}^k$ with $k\ge 2$ and $\mathfrak{m}\subset\mathbb{F}[T_1,\ldots,T_n]$ a maximal ideal.

3. Affinely homogeneous surfaces

For the rest of the paper we consider only nilpotent algebras that are *commutative*. The algebras of finite dimension of this type are precisely the maximal ideals of commutative Artinian local algebras.

It is well-known that the graph $S:=\{(x,t)\in V\oplus\mathbb{F}: t=q(x)\}$ of every quadratic form q on a vector space V is affinely homogeneous. On the other hand, for given vector space

W of finite dimension, the vast majority of smooth algebraic hypersurfaces in W of degree > 2 is not affinely homogeneous. In fact, it is not easy to find an affinely homogeneous hypersurface of higher degree at all. In this section we show that the nil-surfaces associated with a certain class of commutative nilpotent algebras $\mathcal{N}$ are affinely homogeneous varieties and can have arbitrary high degrees. More precisely, we have a positive result in case where $\mathcal{N}$ admits some sort of a $\mathbb{Z}^+$-grading.

3.1 Definition. Let $\mathcal{N}$ be a nilpotent algebra, π a projection on $\mathcal{N}$ and $\mathcal{N} = \bigoplus_{k\in\mathbb{Z}^+} \mathcal{N}_k$ a vector space decomposition. This decomposition is called

- a *grading* if
$$\mathcal{N}_j\mathcal{N}_k \subset \mathcal{N}_{j+k} \quad \text{for all} \quad j,k>0\ ,$$
- a π-*grading* if
$$\mathcal{N}_j\mathcal{N} \ \subset\ \bigoplus_{\ell>j}\mathcal{N}_\ell \quad \text{for all} \quad j>0 \quad \text{and}$$
$$\pi(\mathcal{N}_{j_1}\mathcal{N}_{j_2}\cdots\mathcal{N}_{j_r}) \ \subset\ \pi(\mathcal{N}_{j_1+j_2+\ldots+j_r})$$

holds for every finite sequence $j_1, j_2, \ldots, j_r$ in $\mathbb{Z}^+$.

A quite special sort of grading is what usually is called a *canonical grading:* To every nilpotent algebra $\mathcal{N}$ associate the graded algebra
$$\mathrm{gr}(\mathcal{N}) := \bigoplus_{k>0}\mathcal{N}^k/\mathcal{N}^{k+1}\ ,$$
where for every $j,\,k>0$ and every $x\in\mathcal{N}^j,\ y\in\mathcal{N}^k$ the product of the residue classes $x{+}\mathcal{N}^{j+1}$ and $y{+}\mathcal{N}^{k+1}$ is $xy{+}\mathcal{N}^{j+k+1}$. It is quite rare that $\mathcal{N}$ and $\mathrm{gr}(\mathcal{N})$ are isomorphic as algebras. But if there exists an algebra isomorphism $\varphi:\mathcal{N}\to\mathrm{gr}(\mathcal{N})$, the grading $\mathcal{N}=\bigoplus\mathcal{N}_k$ with $\mathcal{N}_k=\varphi^{-1}(\mathcal{N}^k/\mathcal{N}^{k+1})$ is called a canonical grading. Clearly $\mathcal{N}_k=0$ for all $k>\nu(\mathcal{N})$ in this case.

Gradings and id-gradings on $\mathcal{N}$ coincide. Graded nilpotent commutative algebras exist for every nil-index ν – for instance the maximal ideal of $\mathbb{F}[X]/(X^{\nu+1})$ has an obvious canonical grading. On the other hand, not every nilpotent commutative algebra has a grading, see Section 8 for counterexamples. In general, a gradable nilpotent algebra $\mathcal{N}$ may not have a grading with $\mathcal{N}_1\neq 0$. A simple example with this phenomenon is the commutative algebra $\mathcal{N}=\mathbb{F}x\oplus\mathbb{F}y\oplus\mathbb{F}x^2\oplus\mathbb{F}x^3$ with generators x,y satisfying $x^4=y^3=xy=x^3-y^2=0$. Then we get a grading of $\mathcal{N}$ if we denote the summands successively by $\mathcal{N}_2,\mathcal{N}_3,\mathcal{N}_4,\mathcal{N}_6$. Notice that this $\mathcal{N}$ does not admit a canonical grading. Indeed, the annihilators of $\mathcal{N}$ and $\mathrm{gr}(\mathcal{N})$ have dimension 1 and 2 respectively.

What about affine homogeneity of S_π for arbitrary commutative nilpotent algebras $\mathcal{N}$ and $\mathcal{N}_{[1]}$-ranged projections π? For some time the answer of this question was beyond our reach as all our attempts to prove it for general commutative nilpotent algebras (even when restricted to the special case $\dim(\mathcal{N}_{[1]})=1$) failed in case of nil-index ≥ 5. However, contrary to the expectation (expressed as conjecture in [7], [8a]) counterexamples do exist. Anticipating the answer, which will be extensively discussed in Section 8, we have:

- There exist commutative nilpotent algebras $\mathcal{N}$, such that S_π is not affinely homogeneous (any such algebra cannot be graded).
- There exist commutative nilpotent algebras $\mathcal{N}$ without a grading but still with affinely homogeneous S_π.

Now we resume our investigation by proving the main result of this section. For every pair of vector spaces V,W the affine group $\mathrm{Aff}(V)$ acts from the right on the space of all polynomial mappings $f:V\to W$ and we denote by

$$A_f := \{g \in \mathrm{Aff}(V) : f \circ g^{-1} = f\}$$

the isotropy subgroup at f. Clearly, A_f leaves every level set $f^{-1}(c),\ c \in f(V)$, invariant.

3.2 Theorem. *Let $\mathcal{N}$ be a commutative nilpotent algebra and π a projection on $\mathcal{N}$. Assume that $\mathcal{N} = \bigoplus \mathcal{N}_k$ is a π-grading and that there exists an integer $d > 0$ with $\pi(\mathcal{N}) \subset \mathcal{N}_d$ and $\pi(\mathcal{N}_k) = 0$ for all $k \neq d$. Then for $f := \pi \circ \exp_1$ the affine subgroup $A_f \subset \mathrm{Aff}(\mathcal{N})$ acts transitively on $S = f^{-1}(0)$.*
In case $\pi(\mathcal{N}) \subset \mathrm{Ann}(\mathcal{N})$ the group A_f even acts transitively on every level set $f^{-1}(c) = S + c$ with $c \in \pi(\mathcal{N})$.

Proof. Fix an arbitrary point $a \in S$. For every $j \geq 0$ consider the following condition:

$$(\star) \qquad A_f(a) \cap \bigoplus_{k>j} \mathcal{N}_k \neq \emptyset\ .$$

It is clear that for the first claim in the Theorem we only have to show that $(\star)$ holds for all j since then $0 \in A_f(a)$, or equivalently $a \in A_f(0)$.
We first show by induction over j that $(\star)$ is valid for every $j < d$: For $j = 0$ nothing has to be shown. Now fix an arbitrary integer j with $0 < j < d$. As induction hypothesis we then may assume $a \in \bigoplus_{k \geq j} \mathcal{N}_k$. Set $\mathcal{N}_0 := \mathbb{F}\cdot\mathbb{1}$. Every $x \in \mathcal{N}^0$ has a unique decomposition $x = x_0 + x_1 + \ldots$ with $x_k \in \mathcal{N}_k$ for all $k \geq 0$. In particular, $a = a_j + a_{j+1} + \ldots$ with $a_k \in \mathcal{N}_k$ for all $k \geq j$. We trivially extend f to a function $\widehat{f}$ on $\mathcal{N}^0$, more precisely, $\widehat{f}(s\mathbb{1} + x) := f(x)$ for all $s \in \mathbb{F}$ and $x \in \mathcal{N}$.
Denote by $\mathcal{F}$ the space of all polynomial maps $\mathcal{N}^0 \to \mathcal{N}_d$ of degree $\leq d$. Then $\widehat{f} \in \mathcal{F}$. We identify via $x \leftrightarrow \mathbb{1}+x$ the nilpotent algebra $\mathcal{N}$ with the affine hyperplane $\mathcal{U} := \mathbb{1}+\mathcal{N}$ in $\mathcal{N}^0$ and A_f with the subgroup

$$G := \{g \in \mathrm{GL}(\mathcal{N}^0) : g(\mathcal{U}) = \mathcal{U} \ \text{ and } \ \widehat{f}(gx) = \widehat{f}(x) \ \text{ for all } \ x \in \mathcal{U}\}\ .$$

With $a_j \in \mathcal{N}_j$ from above define $\lambda = \lambda_j \in \mathrm{End}(\mathcal{N}^0)$ by

$$(\star\star) \qquad \lambda(x) := \ a_j\Big(-x_0 + \frac{1}{d-j}\sum_{k=1}^{d-j} k\, x_k\Big)\ .$$

Then $\lambda(\mathbb{1}) = -a_j \in \mathcal{N}$ and $\lambda(\mathcal{N}) \subset \bigoplus_{k>j} \mathcal{N}_k$. For $g := \exp(\lambda) \in \mathrm{GL}(\mathcal{N}^0)$ (λ is nilpotent) we therefore have $g(\mathbb{1}+a) = \mathbb{1}+b$ for some b in $\bigoplus_{k>j} \mathcal{N}_k$. It is enough to show $g \in G$ since then $b \in A_f(a)$ by the above identifications. The identity $g(\mathcal{U}) = \mathcal{U}$ is obvious. It remains to compute $\widehat{f} \circ g$ on $\mathcal{U}$. This can be done in terms of the following nilpotent operator $\xi \in \mathrm{End}(\mathcal{F})$:

$$\xi(f)x := f'(x)(\lambda x) \quad \text{for all} \quad f \in \mathcal{F} \ \text{ and } \ x \in \mathcal{N}^0\ ,$$

where $f'(x) \in \mathrm{Hom}(\mathcal{N}^0, \mathcal{N}_d)$ is the formal derivative of f at x. From $\lambda(\mathcal{N}^0) \subset \mathcal{N}$ we conclude that $\xi(f)$ vanishes on $\mathcal{U}$ as soon as f has the same property. For all $f \in \mathcal{F}$ we have the generalized Taylor's formula

$$f \circ \exp(\lambda) = \exp(\xi)(f) = f + \xi(f) + \frac{1}{2}\xi^2(f) + \ldots\ .$$

It therefore remains to show that $\xi(\widehat{f})$ vanishes on $\mathcal{U}$. Now for every $x \in \mathcal{U}$ we have $x_0 = \mathbb{1}$ and

$$\xi(\widehat{f})x = \pi \sum c_\nu x_1^{(\nu_1)} x_2^{(\nu_2)} \cdots x_d^{(\nu_d)}\ ,$$

where the sum is taken over all multi indices $\nu \in \mathbb{N}^d$ with $\nu_1 + 2\nu_2 + \ldots + d\nu_d = d - j$, and $c_\nu \in \mathcal{N}_j$ are certain factors. Fix such a multi index ν. For simpler notation we put $x^{(-1)} := 0$

for every $x \in \mathcal{N}^0$. Then we have

$$\begin{aligned} c_\nu\, x_1^{(\nu_1)} x_2^{(\nu_2)} \cdots x_d^{(\nu_d)} &= \sum_{k=1}^{d-j} \frac{k}{d-j} a_j x_k\, {}^{\partial}\!/\!_{\partial x_{j+k}} \Big(x_1^{(\nu_1)} \cdots x_k^{(\nu_k - 1)} \cdots x_{j+k}^{(\nu_{j+k}+1)} \cdots x_d^{(\nu_d)} \Big) \\ &\qquad - a_j\, {}^{\partial}\!/\!_{\partial x_j} \Big(x_1^{(\nu_1)} \cdots x_j^{(\nu_j+1)} \cdots x_d^{(\nu_d)} \Big) \\ &= \Big(\sum_{k=1}^{d-j} \frac{k\nu_k}{d-j} - 1 \Big) a_j x_1^{(\nu_1)} x_2^{(\nu_2)} \cdots x_d^{(\nu_d)} \;=\; 0 \end{aligned}$$

since $\nu_k = 0$ for $k > d-j$. This proves the induction step and hence $(\star)$ for all $j < d$, that is, we may assume $a \in \bigoplus_{k\geq d} \mathcal{N}_k$.
To finish the proof of the first statement we notice that $\pi\big((x+b)^k\big) = \pi(x^k)$ holds for all $x \in \mathcal{N}$, $k \geq 1$ and every $b \in \big(\mathrm{id} - \pi\big)\big(\bigoplus_{k\geq d} \mathcal{N}_k\big)$. This implies that the translation $x \mapsto x+b$ for every such b belongs to A_f. As a consequence we may assume $a \in \pi(\mathcal{N}_d)$. But a is also in S by assumption and $S \cap \pi(\mathcal{N}_d) = \{0\}$, that is, $a = 0$.

Now suppose $\pi(\mathcal{N}) \subset \mathrm{Ann}(\mathcal{N})$ and let B be the subgroup of all $g \in A_f$ that commute with all translations $x \mapsto x + c,\ c \in \pi(\mathcal{N}_d)$. In every induction step above the operator λ vanishes on $\pi(\mathcal{N}) \subset \mathcal{N}^0$. This implies that B is already transitive on S and the second claim follows. $\square$

In the proof of 3.2 we have identified $\mathcal{N}$ with $\mathbb{1} + \mathcal{N}$ via the identification $x \leftrightarrow \mathbb{1}+x$. In case $\mathcal{N}_d$ is the annihilator $\mathcal{N}_{[1]}$ of $\mathcal{N}$ in 3.2, the operator λ in $(\star\star)$ corresponds via the identification to the affine transformation $T : \mathcal{N} \to \mathcal{N}$, where

$$T = \frac{1}{(d-j)} D - a_j \ \text{ with } \ D \in \mathfrak{Der}(\mathcal{N}) \ \text{ defined by } \ D(x) = a_j \sum_{k>0} k\, x_k \,. \tag{3.3}$$

The proof in [8] for the special case, where $\mathcal{N}_d$ has dimension 1 and is the annihilator of $\mathcal{N}$, is also based on these nilpotent derivations D.

For base field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ Theorem 3.2 essentially is already contained in [5], see also [6] for a special version with $\mathbb{F} = \mathbb{C}$. For the special case that $\pi(\mathcal{N})$ is the annihilator of $\mathcal{N}$ and this annihilator has dimension 1 see also [8].

In Theorem 3.2 the group A_f is not the full affine group $\mathrm{Aff}(S)$. Indeed,

$$\theta_t := \bigoplus_{k>0} t^k\, \mathrm{id}_{|\mathcal{N}_k} \ \in \ \mathrm{GL}(S) \tag{3.4}$$

satisfies $f \circ \theta_t = t^d f$ for every $t \in \mathbb{F}^*$. As a consequence, if π has 1-dimensional range in $\mathcal{N}_d \cap \mathrm{Ann}(\mathcal{N})$, the group $\mathrm{Aff}(S)$ has at most d orbits in $\mathcal{N}$. In particular, in case $\mathbb{F} = \mathbb{C}$ this group has only two orbits in $\mathcal{N}$, the hypersurface S and its open connected complement $\mathcal{N} \backslash S$. In case $\mathbb{F} = \mathbb{R}$ the connected identity component $\mathrm{Aff}(S)^0$ has three orbits in $\mathcal{N}$, the hypersurface S and both sides of the complement $\mathcal{N} \backslash S$.

In case $\mathcal{N} = \bigoplus \mathcal{N}_k$ is a grading in 3.2, the operator $\lambda := \bigoplus_{k>0} k\, \mathrm{id}_{|\mathcal{N}_k} \in \mathfrak{Der}(\mathcal{N})$ is diagonalizable over $\mathbb{F}$ and has only positive integers as eigen-values. Conversely, if $\mathcal{N}$ is an arbitrary (commutative) nilpotent algebra and $\lambda \in \mathfrak{Der}(\mathcal{N})$ is diagonalizable over $\mathbb{F}$ with spectrum in $\mathbb{Z}^+$, then $\mathcal{N} = \bigoplus \mathcal{N}_k$ is a grading, where $\mathcal{N}_k$ for every k is the k-eigenspace of λ.

As already mentioned, not every nilpotent algebra $\mathcal{N}$ has a grading, compare Section 8 for counterexamples. But there exists always a decomposition

$$\mathcal{N} = \bigoplus_{k\in\mathbb{Z}^+} \mathcal{N}_k\,, \quad \text{with} \quad \mathcal{N}_j \mathcal{N}_k \subset \bigoplus_{\ell \geq j+k} \mathcal{N}_\ell \ \text{ for all } \ j,k > 0\,. \tag{3.5}$$

Indeed, choose $\mathcal{N}_k$ in such a way that $\mathcal{N}^k = \mathcal{N}_k \oplus \mathcal{N}^{k+1}$ for all $k > 0$. In general, a non-gradable nilpotent algebra may have a π-grading with non-trivial projection π. As a trivial example for this phenomenon choose for fixed non-gradable $\mathcal{N}$ a decomposition (3.5) and let π on $\mathcal{N}$ be the canonical projection onto $\mathcal{N}_2$.

The following result gives a lower bound for the size of the 0-orbit under the affine group $\mathrm{Aff}(S)$ in case π has range in the annihilator of $\mathcal{N}$: Fix $k > 0$ and consider the ideal $\mathcal{N}_{[k]}$ as defined in 2.2. Then for $f := \pi \circ \exp_1$ and $S := f^{-1}(0)$ as before the intersection $S \cap \mathcal{N}_{[k]}$ is a smooth subvariety with dimension $\dim(\mathcal{N}_{[k]}/\mathcal{N}_{[1]})$. For every $a \in S \cap \mathcal{N}_{[3]}$ and $\rho := (\mathrm{id} + \pi)/2 \in \mathrm{End}(\mathcal{N})$ define the affine transformation h_a on $\mathcal{N}$ by

$$h_a(x) := x - \rho(ax) + a\ .$$

3.6 Proposition. *Let $\mathcal{N}$ be an arbitrary commutative nilpotent algebra, let π be a projection on $\mathcal{N}$ with range in the annihilator of $\mathcal{N}$ and let $S := S_\pi$. Then $\{h_a : a \in S \cap \mathcal{N}_{[3]}\}$ is contained in A_f and generates a subgroup acting transitively on $S \cap \mathcal{N}_{[3]}$. In particular, S is affinely homogeneous if $\mathcal{N}$ has nil-index ≤ 3.*

Proof. Fix $a \in S \cap \mathcal{N}_{[3]}$. Then $h_a \in \mathrm{Aff}(\mathcal{N})$ since the operator $x \to \rho(ax)$ is nilpotent on $\mathcal{N}$. A simple computation shows $f \circ h_a = f$ and also that h_a, h_a^{-1} leave $\mathcal{N}_{[3]}$ invariant. The first claim follows with $h_a(0) = a$. The second follows from $\mathcal{N}_{[3]} = \mathcal{N}$ in case of $\nu(\mathcal{N}) \leq 3$. □

4. Admissible algebras

For the rest of this paper we deal only with commutative nilpotent algebras $\mathcal{N}$ of finite dimension over $\mathbb{F}$ such that the annihilator $\mathcal{N}_{[1]}$ is of dimension 1. For simplicity we call such algebras *admissible algebras*. These are just the maximal ideals of Gorenstein algebras of finite vector space dimension ≥ 2 over $\mathbb{F}$.

In this and the subsequent sections we construct several objects, universally associated with a given admissible algebra $\mathcal{N}$. These will encode enough information to characterize the admissible algebra up to isomorphy. Roughly speaking, we define a certain family Σ of smooth hypersurfaces $S \subset \mathcal{N}$ such that each of its members determines $\mathcal{N}$. We also establish a natural duality between the points of a given hypersurface $S \in \Sigma$ and the members of Σ itself. In the next section we construct a set of $\mathbb{F}$-valued polynomials, so-called nil-polynomials, closely related to the hypersurfaces $S \in \Sigma$. We also determine how the algebra structure of $\mathcal{N}$ can be reconstructed from an associated nil-polynomial p (in fact the knowledge of the quadratic and cubic terms of p turns out to be sufficient.)

We start with some preparations. We call every projection $\pi = \pi^2 \in \mathrm{End}(\mathcal{N})$ with range $\pi(\mathcal{N}) = \mathcal{N}_{[1]}$ an *admissible projection* on $\mathcal{N}$ and denote by $\Pi(\mathcal{N}) \subset \mathrm{End}(\mathcal{N})$ the subvariety of all admissible projections. Every $\pi \in \Pi(\mathcal{N})$ is uniquely determined by its kernel $\mathcal{K} = \ker(\pi)$ that satisfies $\mathcal{N} = \mathcal{K} \oplus \mathcal{N}_{[1]}$. Further, every projection $\pi \in \Pi(\mathcal{N})$ gives rise to the algebraic smooth hypersurface, compare (2.4),

$$S_\pi = \{x \in \mathcal{N} : \pi \circ \exp_1(x) = 0\} \;=\; \log_1(\ker \pi)$$

that we also call a *nil-hypersurface*. We denote by $\Sigma(\mathcal{N}) := \{S_\pi : \pi \in \Pi(\mathcal{N})\}$ the set of all such hypersurfaces. Note that $\{0\}$ is the intersection of all $S \in \Sigma(\mathcal{N})$. The canonical map $\beta : \Pi(\mathcal{N}) \to \Sigma(\mathcal{N})$, $\pi \mapsto S_\pi$, is clearly surjective. Later on (see 4.6) we will show that β is even bijective.

All the key objects associated with $\mathcal{N}$, such as the bilinear forms $b_\pi : \mathcal{N} \times \mathcal{N} \to \mathcal{N}_{[1]}$, the polynomial maps $f_\pi : \mathcal{N} \to \mathcal{N}_{[1]}$ and the subvarieties $S_\pi \subset \mathcal{N}$ depend on the choice of the projection π. In this section we show that in the admissible case the 'essential' properties of b_π, f_π and S_π do not depend on the projection and can be considered as invariants associated to $\mathcal{N}$ only. Further we prove that every smooth hypersurface S_π determines the admissible algebra $\mathcal{N}$ up to isomorphy.

For every $e \in \mathcal{N}^0$ define the multiplication operator $M_e \in \mathrm{End}(\mathcal{N})$ by $M_e(x) = ex$. Recall that $\exp : \mathcal{N} \to \mathcal{U} = \mathbb{1} + \mathcal{N}$ is a group isomorphism with inverse $\log$.

4.1 Lemma. $\Pi(\mathcal{N})$ *is an affine plane of dimension* $\dim(\mathcal{N}/\mathcal{N}_{[1]})$ *in* $\mathrm{End}(\mathcal{N})$*. In fact*

$$\Pi(\mathcal{N}) = \{\pi \in \ \mathrm{End}(\mathcal{N}) : \pi = \rho \circ \pi,\ \rho = \pi \circ \rho\} \quad \text{for every} \quad \rho \in \Pi(\mathcal{N})$$

and $\mathcal{N}/\mathcal{N}_{[1]} \times \Pi(\mathcal{N}) \longrightarrow \Pi(\mathcal{N})\,,\ (x + \mathcal{N}_{[1]}\,,\ \pi) \longmapsto \pi \circ M_{\exp x}$ *, yields a well-defined simply transitive action of the vector group* $\mathcal{N}/\mathcal{N}_{[1]}$ *on* $\Pi(\mathcal{N})$ *.*

Proof. Clearly, the mapping is well defined, has values in $\Pi(\mathcal{N})$ and gives an action of $\mathcal{N}/\mathcal{N}_{[1]}$. The action is also free – indeed, suppose that $\pi \circ M_{\exp b} = \pi$ for some $\pi \in \Pi(\mathcal{N})$, $b \in \mathcal{N}$. For $c := \exp_1(b)$ then $\pi \circ M_c = 0$, that is, $b_\pi(c, \mathcal{N}) = 0$ and thus $c \in \mathcal{N}_{[1]}$, see (2.5). But then also $b = c \in \mathcal{N}_{[1]}$.
The action is also transitive – indeed, fix arbitrary $\pi, \rho \in \Pi(\mathcal{N})$. Then $\lambda := \rho - \pi$ vanishes on $\mathcal{N}_{[1]}$ and satisfies $\lambda = \pi \circ \lambda$. Hence, again by the non-degeneracy of b_π on $\ker(\pi)$, we conclude $\lambda = \pi \circ M_b$ for some $b \in \ker \pi$. This implies $\rho = \pi + \pi \circ M_b = \pi \circ M_{\mathbb{1}+b} = \pi \circ M_{\exp c}$ for $c := \log(\mathbb{1} + b) \in \mathcal{N}$. □

The algebra automorphism group $\mathrm{Aut}(\mathcal{N})$ acts on $\Pi(\mathcal{N})$ by conjugation, that is, by $L(\pi) := L \circ \pi \circ L^{-1}$ for all $L \in \mathrm{Aut}(\mathcal{N})$ and $\pi \in \Pi(\mathcal{N})$. Then $L(\ker \pi) = \ker L(\pi)$ is obvious. The group $\mathrm{Aut}(\mathcal{N})$ also acts on $\Sigma(\mathcal{N})$ in the obvious way and satisfies $L(S_\pi) = S_{L(\pi)}$ for all $L \in \mathrm{Aut}(\mathcal{N})$ and $\pi \in \Pi(\mathcal{N})$.

The following result generalizes Propositions 2.2. and 2.3 in [6].

4.2 Theorem. *Let* $\mathcal{N}, \widetilde{\mathcal{N}}$ *be arbitrary admissible algebras having not necessarily the same dimension. Also let* $\pi \in \Pi(\mathcal{N})$*,* $\widetilde{\pi} \in \Pi(\widetilde{\mathcal{N}})$ *be arbitrary admissible projections. Then for* $S := S_\pi$*,* $\widetilde{S} := S_{\widetilde{\pi}}$ *and for every linear map* $L : \mathcal{N} \to \widetilde{\mathcal{N}}$ *the following conditions are equivalent, provided* $\dim(\widetilde{\mathcal{N}}) > 1$*.*

(i) L *is an algebra isomorphism.*

(ii) $\widetilde{S} = L(S - c)$ *for some* $c \in \mathcal{N}$*.*

Furthermore, the point $c = c_{L,\pi,\widetilde{\pi}}$ *in* (ii) *is uniquely determined by* $L, \pi, \widetilde{\pi}$ *and coincides with the unique element in* S *satisfying* $\widetilde{\pi} = L \circ (\pi \circ M_{\exp c}) \circ L^{-1}$*. Finally*

$$(*) \qquad S = \big\{ c_{L,\pi,\rho} : \rho \in \Pi\big(\widetilde{\mathcal{N}}\big)\ \big\}$$

holds for every algebra isomorphism $L : \mathcal{N} \to \widetilde{\mathcal{N}}$*.*

Proof. (i) $\Longrightarrow$ (ii) Assume (i). By Lemma 4.1 there exists $c \in \mathcal{N}$ with $L^{-1} \circ \widetilde{\pi} \circ L = \pi \circ M_{\exp c}$. Since $\exp_1(c + a) = \exp_1(c) + a$ for every $a \in \mathcal{N}_{[1]}$, we can assume $\pi(\exp_1 c) = 0$, that is, $c \in S$. Then $\pi(\exp_1 c) = 0$ implies

$$\widetilde{\pi}(\exp_1 L(x)) = \widetilde{\pi} \circ L(\exp_1 x) = L \circ \pi((\exp c)(\exp_1 x)) = L \circ \pi(\exp_1(x + c))$$

for all $x \in \mathcal{N}$, that is, $L(x) \in \widetilde{S}$ if and only if $x + c \in S$.

(ii) $\Longrightarrow$ (i) Since $\dim \widetilde{\mathcal{N}} > 1$ also $\nu(\widetilde{\mathcal{N}}) > 1$ and the linear span of $\widetilde{S}$ is $\widetilde{\mathcal{N}}$. Then L is an epimorphism and also $\nu(\mathcal{N}) > 1$. We show that L is also injective: Note first that since the quadratic part of $\pi \circ \exp_2$ is non-degenerate on $\ker(\pi)$, S is not invariant under any non-trivial translation. If $\ker(L) \neq 0$ then $L^{-1}(\widetilde{S}) = \ker(L) + S - c$ would be Zariski dense in $\mathcal{N}$ and a proper algebraic subset of $\mathcal{N}$ at the same time, a contradiction.
Since $L - L(c)$ provides an affine equivalence between S and $\widetilde{S}$ we can use the analytic proof of Prop. 2.3 in [6] to obtain that L is an $\mathbb{F}$-algebra isomorphism in the special case $\mathbb{F} = \mathbb{C}$. We reduce the case of a general field to this special result by a Lefschetz principle type argument. To begin with we denote by $\boldsymbol{K}$ the set of all subfields $\mathbb{K} \subset \mathbb{F}$ that are obtained by adjoining a finite subset of $\mathbb{F}$ to the prime field of $\mathbb{F}$. It is well known that every $\mathbb{K} \in \boldsymbol{K}$ is isomorphic to

a subfield of $\mathbb{C}$.
Now let $e \in \mathcal{N}$ be an arbitrary but fixed element. Then it is enough to show that $L(e^2) = L(e)^2$: Choose a linear basis B of $\mathcal{N}$ containing a basis of $\pi^{-1}(0)$ and a basis of $\mathcal{N}_{[1]}$. Then there exists a field $\mathbb{K} \in \boldsymbol{K}$ such that the linear span $\mathcal{B} := \langle B \rangle_{\mathbb{K}}$ contains e and is a $\mathbb{K}$-subalgebra of $\mathcal{N}$. By the choice of B the intersection $\mathcal{B} \cap \mathcal{N}_{[1]}$ has dimension 1 over $\mathbb{K}$ and is the annihilator of $\mathcal{B}$. Also, $S \cap \mathcal{B}$ is a smooth hypersurface over $\mathbb{K}$ in $\mathcal{B}$. In the same way choose a linear basis $\widetilde{B}$ of $\widetilde{\mathcal{N}}$ containing a basis of $\widetilde{\pi}^{-1}(0)$ and a basis of $\widetilde{\mathcal{N}}_{[1]}$. Adjoining a suitable finite subset of $\mathbb{F}$ to $\mathbb{K}$ we may assume in addition without loss of generality that $\widetilde{\mathcal{B}} := \langle \widetilde{B} \rangle_{\mathbb{K}}$ contains c and also is a $\mathbb{K}$-subalgebra of $\widetilde{\mathcal{N}}$. Enlarging $\mathbb{K}$ again within $\boldsymbol{K}$ if necessary, we may even assume that the affine transformation $A := L - L(c)$ maps $\mathcal{B}$ onto $\widetilde{\mathcal{B}}$. Clearly A maps $S \cap \mathcal{B}$ onto $\widetilde{S} \cap \widetilde{\mathcal{B}}$. We now consider $\mathbb{K}$ as subfield of $\mathbb{C}$. We then get the complex nilpotent algebras $\mathcal{B} \otimes_{\mathbb{K}} \mathbb{C}$ and $\widetilde{\mathcal{B}} \otimes_{\mathbb{K}} \mathbb{C}$ with annihilators $(\mathcal{B} \cap \mathcal{N}_{[1]}) \otimes_{\mathbb{K}} \mathbb{C}$ and $(\widetilde{\mathcal{B}} \cap \widetilde{\mathcal{N}}_{[1]}) \otimes_{\mathbb{K}} \mathbb{C}$ respectively, each having complex dimension 1 over $\mathbb{C}$. The $\mathbb{K}$-affine map $A_{|\mathcal{B}}$ extends to a $\mathbb{C}$-affine map $\mathcal{B} \otimes_{\mathbb{K}} \mathbb{C} \to \widetilde{\mathcal{B}} \otimes_{\mathbb{K}} \mathbb{C}$ sending the corresponding complex hypersurfaces onto each other. By Proposition 2.3 in [6] then $M_{|\mathcal{B}} \otimes_{\mathbb{K}} \mathrm{id}_{\mathbb{C}}$ is an algebra isomorphism, implying $L(e^2) = L(e)^2$. Together with the first step this proves (i) $\Longleftrightarrow$ (ii).

Next, assume that c in (ii) is not uniquely determined. Then there exists $a \in \mathcal{N}$ with $a \neq 0$ and $S = S + a$. For $\mathcal{K} := \pi^{-1}(0)$ we have $\mathcal{N} = \mathcal{K} \oplus \mathcal{N}_{[1]}$ and $S = \{(y, f(y)) : y \in \mathcal{K}\}$ is the graph of the polynomial map $f : \mathcal{K} \to \mathcal{N}_{[1]}$ given by $f(y) = -\pi(\exp_2(y))$. In particular, $f = (b, f(b))$ for some $b \in \mathcal{K}$ and

$$(**) \qquad f(y + tb) = f(y) + f(tb) \quad \text{for all} \quad y \in \mathcal{K}$$

and all $t \in \mathbb{Z}$. Since f is a polynomial map $(**)$ even holds for all $t \in \mathbb{F}$. Comparing terms that are linear in y as well as in t we get $\pi(by) = 0$ for all $y \in \mathcal{K}$. But the quadratic form $\pi(y^2)$ is non-degenerate on $\mathcal{K}$, implying $b = 0$ in contradiction to $f \neq 0$.

Finally, for the proof of $(*)$ we may assume $\widetilde{\mathcal{N}} = \mathcal{N}$ and $L = \mathrm{id}_{\mathcal{N}}$. Fix an arbitrary $c \in S$ and put $e := \exp_1(c)$. Then $\pi(e) = 0$ and $\rho := \pi + \pi \circ M_e$ is an admissible projection on $\mathcal{N}$. This implies $S_\rho = S - c$ and consequently $c = c_{L,\pi,\rho}$. □

4.3 Corollary. *The algebras $\mathcal{N}, \widetilde{\mathcal{N}}$ are isomorphic if and only if $S, \widetilde{S}$ are affinely equivalent.*

4.4 Corollary. *Under the same assumptions as in* 4.2*, for every linear map $L : \mathcal{N} \to \tilde{\mathcal{N}}$ the following conditions are equivalent.*

(i) $\widetilde{S} = L(S)$.
(ii) *L is an algebra isomorphism with* $\widetilde{\pi} = L(\pi) \quad (= L \circ \pi \circ L^{-1})$.

Proof. Assume (i). Then L is an algebra isomorphism with $\widetilde{\pi} = L(\pi \circ M_{\exp c}) = L(\pi)$ for $c = 0$ by Theorem 4.2. The converse implication is trivial. □

4.5 Corollary. $\mathrm{Aut}(\mathcal{N}) \cap \mathrm{Aff}(S) = \mathrm{GL}(S)$.

Next we show equivalences between the various sets. In particular, for every fixed $S \in \Sigma(\mathcal{N})$, we give a duality between points in S and surfaces in Σ itself.

4.6 Lemma. (Duality) *Let $\mathcal{N}$ be an admissible algebra and $\pi \in \Pi(\mathcal{N})$. Then the mappings*

$$\begin{array}{rcl} \alpha_\pi : S_\pi & \longrightarrow & \Pi(\mathcal{N}) \\ s & \longmapsto & \pi \circ M_{\exp s} \end{array}, \qquad \begin{array}{rcl} \beta : \Pi(\mathcal{N}) & \longrightarrow & \Sigma(\mathcal{N}) \\ \rho & \longmapsto & S_\rho \end{array}$$

are bijective and satisfy $\quad \beta \circ \alpha_\pi(s) = S_\pi - s \quad$ *for all $s \in S_\pi$.*

Proof. Let $\rho = \pi \circ M_{\exp s}$ with $s \in S_\pi$. Then $x \in S_\rho$ is equivalent to

$$0 = \pi\big(\exp(s)\exp_1(x)\big) = \pi\big(\exp_1(x+s) - \exp_1(s)\big)$$

and hence to $(x + s) \in S_\pi$ since $\pi(\exp_1(s)) = 0$. Bijectivity of α_π follows from the proof of 4.1 and the fact that S_π as graph has the following property: Every $x \in \mathcal{N}$ is a unique sum

$x = a + b$ with $a \in \mathcal{N}_{[1]}$ and $b \in \mathrm{S}_\pi$. Surjectivity of β holds by definition and injectivity follows from 4.4. $\square$

Our next goal is to investigate the behavior of the above sets under algebra isomorphisms. Obviously, algebra isomorphisms are functorial in the sense that for every such isomorphism $L : \mathcal{N} \to \widetilde{\mathcal{N}}$ one has the following well-defined maps (also denoted by the same letter L):

$$\begin{array}{rcl} L : \Sigma(\mathcal{N}) & \longrightarrow & \Sigma(\widetilde{\mathcal{N}}) \\ S & \longmapsto & L(S) \end{array}, \qquad \begin{array}{rcl} L : \Pi(\mathcal{N}) & \longrightarrow & \Pi(\widetilde{\mathcal{N}}) \\ \pi & \longmapsto & L \circ \pi \circ L^{-1} \end{array} \tag{4.7}$$

$$\text{with} \quad L(S_\pi) = S_{L(\pi)} \ . \tag{4.8}$$

In particular, we have in case of $\mathcal{N} = \widetilde{\mathcal{N}}$ the group action of the algebra automorphism group $\mathrm{Aut}(\mathcal{N}) \subset \mathrm{GL}(\mathcal{N})$ on the affine plane $\Pi(\mathcal{N})$ by conjugation, that is, by $L(\pi) = L \circ \pi \circ L^{-1}$ for all $L \in \mathrm{Aut}(\mathcal{N})$ and $\pi \in \Pi(\mathcal{N})$. Also the affine group $\mathrm{Aff}(S)$ acts canonically on the hypersurface $S \in \Sigma(\mathcal{N})$ and we show next that both group actions are equivariantly equivalent, more precisely, define the following map

$$\gamma : \mathrm{Aut}(\mathcal{N}) \longrightarrow \mathrm{Aff}(\mathcal{N}), \qquad L \mapsto \gamma(L) := L - L(c_{L,\pi,\pi}) \ ,$$

with $c_{L,\pi,\pi} \in S$ as in Theorem 4.2, see also the first part of its proof.

4.9 Proposition. *Let $\mathcal{N}$ be an admissible algebra, $\pi \in \Pi(\mathcal{N})$, $S := S_\pi$ and γ as above. Then γ induces a group isomorphism $\mathrm{Aut}(\mathcal{N}) \to \mathrm{Aff}(S)$. Furthermore, the diagram*

$$\begin{array}{ccccc} \mathrm{Aut}(\mathcal{N}) & \times & \Pi(\mathcal{N}) & \longrightarrow & \Pi(\mathcal{N}) \\ \downarrow \gamma & & \downarrow \alpha_\pi^{-1} & & \downarrow \alpha_\pi^{-1} \\ \mathrm{Aff}(S) & \times & S & \longrightarrow & S \end{array}$$

commutes and has bijective vertical arrows, while the horizontal arrows represent the respective group actions.

Proof. For $A := \gamma(L)$ we have $A(S) = L(S - c_{L,\pi,\pi}) = S$ by (ii) of 4.2, i.e., γ yields a map $\mathrm{Aut}(\mathcal{N}) \to \mathrm{Aff}(S)$, which by Theorem 4.2 is a bijection onto $\mathrm{Aff}(S)$. The inverse of γ is just the mapping that associates to every $A \in \mathrm{Aff}(S)$ its linear part $L := A - A(0)$. This implies that γ is a group isomorphism. The commutativity of the diagram can be seen as follows: Direct consequence of 4.8 is the commutativity of the diagram

$$\begin{array}{ccccc} \mathrm{Aut}(\mathcal{N}) & \times & \Sigma(\mathcal{N}) & \longrightarrow & \Sigma(\mathcal{N}) \\ \| & & \uparrow \beta & & \uparrow \beta \\ \mathrm{Aut}(\mathcal{N}) & \times & \Pi(\mathcal{N}) & \longrightarrow & \Pi(\mathcal{N}) \end{array}$$

with bijective vertical maps. Hence, it suffices to prove the commutativity of

$$\begin{array}{ccccc} \mathrm{Aut}(\mathcal{N}) & \times & \Sigma(\mathcal{N}) & \longrightarrow & \Sigma(\mathcal{N}) \\ \downarrow \gamma & & \uparrow \beta \circ \alpha_\pi & & \uparrow \beta \circ \alpha_\pi \\ \mathrm{Aff}(S) & \times & S & \longrightarrow & S \end{array} \ .$$

According to 4.6 and 4.2 we have for arbitrary $L \in \mathrm{Aut}(\mathcal{N})$ and $t \in S = S_\pi$

$$\begin{array}{ccccc} \big(\quad L & , & S - t \big) & \longrightarrow & L(S) - L(t) = S + L(c_{L,\pi,\pi}) - L(t) \\ \downarrow \gamma & & \uparrow \beta \circ \alpha_\pi & & \uparrow \beta \circ \alpha_\pi \\ \big(L - L(c_{L,\pi,\pi}) & , & t \quad \big) & \longrightarrow & L(t) - L(c_{L,\pi,\pi}) \end{array} \ .$$

$\square$

Remark. Proposition 4.9 remains true also in the trivial case $\dim(\mathcal{N}) = 1$, although then S consists of a single point and $\mathrm{Aut}(\mathcal{N}) = \mathrm{GL}(\mathcal{N})$ holds. Indeed, by our definition in this case $\mathrm{Aff}(S) = \mathrm{GL}(\mathcal{N})$ as well. In all other cases, S is total in $\mathcal{N}$ and $\mathrm{Aff}(S)$ acts effectively on S. The proposition implies that the orbit structure for $\mathrm{Aff}(S)$ in S is isomorphic to the orbit structure for $\mathrm{Aut}(\mathcal{N})$ in $\Pi(\mathcal{N})$. In particular, both group actions of $\mathrm{Aut}(\mathcal{N})$ and $\mathrm{Aff}(S)$ are transitive as soon as one of it has this property. For the special case $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ this is essentially the content of Theorem 2.3 in [8]: There the space $\mathbb{T}$ of all hyperplanes in $\mathcal{N}$ transversal to $\mathcal{N}_{[1]}$ is introduced, which via $\pi \leftrightarrow \ker(\pi)$ can be canonically identified with our space $\Pi(\mathcal{N})$. In addition a certain subgroup $G_\pi \subset \mathrm{Aff}(S)$ is introduced, and as Theorem 2.2 in [8] it is proved that $\mathrm{Aut}(\mathcal{N})$ acts transitively on $\mathbb{T}$ if and only if G_π acts transitively on S. Then Theorem 2.3 says $G_\pi = \mathrm{Aff}(S)$ in case of base field $\mathbb{R}$ or $\mathbb{C}$.

Theorem 4.2 together with Proposition 4.9 implies the following result.

4.10 Corollary. *For every admissible algebra $\mathcal{N}$ the following conditions are equivalent.*

(i) *For some (and hence every) $\pi \in \Pi(\mathcal{N})$ the hypersurface S_π is affinely homogeneous.*
(ii) *For all $\pi, \widetilde{\pi} \in \Pi(\mathcal{N})$ the hypersurfaces $S_\pi, S_{\widetilde{\pi}}$ are linearly equivalent.*
(iii) *The group $\mathrm{Aut}(\mathcal{N})$ acts transitively on $\Pi(\mathcal{N})$.*
(iv) *The group $\mathrm{Aut}(\mathcal{N})$ acts transitively on $\Sigma(\mathcal{N})$.*

Proof. (i) $\Longrightarrow$ (ii) Let $\pi, \widetilde{\pi}$ be admissible projections and assume that S_π is affinely homogeneous. Then $S_{\widetilde{\pi}} = S_\pi - c$ for some $c \in S$ by Theorem 4.2. For every $A \in \mathrm{Aff}(S_\pi)$ with $A(0) = c$, the linear transformation $x \mapsto \alpha(x) - c$ maps S_π onto $S_{\widetilde{\pi}}$.

(ii) $\Longrightarrow$ (i) Assume (ii) and fix $\pi \in \Pi(\mathcal{N})$ together with an arbitrary point $c \in S_\pi$. By Theorem 4.2 there exists $\widetilde{\pi} \in \Pi(\mathcal{N})$ with $S_{\widetilde{\pi}} = S_\pi - c$. By assumption there exists $g \in \mathrm{GL}(\mathcal{N})$ with $g(S_\pi) = S_{\widetilde{\pi}}$. The transformation $x \mapsto g(x) + c$ is in $\mathrm{Aff}(S_\pi)$ and maps the origin to c.

(i) $\Longleftrightarrow$ (iii) This follows immediately from Proposition 4.9. (iii) $\Longleftrightarrow$ (iv) is trivial. $\square$

As an immediate consequence of Theorem 3.2 we state:

4.11 Corollary. *For every graded admissible algebra $\mathcal{N}$ conditions* (i) - (iv) *in* 4.10 *hold.* $\square$

Proposition 4.9 says, in particular, that the groups $\mathrm{Aut}(\mathcal{N})$ and $\mathrm{Aff}(S)$ are isomorphic. A careful inspection of the corresponding proofs reveals that under the assumptions of Theorem 3.2 and of Proposition 3.6 these groups contain unipotent subgroups of dimension $\dim(\mathcal{N}) - 1$. In case $\mathcal{N}$ has a grading, we even get $\dim \mathrm{Aut}(\mathcal{N}) \geq \dim(\mathcal{N})$ since then $\theta_s \in \mathrm{Aut}(\mathcal{N})$ as defined in (3.4). The same argument gives $\dim \mathfrak{der}(\mathcal{N}) \geq \dim(\mathcal{N}) = \dim(\mathcal{N}^0) - \dim(\mathcal{N}_{[1]})$ in the graded case, compare also Proposition 2.3 in [12] in case $\mathbb{F} = \mathbb{C}$. For every cyclic nilpotent algebra $\mathcal{N}$ equality holds.

The infinitesimal analogon. As shown in 4.9 the groups $\mathrm{Aut}(\mathcal{N})$, $\mathrm{Aff}(S)$ are always isomorphic. In case $\mathbb{F} = \mathbb{R}, \mathbb{C}$ these groups are even isomorphic as Lie groups, implying that then also the corresponding Lie algebras $\mathfrak{der}(\mathcal{N})$, $\mathfrak{aff}(S)$ are isomorphic. Besides $\mathfrak{der}(\mathcal{N})$ also a Lie algebra $\mathfrak{aff}(S)$ can be canonically defined for arbitrary base fields, but a priori there is no reason why these Lie algebras should be isomorphic also in case $\mathbb{F} \neq \mathbb{R}, \mathbb{C}$:
Fix an admissible algebra $\mathcal{N}$ and an admissible projection π on $\mathcal{N}$. Put $S := S_\pi$, that is $S = f^{-1}(0)$ for $f := \pi \circ \exp_1$. For every $x \in S$ then $T_x(S) := \ker(f'(x))$ is the *tangent space* at x, where $f'(x) = \pi \circ M_{\exp x} \in \mathrm{End}(\mathcal{N})$ is the formal derivative of f at x and, as defined above, $M_y \in \mathrm{End}(\mathcal{N})$ is the multiplication operator $z \mapsto yz$.
Denote by $\mathfrak{aff}(S)$ the linear space of all affine transformations $A : \mathcal{N} \to \mathcal{N}$ that are 'tangent' to S, that is, satisfy $A(x) \in T_x(S)$ for all $x \in S$. Then $\mathfrak{aff}(S)$ is a Lie algebra with respect to $[A, B] = A' \circ B - B' \circ A$, where the derivative $A' = A - A(0)$ is the linear part of A. A subalgebra is $\mathfrak{gl}(S) := \mathfrak{gl}(\mathcal{N}) \cap \mathfrak{aff}(S)$.

4.12 Proposition. *For every $D \in \mathrm{End}(\mathcal{N})$ the following conditions are equivalent.*

(i) $D \in \mathfrak{der}(\mathcal{N})$.

(ii) $D - v \in \mathfrak{aff}(S)$ *for some* $v \in \mathcal{N}$.

Furthermore, the vector $v = v_{D,\pi}$ *in* (ii) *is uniquely determined by* D *and coincides with the unique element* v *in* $T_0 S = \ker \pi$ *satisfying* $[\pi, D] = \pi \circ M_v$. *Also* $D \mapsto D - v_{D,\pi}$ *induces a Lie algebra isomorphism* $\mathfrak{der}(\mathcal{N}) \overset{\approx}{\to} \mathfrak{aff}(S)$.

Proof. Assume (i). Then $D(\mathcal{N}_{[1]}) \subset \mathcal{N}_{[1]}$ for the annihilator $\mathcal{N}_{[1]}$ and hence $\pi + [\pi, D] \in \Pi(\mathcal{N})$. By Lemma 4.1 there exists $c \in \mathcal{N}$ with $\pi + [\pi, D] = \pi \circ M_{\exp c}$, that is, $[\pi, D] = \pi \circ M_v$ for $v = \exp_1(c)$. It is no restriction to assume $v \in \ker(\pi) = T_0 S$. Consider the affine transformation $A := D - v$ on $\mathcal{N}$. Then $(\exp x) D(x) = D(\exp_1 x)$ and $\pi(v \exp x) = \pi(v \exp_1 x)$ imply

$$\left(\pi \circ M_{\exp x}\right) A(x) = \pi \circ (D - M_v)(\exp_1 x) = D \circ \pi(\exp_1 x) = 0$$

for all $x \in S$. This means $A(x) \in T_x S$ and (ii) is proved.

Assume conversely (ii). We have to show $D(c^2) = 2cD(c)$ for all $c \in \mathcal{N}$. In case $\mathbb{F} = \mathbb{C}$ this follows with the affine vector field $\xi := (D(x) - v)\, {}^{\partial}\!/_{\partial x}$ on $\mathcal{N}$ and applying Theorem 4.2 to the 1-parameter subgroup $\exp(t\xi)$ of $\mathrm{Aff}(S)$. The case of general base field can be reduced to $\mathbb{C}$ by a Lefschetz type argument similar to the one used in the proof of 4.2, we omit the details. Also the remaining claims follow as in the proof of 4.2. $\square$

5. Nil-polynomials

For every admissible algebra $\mathcal{N}$ with annihilator $\mathcal{N}_{[1]}$ we call a linear form $\omega : \mathcal{N} \to \mathbb{F}$ a *pointing on* $\mathcal{N}$ if $\omega(\mathcal{N}_{[1]}) = \mathbb{F}$ (in analogy to function spaces where points in the underlying geometric space induce linear forms with certain properties). Also, $\mathcal{N}$ with a fixed pointing is called a *pointed algebra*. For every vector space W of finite dimension we denote by $\mathbb{F}[W]$ the algebra of all ($\mathbb{F}$-valued) polynomials on W. Since in characteristic zero every field is infinite, we do not distinguish between polynomials in $\mathbb{F}[W]$ and the polynomial functions $W \to \mathbb{F}$ they induce.

5.1 Definition. $p \in \mathbb{F}[W]$ is called a *nil-polynomial* associated to the admissible algebra $\mathcal{N}$ if there exists a pointing ω on $\mathcal{N}$ and a linear isomorphism $\varphi : W \to \ker(\omega) \subset \mathcal{N}$ such that $p = \omega \circ \exp_2 \circ \varphi$.

Notice that we do not exclude the trivial case $W = 0$ with nil-polynomial $p = 0$. Let us agree that this p has degree $-\infty$.

To every pointing ω on the admissible algebra $\mathcal{N}$ there exists a unique admissible projection π on $\mathcal{N}$ and a unique linear isomorphism $\psi : \mathcal{N}_{[1]} \to \mathbb{F}$ with $\omega(x) = \psi(\pi x)$ for all $x \in \mathcal{N}$, and conversely, every pointing on $\mathcal{N}$ is obtained this way. To π we have associated the hypersurface $S_\pi \subset \mathcal{N}$, compare (2.4). It is easy to see that for the nil-polynomial p occurring in 5.1 the hypersurface S_π is linearly equivalent to the graph

$$\Gamma_p := \{(x, t) \in W \oplus \mathbb{F} : t = p(x)\}\,.$$

5.2 Definition. We say that an admissible algebra $\mathcal{N}$ has **Property (AH)** if for some (and hence every) nil-polynomial p associated to $\mathcal{N}$ the graph Γ_p is affinely homogeneous, or equivalently, if one of the equivalent conditions (i) - (iv) in Corollary 4.10 is satisfied.

5.3 Definition. Two nil-polynomials $p \in \mathbb{F}[W]$, $\widetilde{p} \in \mathbb{F}[\widetilde{W}]$ are called *linearly (affinely) equivalent* if there exists a linear (affine) isomorphism $g : W \oplus \mathbb{F} \to \widetilde{W} \oplus \mathbb{F}$ mapping Γ_p onto $\Gamma_{\widetilde{p}}$.

5.4 Proposition. *The nil-polynomials* $p \in \mathbb{F}[W]$, $\widetilde{p} \in \mathbb{F}[\widetilde{W}]$ *are linearly equivalent if and only if there exists a linear isomorphism* $\alpha : W \to \widetilde{W}$ *and an* $\varepsilon \in \mathrm{GL}(\mathbb{F}) \cong \mathbb{F}^*$ *with* $\widetilde{p} = \varepsilon \circ p \circ \alpha^{-1}$.

Proof. Assume that p, $\widetilde{p}$ are linearly equivalent. Then there exist $\alpha \in \mathrm{Hom}(W, \widetilde{W})$, $\beta \in \mathrm{Hom}(\mathbb{F}, \widetilde{W})$ as well as $\gamma \in \mathrm{Hom}(W, \mathbb{F})$, $\delta \in \mathbb{F}$ such that $(x, t) \mapsto (\alpha x + \beta t, \gamma x + \delta t)$

establishes a linear equivalence $\Gamma_p \to \Gamma_{\widetilde{p}}$. The ideal in $\mathbb{F}(W \oplus \mathbb{F})$ of all polynomials vanishing on Γ_p is generated by $t - p(x)$. As a consequence we have for a suitable $\varepsilon \in \mathbb{F}^*$

$$(\gamma x + \delta t) - \widetilde{p}(\alpha x + \beta t) = \varepsilon(t - p(x)) \quad \text{for all} \quad (x,t) \in W \oplus \mathbb{F}\,.$$

Then $\gamma = 0$ and hence α is invertible. Denote by $q, \widetilde{q}$ the quadratic parts of $p, \widetilde{p}$. Then $\widetilde{q}(\alpha x + \beta t) = \varepsilon q(x)$ for all x, t implies $\beta = 0$ since the quadratic form $\widetilde{q}$ is non-degenerate on $\widetilde{W}$. Then $\widetilde{p}(\alpha x) = \varepsilon p(x)$ proves the first claim. The converse is obvious. □

As a consequence, every equivalence class of nil-polynomials in $\mathbb{F}[W]$ is an orbit of the group $\mathbb{F}^* \times \mathrm{SL}(W)$ acting in the obvious way on $\mathbb{F}[W]$.

Corollaries 4.3 and 4.10 immediately imply the following result.

5.5 Proposition. *Let $p, \widetilde{p}$ be nil-polynomials associated to the admissible algebras $\mathcal{N}, \widetilde{\mathcal{N}}$. Then*

(i) *$\mathcal{N}, \widetilde{\mathcal{N}}$ are isomorphic if and only if $p, \widetilde{p}$ are affinely equivalent.*

(ii) *In case $\mathcal{N}$ has* property (AH) *(for instance, if $\mathcal{N}$ admits a grading) then* (i) *remains true with 'affinely' replaced by 'linearly'.*

For any pair of admissible algebras $\mathcal{N}, \widetilde{\mathcal{N}}$ with nil-polynomials $p \in \mathbb{F}[W]$, $\widetilde{p} \in \mathbb{F}[\widetilde{W}]$ Proposition 5.5 has the following obvious consequence.

5.6 Corollary. *If $\mathcal{N}$ has* property (AH) *and $\mathcal{N}, \widetilde{\mathcal{N}}$ are isomorphic, then there exists an $\varepsilon \in \mathbb{F}^*$ and a linear isomorphism $\alpha : W \to \widetilde{W}$ with $\widetilde{p}^{[k]} \circ \alpha = \varepsilon\, p^{[k]}$ for all k, where $p^{[k]}$ is the homogeneous part of degree k in p.*

For $2 \le k \le \nu(\mathcal{N})$ the homogeneous polynomial $p^{[k]}$ is non-zero and gives a projective subvariety in the projective space $\mathbb{P}(\mathcal{N})$ associated to $\mathcal{N}$. All these varieties then are invariants for the algebra structure of $\mathcal{N}$, provided $\mathcal{N}$ has property (AH). It is worthwhile to mention that this remains true for the leading homogeneous part also in the general situation, more precisely:

5.7 Proposition. *The statement of* Corollary 5.6 *remains true for $k = \nu(\mathcal{N})$ even without requiring that $\mathcal{N}$ has* property (AH).

Proof. Assume that $L : \mathcal{N} \to \widetilde{\mathcal{N}}$ is an algebra isomorphism. Then $\mathcal{N}$, $\widetilde{\mathcal{N}}$ have the same nil-index, say $\nu \ge 2$. Without loss of generality we may assume that there are pointings ω, $\widetilde{\omega}$ on $\mathcal{N}, \widetilde{\mathcal{N}}$ with $W = \ker(\omega)$, $\widetilde{W} = \ker(\widetilde{\omega})$ and $p = \omega \circ \exp_2$, $\widetilde{p} = \widetilde{\omega} \circ \exp_2$ on $W, \widetilde{W}$. Because of $L(\mathcal{N}_{[1]}) = \widetilde{\mathcal{N}}_{[1]}$ there is an $\varepsilon \in \mathbb{F}^*$ with $\widetilde{\omega}(L(a)) = \varepsilon\omega(a)$ for every $a \in \mathcal{N}_{[1]}$. Further, there exists a linear isomorphism $\alpha : W \to \widetilde{W}$ and a linear map $\lambda : W \to \widetilde{\mathcal{N}}_{[1]}$ with $L(x) = \alpha(x) + \lambda(x)$ for all $x \in W$. But then $\widetilde{p}^{[\nu]}(\alpha x) = \widetilde{\omega}((\alpha x)^{(\nu)}) = \widetilde{\omega}((Lx)^{(\nu)}) = \widetilde{\omega}(L(x^{(\nu)})) = \varepsilon\omega(x^{(\nu)}) = \varepsilon p^{[\nu]}(x)$ since $x^{(\nu)} \in \mathcal{N}_{[1]}$. □

We illustrate by an example how 5.7 can be applied to prove that two given admissible algebras are not isomorphic: Anticipating notation of Section 8, see 8.2, consider $\mathcal{M}(Z^3 + Y^4 + X^3Z + X^3Y^2 + X^5YZ)$ and $\mathcal{M}(Z^3 + Y^4 + X^3Z + X^2YZ + X^3Y^2)$. These are admissible algebras of dimension 20 with nil-index 6, both having the same Hilbert function $\{1,3,5,5,5,4,2,1\}$. It can be seen[2] that both algebras do not have property (AH), compare also with Section 8. Leading terms of nil-polynomials $p, \widetilde{p} \in \mathbb{F}[x_1, \ldots, x_{20}]$ are, for instance, $p^{[6]} = x_1^4 x_2^2$ (for the first algebra) and $\widetilde{p}^{[6]} = x_1^4(19x_1^2 - 90x_1x_2 + 135x_2^2)$ (for the second). Since the quadratic factor in $\widetilde{p}^{[6]}$ is not the square of a linear form we conclude with Proposition 5.7 that the algebras are not isomorphic.

Remark. There is a geometric interpretation of the leading homogeneous term $p^{[\nu]}$: Identify W in the standard way with an affine open subset in the projective space $\mathbb{P}(\mathbb{F} \oplus W)$. Hence $\mathbb{P}(\mathbb{F} \oplus W) = W \;\dot\cup\; \mathbb{P}(W)$, where $\mathbb{P}(W)$ is the projective hyperplane at infinity. The zero set $T := \{p = 0\} \subset W$ is linearly equivalent to $\{f = 0\} \cap \varphi(W)$ where $\varphi : W \to \ker(\omega) \subset \mathcal{N}$ is the linear isomorphism from definition 5.1. Consider the Zariski closure $C\ell(T) \subset \mathbb{P}(\mathbb{F} \oplus W)$. Then the set of points at infinity, $T^\infty := C\ell(T) \cap \mathbb{P}(W)$, coincides with $\{[z] \in \mathbb{P}(W) : p^{[\nu]}(z) = 0\}$. For a not algebraically closed field $\mathbb{F}$ then T^∞

encodes in general less information then the homogeneous part $p^{[\nu]}$. Indeed, for instance in case $\mathbb{F} = \mathbb{R}$ the quadratic factor in $\widetilde{p}^{[6]}$ above is positive definite on $\mathbb{R}^2$, that is, the zero locus of $\widetilde{p}^{[6]}$ in the real projective space $\mathbb{P}_{19}(\mathbb{R})$ is the hyperplane $\{x_1 = 0\}$. This suggests to consider projective varieties defined by the $p^{[\nu]}$ (or $p^{[k]}$, $k < \nu$) only in case of algebraically closed base fields. In such a situation the corresponding divisor rather then the mere zero set is an invariant equivalent to $p^{[\nu]}$.

Every nil-polynomial p associated to $\mathcal{N}$ depends on $\dim(\mathcal{N}) - 1$ variables. Another type of polynomial, closer to (2.3), can be defined as follows:

5.8 Definition. The polynomial $f \in \mathbb{F}[V]$ is called an *extended nil-polynomial* associated to $\mathcal{N}$, if there exists a linear isomorphism $\varphi : V \to \mathcal{N}$ and a pointing ω on $\mathcal{N}$ such that $f = \omega \circ \exp_1 \circ\, \varphi$.

It is clear that for the linear part $f^{[1]} \in \mathbb{F}[V]$ of f the restriction of f to $W := (f^{[1]})^{-1}(0)$ is a nil-polynomial associated to $\mathcal{N}$ and that the graph $\Gamma_p \subset W \oplus \mathbb{F}$ is linearly equivalent to the hypersurface $f^{-1}(0) \subset V$. Conversely, every nil-polynomial $p \in \mathbb{F}[W]$ associated to $\mathcal{N}$ can be extended by $f(x,t) := p(x) + t$ to an extended nil-polynomial $f \in \mathbb{F}(W \oplus \mathbb{F})$.

For a given pointed algebra $(\mathcal{N}, \omega)$ fix a nil-polynomial $p = \omega \circ \exp_2 \circ\, \varphi \in \mathbb{F}[W]$ in the following and define the symmetric k-form ω_k on W by

$$\omega_k(x_1, x_2, \ldots, x_k) = \omega\big((\varphi x_1)(\varphi x_2)\cdots(\varphi x_k)\big)\,. \tag{5.9}$$

Then we have the expansion $p = \sum_{k \geq 2} p^{[k]}$ into homogeneous parts, where

$$p^{[k]}(x) = \frac{1}{k!}\,\omega_k(x, \ldots, x) \tag{5.10}$$

and ω_2 is non-degenerate on W. Using $p^{[2]}$ and $p^{[3]}$ we define a commutative (a priori not necessarily associative) product $(x, y) \mapsto x{\cdot}y$ on W by

$$\omega_2(x{\cdot}y, z) = \omega_3(x, y, z) \;\text{ for all }\; z \in W \tag{5.11}$$

and also a commutative product on $W \oplus \mathbb{F}$ by

$$(x, s)(y, t) := (x{\cdot}y, \omega_2(x, y))\,. \tag{5.12}$$

For $\mathcal{K} := \ker(\omega)$ there is a unique linear isomorphism $\psi : \mathbb{F} \to \mathcal{N}_{[1]}$ such that $\pi = \psi \circ \omega$ is the canonical projection $\mathcal{K} \oplus \mathcal{N}_{[1]} \to \mathcal{N}_{[1]}$. With these ingredients we have

5.13 Proposition. *With respect to the product* (5.12) *the linear map*

$$W \oplus \mathbb{F} \;\to\; \mathcal{N}\,, \quad (x, s) \mapsto \varphi(x) + \psi(s)\,,$$

is an isomorphism of algebras. In particular, W with product $x{\cdot}y$ is isomorphic to the nilpotent algebra $\mathcal{N}/\mathcal{N}_{[1]}$ and has nil-index $\nu(\mathcal{N}){-}1$.

Proof. For all $x, y \in W$ we have

$$(\varphi(x) + \psi(s))(\varphi(y) + \psi(t)) = (N - A) + A \text{ with}$$
$$N := \varphi(x)\varphi(y) \in \mathcal{N} \;\text{ and }\; A := \pi(\varphi(x)\varphi(y)) = \psi(\omega_2(x, y)) \in \mathcal{N}_{[1]}.$$

It remains to show $N - A = \varphi(x{\cdot}y)$. But this follows from

$$N - A \in \mathcal{K} \quad\text{and}\quad \omega\big(\varphi(x{\cdot}y)\varphi(z)\big) = \omega\big(\varphi(x)\varphi(y)\varphi(z)\big) = \omega\big((N - A)\varphi(z)\big)$$

for all $z \in W$. □

5.14 Corollary. *Every nil-polynomial p on W is uniquely determined by its quadratic and cubic term, $p^{[2]}$ and $p^{[3]}$. In fact, the other $p^{[k]}$ are recursively determined by* (5.10) *and*

$$\omega_{k+1}(x_0, x_1, \dots, x_k) = \omega_k(x_0{\cdot}x_1, x_2, \dots, x_k) \tag{5.15}$$

for all $k \geq 2$ and $x_0, x_1, \dots, x_k \in W$.

Corollary 5.14 suggests the following question: Given a non-degenerate quadratic form q and a cubic form c on W. When does there exist a nil-polynomial $p \in \mathbb{F}[W]$ with $p^{[2]} = q$ and $p^{[3]} = c$? Using q, c we can define as above for $k = 2, 3$ the symmetric k-linear form ω_k on W and with it the commutative product $x{\cdot}y$ on W. A necessary and sufficient condition for a positive answer is that W with this product is a nilpotent and associative algebra. As a consequence we get for every fixed non-degenerate quadratic form q on W the following structural information on the space of all nil-polynomials p on W with $p^{[2]} = q$: Denote by C the set of all cubic forms on W. Then C is a linear space of dimension $\binom{n+2}{3}$, $n = \dim W$, and

$$C_q := \{c \in C : \exists \text{ nil-polynomial } p \text{ on } W \text{ with } p^{[2]} = q, p^{[3]} = c\} \tag{5.16}$$

is an algebraic subset.

6. Representations of nil-algebras and adapted decompositions

In the following let A be an arbitrary commutative nilpotent algebra and $\mathbb{E}$ a vector space. Also let

$$N : A \to \operatorname{End}(\mathbb{E}) \ , \qquad x \mapsto N_x \ ,$$

be an algebra homomorphism. For example, every commutative algebra A admits the faithful left-regular representation $L : A \to \operatorname{End}(A^0)$, where A^0 is the unital extension of A. Consider the following characteristic subspaces of $\mathbb{E}$:

$$\mathbb{B} := \langle N_x(\mathbb{E}) : x \in A \rangle_{\mathbb{F}} \quad \text{and} \quad \mathbb{K} := \bigcap_{x \in A} \ker(N_x) \ .$$

Let us call every decomposition

$$\mathbb{E} = \mathbb{E}_0 \oplus \mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3 \quad \text{with} \quad \mathbb{B} = \mathbb{E}_2 \oplus \mathbb{E}_3 \ , \ \mathbb{K} = \mathbb{E}_0 \oplus \mathbb{E}_3 \tag{6.1}$$

an *N-adapted decomposition* of $\mathbb{E}$. It is obvious that starting with $\mathbb{E}_3 := \mathbb{B} \cap \mathbb{K}$ and choosing successively suitable linear complements $\mathbb{E}_2$, $\mathbb{E}_0$ and $\mathbb{E}_1$ one always obtains an N-adapted decompositions for $\mathbb{E}$. Clearly, $\mathbb{E}_0 = 0$ if the image algebra $N(A)$ is maximal among all commutative nilpotent subalgebras of $\operatorname{End}(\mathbb{E})$.

Now assume that $\mathbb{E}$ has finite dimension and that a non-degenerate symmetric bilinear form $h : \mathbb{E} \times \mathbb{E} \to \mathbb{F}$ is fixed such that every N_x is selfadjoint with respect to h: For every $T \in \operatorname{End}(\mathbb{E})$ the adjoint $T^\star$ is defined by $h(Tv, w) = h(v, T^*w)$ for all $v, w \in \mathbb{E}$. The orthogonal 'complement' of every linear subspace $\mathbb{L} \subset \mathbb{E}$ is $\mathbb{L}^\perp := \{v \in \mathbb{E} : h(v, \mathbb{L}) = 0\}$. The linear subspace $\mathbb{L}$ is called *totally isotropic* if $\mathbb{L} \subset \mathbb{L}^\perp$.

6.2 Proposition. *There exists an N-adapted decomposition* (6.1) *that is related to h in the following way:*

(i) *The three subspaces $\mathbb{E}_0$, $\mathbb{E}_1 \oplus \mathbb{E}_3$ and $\mathbb{E}_2$ are mutually orthogonal with respect to h.*
(ii) *The subspaces $\mathbb{E}_1$ and $\mathbb{E}_3$ are totally isotropic, and hence have the same dimension.*

Proof. Choose an arbitrary N-adapted decomposition $\mathbb{E} = \mathbb{E}_0 \oplus \widetilde{\mathbb{E}}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3$. Since every N_x is selfadjoint we have $\mathbb{K} = \mathbb{B}^\perp$. In particular, $\mathbb{E}_3$ is totally isotropic. We get further $\dim(\mathbb{E}) = \dim(\mathbb{K}) + \dim(\mathbb{B})$, $\dim(\widetilde{\mathbb{E}}_1) = \dim(\mathbb{E}_3)$, $\mathbb{E}_0 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3 \ = \ \mathbb{E}_3^\perp$ and that the three spaces $\mathbb{E}_0$, $\mathbb{E}_2$, $\mathbb{E}_3$ are mutually orthogonal.

Further we conclude from $\mathbb{E}_2 \cap \mathbb{E}_2^\perp \subset \mathbb{B}^\perp = \mathbb{K}$ that $\mathbb{E}_2 \cap \mathbb{E}_2^\perp \subset \mathbb{E}_2 \cap \mathbb{K} = 0$. In the same way we conclude from $\mathbb{E}_0 \cap \mathbb{E}_0^\perp \subset \mathbb{K}^\perp = \mathbb{B}$ that $\mathbb{E}_0 \cap \mathbb{E}_0^\perp \subset \mathbb{E}_0 \cap \mathbb{B} = 0$. As a consequence we get $(\mathbb{E}_0^\perp \cap \mathbb{E}_2^\perp) \cap \mathbb{E}_3^\perp = \mathbb{E}_3$ and $\mathbb{E}_0^\perp + \mathbb{E}_2^\perp = \mathbb{E}$. Now choose a linear subspace $\mathbb{E}_1 \subset (\mathbb{E}_0^\perp \cap \mathbb{E}_2^\perp)$ with $\mathbb{E}_1 \oplus \mathbb{E}_3 = \mathbb{E}_0^\perp \cap \mathbb{E}_2^\perp$. Counting dimensions we get $\dim(\mathbb{E}_1) = \dim(\widetilde{\mathbb{E}}_1)$ from $\dim(\mathbb{E}_0^\perp \cap \mathbb{E}_2^\perp) = \dim(\mathbb{E}_0^\perp) + \dim(\mathbb{E}_2^\perp) - \dim(\mathbb{E}_0^\perp + \mathbb{E}_2^\perp)$ and thus that $\mathbb{E} = \mathbb{E}_0 \oplus \mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3$ is an N-adapted decomposition satisfying (i). The form h is non-degenerate on $\mathbb{E}_1 \oplus \mathbb{E}_3$. Because of $\dim(\mathbb{E}_1) = \dim(\mathbb{E}_3)$ we finally may assume without loss of generality that also $\mathbb{E}_1$ is totally isotropic. □

We call every N-adapted decomposition satisfying (i), (ii) above an *(N, h)-adapted decomposition* of the representation space $\mathbb{E}$. In the following we give two applications:

Let $\mathcal{N}$ be an admissible algebra with pointing ω. Clearly, in general the quotient $B := \mathcal{N}/\mathcal{N}_{[1]}$ is a non-admissible nilpotent algebra, say with product $(x, y) \mapsto x \bullet y$. Left multiplication yields a (non-faithful) representation $N : B \to \mathrm{End}(B)$ in terms of the multiplication operator $N_x : y \mapsto x \bullet y$. Further, the symmetric bilinear form $b(x, y) = \omega(xy)$ on $\mathcal{N}$ factors to a non-degenerate symmetric bilinear form h on $\mathcal{N}/\mathcal{N}_{[1]}$ and all N_x are selfadjoint with respect to h. Note that $\mathcal{N}$ is isomorphic to $(\mathcal{N}/\mathcal{N}_{[1]}) \times \mathbb{F}$ with multiplication given by

$$(6.3) \qquad (x, a) \diamond (y, b) := (N_x(y), h(x, y)) = (N_y(x), h(y, x))\,.$$

Instead of $\mathcal{N}/\mathcal{N}_{[1]}$ we use the isomorphic algebra $W := \ker(\omega)$ with product $x{\cdot}y$, as given in Proposition 5.13. Then the form h is the restriction of b to W. If $\mathcal{N}$ has nil-index $\nu \geq 2$ then the subalgebra $N(W) \subset \mathrm{End}(W)$ has nil-index $\nu{-}2$. If $W = W_0 \oplus W_1 \oplus W_2 \oplus W_3$ is a (W, h)-adapted decomposition, then $\mathcal{N}' := W_1 \oplus W_2 \oplus W_3 \oplus \mathcal{N}_{[1]}$ and $\mathcal{N}'' := W_0 \oplus \mathcal{N}_{[1]}$ are admissible subalgebras with $\nu(\mathcal{N}'') \leq 2$, and $\mathcal{N}$ is a smash product of $\mathcal{N}'$ with $\mathcal{N}''$, as defined in Section 7.

6.4 Proposition. *Every admissible algebra of nil-index ≤ 3 has a grading.*

Proof. As indicated above $\mathcal{N}$ is isomorphic to $W \times \mathbb{F}$ with product 6.3 where $W = \ker(\omega) \subset \mathcal{N}$ is the nilpotent subalgebra isomorphic to $\mathcal{N}/\mathcal{N}_{[1]}$. Let $N : W \to \mathrm{End}(W)$ as above and consider the subalgebra $N(W) \subset \mathrm{End}(W)$. Let $W = W_0 \oplus W_1 \oplus W_2 \oplus W_3$ be a (W, h)-adapted decomposition. Since $N(W)$ has nil-index ≤ 1 we have $W_2 = 0$. Put $\mathcal{N}_2 := W_1$, $\mathcal{N}_4 := W_3$, $\mathcal{N}_6 := \mathrm{Ann}(\mathcal{N})$ and $\mathcal{N}_3 := W_0$. Then $\mathcal{N} = \mathcal{N}_2 \oplus \mathcal{N}_3 \oplus \mathcal{N}_4 \oplus \mathcal{N}_6$ is a grading of $\mathcal{N}$. □

The estimate $\nu(\mathcal{N}) \leq 3$ in Proposition 6.4 is sharp as a counterexample in Section 8 with nil-index 4 and dimension 8 will show.

The next result improves Proposition 3.6 in the case of admissible algebras.

6.5 Proposition. *For every admissible algebra $\mathcal{N}$ and every $\pi \in \Pi(\mathcal{N})$ there exists a subgroup of $\mathrm{Aff}(S_\pi)$ acting transitively on $S_\pi \cap \mathcal{N}_{[4]}$. In particular, $\mathcal{N}$ has* Property (AH) *if $\mathcal{N}$ has nil-index ≤ 4.*

Proof. Put $S := S_\pi$ as shorthand and denote by h the restriction of b_π to $W := \ker(\pi)$. As above, $N_x \in \mathrm{End}(W)$ is the multiplication operator $y \mapsto x{\cdot}y$. Choose a (W, h)-adapted decomposition $W = W_0 \oplus W_1 \oplus W_2 \oplus W_3$ and denote by $\pi_k \in \mathrm{End}(W)$ the canonical projection with range W_k for $0 \leq k \leq 3$. Then $(W_0 + W_3){\cdot}W = 0$, $W{\cdot}W \subset W_2 \oplus W_3$ and $W'{\cdot}W_2 \subset W_3$, where $W' := \mathcal{N}_{[4]} \cap W$.

Now fix a point $a \in S \cap W'$. Because of Proposition 3.6 it is enough to show $g(a) \in \mathcal{N}_{[3]}$ for some $g \in \mathrm{Aff}(S) \cap \mathrm{Aff}(\mathcal{N}_{[4]})$: Put $P := \pi_3 \circ N_c \circ \pi_2 - \pi_2 \circ N_c \circ \pi_1$ for $c := a - \pi(a) \in W'$. Then $P^* = -P$ and $Q := \frac{1}{2} N_c + \frac{1}{6} P \in \mathrm{End}(W)$ is nilpotent with $Q(W) \subset \mathcal{N}_{[3]}$. Set $\mathcal{N}^0 := \mathbb{F}\mathbb{1} \oplus W \oplus \mathcal{N}_{[1]}$ and define $\lambda \in \mathrm{End}(\mathcal{N}^0)$ by

$$(s\mathbb{1}, x, t) \mapsto (0, Qx - sc, h(c, x)) \ \text{ for all } \ s \in \mathbb{F}, x \in W, t \in \mathcal{N}_{[1]}\,.$$

λ is nilpotent and maps $\mathcal{N}^0$ to $\mathcal{N}_{[4]}$. Therefore the unipotent operator $g := \exp(\lambda) \in \mathrm{GL}(\mathcal{N}^0)$ exists. Clearly, $\mathcal{U} := \mathbb{1} + \mathcal{N}$ and $\mathbb{1} + \mathcal{N}_{[4]}$ are g-invariant.

We proceed as in the proof of Theorem 3.2. Denote by $\mathcal{F}$ the space of all polynomial maps $\mathcal{N}^0 \to \mathcal{N}_{[1]}$ of degree ≤ 4 and define the nilpotent operator $\xi \in \mathrm{End}(\mathcal{F})$ by

$$(\xi f)(z) := f'(z)(\lambda z) \quad \text{for all} \quad f \in \mathcal{F} \;\text{ and }\; z \in \mathcal{N}^0 \,.$$

We claim that g leaves $\mathbb{1}+S$ invariant. For this we only have to show that $\xi\widehat{f}$ vanishes on $\mathbb{1}+S$, where $\widehat{f} \in \mathcal{F}$ is defined by $\widehat{f}(s\mathbb{1},x,t) := t + \pi(\exp_2 x)$. But this just means that

$$h(c - Qx, x + \tfrac{1}{2}N_x x + \tfrac{1}{6}N_x^2 x) = h(c,x)$$

holds for all $x \in W$, or equivalently

$$h(2Qx - N_c x, x) \;=\; h(3Qx - N_c x, N_x x) \;=\; h(Qx, N_x^2 x) \;= 0 \;.$$

The first term vanishes since $2Q - N_c$ is skew adjoint. The two other terms vanish since $3Qx - N_c x \in \mathcal{N}_{[2]}$ and $Qx \in \mathcal{N}_{[3]}$. This proves the claim, and as a consequence we get $g(\mathbb{1}+a) = \mathbb{1}+b$ with $b \in (c{\cdot}W + W_3 + \mathcal{N}_{[1]}) \subset \mathcal{N}_{[3]}$. □

We conclude the section with a lower bound for the dimension of $\mathfrak{Der}(\mathcal{N})$ and $\mathrm{Aut}(\mathcal{N})$. For fixed $\pi \in \Pi(\mathcal{N})$, by the proof of Lemma 4.1 every $\lambda \in \mathrm{Hom}(\mathcal{N},\mathcal{N}_{[1]})$ with $\lambda(\mathcal{N}_{[1]}) = 0$ is of the form $\lambda = \pi \circ M_b$, $b \in \ker \pi$. Therefore $c \mapsto \pi \circ M_c$ induces for every $k > 0$ a linear isomorphism $\mathcal{N}_{[k]}/\mathcal{N}_{[1]} \;\cong\; \mathrm{Hom}(\mathcal{N}/\mathcal{N}^k, \mathcal{N}_{[1]})$. This implies

$$\dim \mathcal{N}_{[k]} + \dim \mathcal{N}^k = \dim \mathcal{N} + 1 \quad \text{for all} \quad 0 \leq k \leq \nu(\mathcal{N})\,.$$

As a consequence we get from Proposition 6.5 the

6.6 Corollary. $\dim \mathrm{Aut}(\mathcal{N}) \;\geq\; \dim \mathcal{N}/\mathcal{N}^4$ *for every admissible algebra* $\mathcal{N}$. *The same lower bound holds for the dimension of* $\mathfrak{Der}(\mathcal{N})$.

Proof. In case $\nu(\mathcal{N}) < 4$ the algebra $\mathcal{N}$ is gradable by Proposition 6.4 proving the claim. In case $\nu(\mathcal{N}) \geq 4$ Proposition 6.5 together with $\dim(\mathcal{N}_{[4]}/\mathcal{N}_{[1]}) = \dim(\mathcal{N}/\mathcal{N}^4)$ implies

$$\dim \mathrm{Aff}(S_\pi) \;\geq\; \dim \mathcal{N}/\mathcal{N}^4 \;+\; \dim \mathrm{GL}(S_\pi)\,, \quad \text{that is}$$

$$\dim \mathrm{Aut}(\mathcal{N}) \;\geq\; \dim \mathcal{N}/\mathcal{N}^4 + \dim\{g \in \mathrm{Aut}(\mathcal{N}) : g \circ \pi = \pi \circ g\}$$

as a consequence of Proposition 4.9. The estimate for $\mathfrak{Der}(\mathcal{N})$ follows with 4.12. □

7. Some examples

For every admissible algebra $\mathcal{N}$ and associated nil-polynomial p the degree of p is the nil-index of $\mathcal{N}$, provided $\dim \mathcal{N} > 1$. In the following we give a method how large classes of nil-polynomials of degrees 3 and 4 can be constructed.

Nil-polynomials of degree 2 on a given vector space $W \neq 0$ are quite obvious – these are precisely all non-degenerate quadratic forms q on W. Then q is associated to the admissible algebra $\mathcal{N} := W \oplus \mathbb{F}$ whose commutative product is uniquely determined by $(x,t)^2 = (0, q(x))$ for all $x \in W$ and $t \in \mathbb{F}$ (special case of 6.3). Clearly, $\mathcal{N}$ is canonically graded by $\mathcal{N}_1 = W$ and $\mathcal{N}_2 = \mathbb{F}$. Notice that in every dimension $n \geq 2$ the number of isomorphy classes for admissible algebras with nil-index 2 is: 1 if $\mathbb{F}$ is algebraically closed; $[n/2]$ if $\mathbb{F}$ is the real field; infinite if $\mathbb{F}$ is the rational field.

For every pair of nil-polynomials $p' \in \mathbb{F}[W']$, $p'' \in \mathbb{F}[W'']$ we get a new nil-polynomial $p := p' \oplus p'' \in \mathbb{F}[W' \oplus W'']$ by setting $p(x',x'') := p'(x') + p''(x'')$ for all $x' \in W'$ and $x'' \in W''$. Let us call a nil-polynomial *reduced* if it is not affinely equivalent to a direct sum $p' \oplus p''$ of nil-polynomials with p'' of degree two. Then it is clear that every nil-polynomial

p is affinely equivalent to a direct sum $p' \oplus p''$ with p' reduced and p'' of degree ≤ 2. The nil-polynomials p', p'' are uniquely determined up to affine equivalence.

On the level of algebras the direct sum of nil-polynomials corresponds to the following construction: Let $(\mathcal{N}', \omega')$, $(\mathcal{N}'', \omega'')$ be pointed algebras. Then $\mathcal{I} := \{(s,t) \in \mathcal{N}'_{[1]} \times \mathcal{N}''_{[1]} : \omega'(s) + \omega''(t) = 0\}$ is an ideal in $\mathcal{N}' \times \mathcal{N}''$ and $\mathcal{N} := (\mathcal{N}' \times \mathcal{N}'')/\mathcal{I}$ becomes a pointed algebra with respect to the pointing $(s,t) + \mathcal{I} \ \mapsto \ \omega'(s) + \omega''(t)$. We write $\mathcal{N} = \mathcal{N}' \vee \mathcal{N}''$ and call it the *smash product* of the pointed algebras $\mathcal{N}_j$. This product is commutative and associative in the sense that $(\mathcal{N}_1 \vee \mathcal{N}_2) \vee \mathcal{N}_3$ and $\mathcal{N}_1 \vee (\mathcal{N}_2 \vee \mathcal{N}_3)$ are canonically isomorphic. We call the admissible algebra $\mathcal{N}$ *reduced* if it is not isomorphic to a smash product $\mathcal{N}_1 \vee \mathcal{N}_2$ with $\nu(\mathcal{N}_2) = 2$. Notice that every admissible algebra $\mathcal{N}'$ of dimension one is neutral with respect to the smash product, that is, $\mathcal{N} \vee \mathcal{N}' \cong \mathcal{N}$. Notice also that any smash product of gradable admissible algebras is also gradable.

Nil-polynomials of degree 3

Let $p \in \mathbb{F}[V]$ be a homogeneous polynomial of degree $d \geq 2$, that is, $p(x) = q(x, \ldots, x)$ for all $x \in V$ and a uniquely determined symmetric d-linear form $q : V^d \to \mathbb{F}$. Then p is called *non-degenerate* if $V_p^0 = 0$ for

$$V_p^0 := \{a \in V : q(a, V, \ldots, V) = 0\} \ = \ \{a \in V : p(x+a) = p(x) \ \text{for all} \ x \in V\} \, .$$

7.1 Proposition. *Let W be an $\mathbb{F}$-vector space of finite dimension and q a non-degenerate quadratic form on W. Suppose furthermore that $W = W_1 \oplus W_2$ for totally isotropic (with respect to q) linear subspaces W_k and that c is a cubic form on W_1. Then, if we extend c to W by $c(x+y) = c(x)$ for all $x \in W_1$, $y \in W_2$, the sum $p := q + c$ is a nil-polynomial on W, and p is reduced if and only if c is non-degenerate on W_1. For all cubic forms c, $\widetilde{c}$ on W_1 with nil-polynomials $p = q + c$, $\widetilde{p} = q + \widetilde{c}$ and associated admissible algebras $\mathcal{N}, \widetilde{\mathcal{N}}$ the following conditions are equivalent:*

(i) $\widetilde{c} = c \circ g$ *for some* $g \in \mathrm{GL}(W_1)$.

(ii) $\mathcal{N}, \widetilde{\mathcal{N}}$ *are isomorphic as algebras.*

Proof. $\omega_3(x,y,t) = 0$ for all $t \in W_2$ implies $W{\cdot}W \subset W_2$ and $W{\cdot}W_2 = 0$, that is, $(x{\cdot}y){\cdot}z = 0$ for all $x,y,z \in W$, see (5.11). This implies that $p := q + c$ is a nil-polynomial. Fix a decomposition $W_1 = W_1' \oplus W_1''$ with $W_1'' := \{a \in W_1 : c(x+a) = c(x) \ \text{for all} \ x \in W_1\}$. Then also $W_1'' = \{a \in W_1 : a{\cdot}W_1 = 0\}$ and we put

$$W_2' := \{y \in W_2 : y \perp W_1''\} \quad \text{and} \quad W_2'' := \{y \in W_2 : y \perp W_1'\} \, .$$

Then for $W' := W_1' \oplus W_2'$ and $W'' := W_1'' \oplus W_2''$ we have the orthogonal decomposition $W = W' \oplus W''$ with $W{\cdot}W'' = 0$ and $W{\cdot}W \subset W_2'$. Denote by p' and p'' the restriction of p to W' and W'' respectively. Then $p = p' \oplus p''$ as direct sum of nil-polynomials and p'' has degree ≤ 2. These considerations show that p is reduced if and only if $W_1'' = 0$. It remains to verify the equivalence of (i) and (ii).

(i) $\Longrightarrow$ (ii) There exists a unique $g^\sharp \in \mathrm{GL}(W_2)$ with $\omega_2(gx, y) = \omega_2(x, g^\sharp y)$ for all $x \in W_1$ and $y \in W_2$. But then $\widetilde{p} = p \circ h$ for $h := g \times (g^\sharp)^{-1} \in \mathrm{O}(q) \subset \mathrm{GL}(W)$. By Proposition 5.5 the algebras $\mathcal{N}, \widetilde{\mathcal{N}}$ are isomorphic.

(ii) $\Longrightarrow$ (i) By Proposition 7.1 the algebra $\mathcal{N}/\mathcal{N}_{[1]}$ is isomorphic to W with the product $x{\cdot}y$ determined by $p^{[2]} = q$ and $p^{[3]} = c$. Choose, as above, decompositions $W = W' \oplus W''$, $W' = W_1' \oplus W_2'$ and $W'' = W_1'' \oplus W_2''$. Then the cubic form c 'essentially lives' on the subspace $W_1' \subset W_1$ and $W_1'' \oplus W_2$ is the annihilator of W. In the same way the product $x \tilde{\cdot} y$ determined by $\widetilde{p}^{[2]} = q$ and $\widetilde{p}^{[3]} = \widetilde{c}$ on W gives an algebra $\widetilde{W}$ isomorphic to $\widetilde{\mathcal{N}}/\widetilde{\mathcal{N}}_{[1]}$. We choose again decompositions $\widetilde{W} = \widetilde{W}' \oplus \widetilde{W}''$, $\widetilde{W}' = \widetilde{W}_1' \oplus \widetilde{W}_2'$ and $\widetilde{W}'' = \widetilde{W}_1'' \oplus \widetilde{W}_2''$ and have $\widetilde{W}_1'' \oplus \widetilde{W}_2 = \mathrm{Ann}(\widetilde{W})$.

Now assume (ii) and fix an algebra isomorphism $h : W \to \widetilde{W}$. Then there exists a linear isomorphism $\alpha : W_1' \to \widetilde{W}_1'$ with $\ h(x) \equiv \alpha(x) \bmod \mathrm{Ann}(W) \ $ for all $x \in W_1$. Let β :

$W_1'' \to \widetilde{W}_1''$ be an arbitrary linear isomorphism and consider $g := \alpha \oplus \beta$ as an element of $\mathrm{GL}(W_1)$. Replacing $\widetilde{c}$ by $\widetilde{c} \circ g^{-1}$ we may assume without loss of generality that $g = \mathrm{id}_{W_1}$. But then $x{\cdot}x = x\tilde{\cdot}x$ and thus $6\,c(x) = \omega_2(x{\cdot}x, x) = \omega_2(x\tilde{\cdot}x, x) = 6\,\widetilde{c}(x)$ for all $x \in W_1$. $\square$

Suppose that $W_1 \cong \mathbb{F}^m$ with coordinates $x = (x_1, \ldots, x_m)$ has dimension $m > 0$ in Proposition 7.1. Then $W \cong \mathbb{F}^{2m}$ with coordinates $(x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_m)$ and we may assume $q(x, y) = x_1 y_1 + \ldots + x_m y_m$. As already mentioned, the linear space C of all cubic forms on W_1 has dimension $\binom{m+2}{3}$. The group $\mathrm{GL}(W_1)$ acts on C from the right and has dimension m^2 over $\mathbb{F}$. The difference of dimensions is $\binom{m}{3}$. But this number is also the cardinality of the subset $J \subset \mathbb{N}^3$, consisting of all triples $j = (j_1, j_2, j_3)$ with $1 \le j_1 < j_2 < j_3 \le m$. Consider the affine map

$$\alpha : \mathbb{F}^J \to C\,, \quad (t_j) \mapsto c_0 + \sum_{j \in J} t_j c_j\,, \tag{7.2}$$

where $c_0 := x_1^3 + \ldots + x_m^3$ and $c_j := x_{j_1} x_{j_2} x_{j_3}$ for all $j \in J$. Notice that the nil-polynomial $q + c_0 = \bigoplus_{k=1}^{m} (x_k y_k + x_k^3)$ is a direct sum of m nil-polynomials, that is, the corresponding admissible algebra is an m-fold smash power of the cyclic algebra $\mathbb{F}t \oplus \mathbb{F}t^2 \oplus \mathbb{F}t^3$, where $t^4 = 0$.

In case $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, for a suitable neighbourhood U of $0 \in \mathbb{F}^J$ the map $\alpha : U \to C$ intersects all $\mathrm{GL}(n, \mathbb{F})$-orbits in C transversally. Indeed, since all partial derivatives of c_0 are monomials containing a square, the tangent space at c_0 of its $\mathrm{GL}(n, \mathbb{F})$-orbit is transversal to the linear subspace $\langle c_j : j \in J \rangle_{\mathbb{F}}$ of C. In particular, in case $m \ge 3$ there is a family of dimension $\binom{m}{3} \ge 1$ over $\mathbb{F}$ $(= \mathbb{R}$ or $\mathbb{C})$ of pairwise different $\mathrm{GL}(n, \mathbb{F})$-orbits and thus of non-equivalent nil-polynomials of degree 3 on W. Notice that in case $m = 3$ the mapping α in (7.2) reduces to

$$\alpha : \mathbb{F} \to C, \quad t \mapsto x_1^3 + x_2^3 + x_3^3 + t x_1 x_2 x_3\,. \tag{7.3}$$

7.4 Corollary. *Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. Then in every dimension $n \ge 7$ there is an infinite (in fact uncountable) number of isomorphy classes of admissible algebras of dimension n and nil-index 3.*

Proposition 6.2 together with Proposition 7.1 generalizes Theorems 3.3 and 4.1 in [3] from the case of algebraically closed base fields to arbitrary fields of characteristic 0, see Proposition 7.5 below. For every admissible algebra $\mathcal{N}$ with nil-index ν the non-degeneracy of b_π on $\mathcal{N}/\mathcal{N}^\nu$ implies $\dim(\mathcal{N}^k/\mathcal{N}^\nu) \le \dim(\mathcal{N}/\mathcal{N}^j)$ for all $\nu/2 < k < \nu$ and $j = \nu - k + 1$ and thus $H(\nu - 1) \le H(1)$ for the Hilbert function $H = H_{\mathcal{N}}$. This means in the special case of nil-index $\nu = 3$ that the Hilbert function of $\mathcal{N}$ has the form $\{1, m, n, 1\}$ with $m \ge n \ge 1$.

7.5 Proposition. (Classification of admissible algebras with nil-index 3) *The admissible algebras $\mathcal{N}$ of nil-index 3 are, up to isomorphism, precisely the smash products $\mathcal{N}' \vee \mathcal{N}''$, where $\mathcal{N}'$ is a reduced algebra of dimension > 1 as described in Proposition 7.1 and $\mathcal{N}''$ has nil-index at most two. If $\mathcal{N}$ has Hilbert function $\{1, m, n, 1\}$ then $\mathcal{N}'$ has symmetric Hilbert function $\{1, n, n, 1\}$ and admits a canonical grading. Furthermore, the Hilbert function of $\mathcal{N}''$ is $\{1, m - n, 1\}$ if $m > n$, and is $\{1, 1\}$ if $m = n$ (that is, $\dim(\mathcal{N}'') = 1$, or equivalently, $\mathcal{N} = \mathcal{N}'$ in this case).*

Notice that every admissible algebra with a canonical grading has symmetric Hilbert function (2.6). The converse is not true, see e.g. the example of dimension 23 in Section 8.

Moduli algebras of type $\widetilde{E}_6$. With Proposition 5.13 it is possible to compute for a given nil-polynomial an admissible algebra it is associated to. For the nil-polynomials in Proposition 7.1 in case $\dim W = 6$ there is a connection to moduli algebras associated to simple elliptic singularities $\widetilde{E}_6$, see [2], p. 306: Let $\mathbb{F} = \mathbb{C}$ and for $t \in \mathbb{C}$ with $t^3 + 27 \neq 0$ consider the nilpotent algebra $\mathcal{N}_t = \mathcal{M}(X^3+Y^3+Z^3+tXYZ)$, compare the notation (8.2) below. Then, with x,y,z being the residue classes of X,Y,Z a basis for $\mathcal{N}_t$ is x,y,z,yz,xz,xy,xyz with $\mathrm{Ann}(\mathcal{N}_t)=\mathbb{C}xyz$ and

$$p_t = tx_1^3+tx_2^3+tx_3^3-18x_1x_2x_3+x_1x_4+x_2x_5+x_3x_6 \tag{7.6}$$

is a nil-polynomial associated to $\mathcal{N}_t$. Notice that the cubic part of p_t occurs already in (3.1) of [2]. Notice also that $W_1=\langle x,y,z\rangle_{\mathbb{C}}$ and $W_3=\langle yz,xz,xy\rangle_{\mathbb{C}}$ are two totally isotropic subspaces as occurring in Proposition 7.1. Although for the moduli algebras the three parameters t with $t^3+27=0$ have to be excluded (for these three values of t the singularity of $\{X^3+Y^3+Z^3+tXYZ=0\}$ is not isolated), p_t is a nil-polynomial also for these t and is associated to an admissible algebra of nil-index 3 as well. For $t\neq 0$ and $s:=-18/t$ it is easy to see that the nil-polynomial p_t in (7.6) is linearly equivalent to

$$x_1^3+x_2^3+x_3^3+sx_1x_2x_3+x_1x_4+x_2x_5+x_3x_6\,,\ \text{compare with (7.3)}\,.$$

Nil-polynomials of degree 4

The method in Proposition 7.1 can be generalized to get nil-polynomials of higher degrees, say of degree 4 for simplicity. Throughout the subsection we use the notation (2.1). Because of Propositions 5.5 and 6.5 it is not necessary to distinguish between linear and affine equivalence for nil-polynomials of degree 4.

For fixed $n,m\geq 1$ let $W=W_1\oplus W_2\oplus W_3$ be a vector space with $W_1=\mathbb{F}^n$, $W_2=\mathbb{F}^m$ and let q be a fixed non-degenerate quadratic form on W in the following. Assume that W_1, W_3 are totally isotropic and that $W_1\oplus W_3$, W_2 are orthogonal with respect to q. Then W has dimension $2n+m$, and without loss of generality we assume that for suitable $\varepsilon_1,\dots,\varepsilon_m\in\mathbb{F}^*$

$$q(y)=\sum_{k=1}^{m}\varepsilon_k y_k^{(2)}\quad\text{if}\quad y\in W_2\,.$$

As before let C be the space of all cubic forms on W. Our aim is to find cubic forms $c\in C_q$ that are the cubic part of a nil-polynomial of degree 4.

Denote by C' the space of all cubic forms c on $W_1\oplus W_2$ such that $c(x+y)$ is quadratic in $x\in W_1$ and linear in $y\in W_2$, or equivalently, which are of the form

$$c(x+y)=\frac{1}{2}\sum_{k=1}^{m}\sum_{i,j=1}^{n}c_{ijk}x_ix_jy_k\quad\text{for all}\quad x\in W_1, y\in W_2$$

with suitable coefficients $c_{ijk}=c_{jik}\in\mathbb{F}$. Extending every $c\in C'$ trivially to a cubic form on W we consider C' as a subset of C.

For fixed $c\in C'$ the symmetric 2- and 3-linear forms ω_2,ω_3 on W are defined by $\omega_2(x,x)=2q(x)$ and $\omega_3(x,x,x)=6c(x)$ for all $x\in W$. With the commutative product $x{\cdot}y$ on W, see (5.11), define in addition also the k-linear forms ω_k by (5.15) for all $k\geq 4$. Then, for every $x,y\in W_1$ the identity $\omega_2(x{\cdot}y,t)=\omega_3(x,y,t)=0$ for all $t\in W_1\oplus W_3$ implies $x{\cdot}y\in W_2$, that is $W_1{\cdot}W_1\subset W_2$. In the same way $\omega_2(x{\cdot}y,t)=0$ for all $x\in W_1$, $y\in W_2$ and $t\in W_2\oplus W_3$ implies $W_1{\cdot}W_2\subset W_3$. Also $W_j{\cdot}W_k=0$ follows for all j,k with $j+k\geq 4$. Therefore c belongs to C_q if and only if $(a{\cdot}b){\cdot}c$ is symmetric in $a,c\in W_1$ for every $b\in W_1$.

In terms of the standard basis $e_1,\dots,e_m$ of $W_2=\mathbb{F}^m$ we have

$$a{\cdot}b=\sum_{k=1}^{m}\Big(\sum_{i,j=1}^{n}\varepsilon_k^{-1}c_{ijk}a_ib_j\Big)e_k\quad\text{for all}\quad a,b\in W_1$$

and thus with $\Theta_{i,j,r,s} := \sum_{k=1}^{m} \varepsilon_k^{-1} c_{ijk} c_{rsk}$ we get the identity

$$\omega_2\big((a\cdot b)\cdot c, d\big) = \sum_{i,j,r,s=1}^{n} \Theta_{i,j,r,s}\, a_i b_j c_r d_s \quad \text{for all} \quad a,b,c,d \in W_1\,, \text{ implying}$$

(7.7) $$A := C' \cap C_q = \{c \in C' : \Theta_{i,j,r,s} \text{ is symmetric in } i,r\}\,.$$

Notice that the condition in (7.7) implies that $\Theta_{i,j,r,s}$ is symmetric in all indices. A is a rational subvariety of the linear space C', it consists of all those c for which the corresponding product $x\cdot y$ on W is associative. The group $\Gamma := \mathrm{GL}(W_1) \times \mathrm{O}(q_{|W_2}) \subset \mathrm{GL}(W_1 \oplus W_2)$ acts on C' by $c \mapsto c \circ \gamma^{-1}$ for every $\gamma \in \Gamma$. Furthermore, $(g,h) \mapsto (g,h,(g^\sharp)^{-1})$ embeds Γ into $\mathrm{O}(q)$, compare the proof of Proposition 7.1. As a consequence, the subvariety $A \subset C'$ is invariant under Γ.

We are only interested in the case where $c \in A$ is non-degenerate on $W_1 \oplus W_2$. Then $W_{k+1} = \langle W_1\cdot W_k\rangle_{\mathbb{F}}$ holds for $k = 1,2$, implying $m \le \binom{n+1}{2}$. Put $\mathcal{N} := \bigoplus_{k>0} W_k$ with $W_4 := \mathbb{F}$ and $W_k := 0$ for all $k > 4$. The product (5.12) realizes $\mathcal{N}$ as graded admissible algebra. The non-degeneracy of c gives in addition $\mathcal{N}^k = \bigoplus_{\ell\ge k} W_\ell$ for all $k > 0$, that is, the grading is canonical. Conversely, every canonically graded admissible algebra of nil-index 4 occurs (up to isomorphism) this way with a non-degenerate c as above. Further admissible algebras with nil-index 4 can be obtained from $\mathcal{N}$ as above by taking $\mathcal{N} \vee \mathcal{N}'$ with $\mathcal{N}'$ an arbitrary admissible algebra of nil-index ≤ 3. But all these algebras are gradable by Proposition 6.4. Since there exist non-gradable admissible algebras of nil-index 4 (see Section 8) the classification problem in the nil-index 4 case must be more involved than the one in Proposition 7.5.

Let us consider the special case $n = 2$ with $m = \binom{n+1}{2} = 3$ in more detail (among these are in case $\mathbb{F} = \mathbb{C}$ also all nil-polynomials of maximal ideals of moduli algebras associated to singularities of type $\widetilde{E}_7$, see [2], p. 307). For simplicity we assume that for suitable coordinates (x_1,x_2) of W_1, (y_1,y_2,y_3) of W_2 and (z_1,z_2) of W_3 the quadratic form q is given by

(7.8) $$q = x_1z_1 + x_2z_2 + y_1^{(2)} + y_2^{(2)} + \varepsilon y_3^{(2)} \quad \text{for fixed} \quad \varepsilon \in \mathbb{F}^*$$

(in case $\mathbb{F} = \mathbb{R}, \mathbb{C}$ this is not a real restriction). For every $t \in \mathbb{F}$ consider the cubic form

$$c_t := (x_1^{(2)} + x_2^{(2)})y_1 + x_1x_2y_2 + tx_2^{(2)}y_3$$

on $W_1 \oplus W_2$, which is non-degenerate if $t \ne 0$. A simple computation reveals that every c_t is contained in $A = C' \cap C_q$. The corresponding nil-polynomial (depending on the choice of ε) then is

(7.9) $$p_t = q + c_t + d_t \quad \text{with} \quad d_t := x_1^{(4)} + x_1^{(2)}x_2^{(2)} + (1+\varepsilon^{-1}t^2)x_2^{(4)}\,.$$

For every $t \in \mathbb{F}^*$ an invariant of d_t is $\phi(t) := g_2(d_t)^3/g_3(d_t)^2 = \varepsilon^2 t^{-4}(4+\varepsilon^{-1}t^2)^3 \in \mathbb{F}$, where g_2, g_3 are the classical invariants of binary quartics, compare [9] p. 27. Since every fiber of $\phi : \mathbb{F}^* \to \mathbb{F}$ contains at most 6 elements we conclude

7.10 Proposition. *For every field $\mathbb{F}$ and every fixed $\varepsilon \in \mathbb{F}^*$ the set of all equivalence classes given by all nil-polynomials p_t, $t \in \mathbb{F}^*$, has the same cardinality as $\mathbb{F}$ and, in particular, is infinite.*

Remarks 1. In case $\mathbb{F} = \mathbb{Q}$ is the rational field there are infinitely many choices of $\varepsilon \in \mathbb{Q}^*$ leading to pairwise non-equivalent quadratic forms q in (7.8). For each such choice there is an infinite number of pairwise non-equivalent nil-polynomials p_t of degree 4 over $\mathbb{Q}$.

2. In case $\mathbb{F} = \mathbb{R}$ is the real field there are essentially the two choices $\varepsilon = \pm 1$. In case $\varepsilon = 1$ the form q has type $(5,2)$ and all nil-polynomials p_t with $0 < t \le \sqrt{8}$ are pairwise non-equivalent. In case $\varepsilon = -1$ the form q has type $(4,3)$ and all p_t with $t > 0$ are pairwise non-equivalent.

3. Nil-polynomials of degree ≥ 5 can be constructed just as in the case of degrees 3 and 4 as before. As an example we briefly touch the case of degree 5: Fix a vector space W of finite dimension over $\mathbb{F}$ together with a non-degenerate quadratic form q on W. Assume furthermore that there is a direct sum decomposition $W = W_1 \oplus W_2 \oplus W_3 \oplus W_4$ into non-zero totally isotropic subspaces such that $W_1 \oplus W_4$ and $W_2 \oplus W_3$ are orthogonal. Then consider a non-degenerate cubic form c on $W_1 \oplus W_2 \oplus W_3$ (trivially extended to W) that can be written as a sum $c = c' + c''$ of cubic forms with the following properties: c' is a cubic form on $W_1 \oplus W_3$ that is linear in the variables of W_3 while c'' is a cubic form on $W_1 \oplus W_2$ that is linear in the variables of W_1. Denote by $x{\cdot}y$ the commutative product on W determined by q and c. Then $W_2 = \langle W_1{\cdot}W_1\rangle$, $W_3 = \langle W_1{\cdot}W_2\rangle$ and $W_4 = \langle W_1{\cdot}W_3 + W_2{\cdot}W_2\rangle$. If we assume $c \in C_q$, we get with $W_5 := \mathbb{F}$ as in the nil-index 4 case above that $\mathcal{N} := \bigoplus_{k>0} W_k$ is a canonically graded admissible algebra of nil-index 5. Conversely, every canonically graded admissible algebra of nil-index 5 is obtained this way.

8. Some counterexamples

In this section we give examples of admissible algebras *without* Property (AH). By Theorem 3.2 such an algebra cannot be graded. We also give examples of non-gradable algebras *with* Property (AH).

A test for non-gradability (especially with computer aid) involving nil-polynomials is the following

8.1 Proposition. *Let $\mathcal{N}$ be an admissible algebra with nil-polynomial $p \in \mathbb{F}[x_1, \ldots, x_n]$ and nil-index ν. Then, if $\mathcal{N}$ has a grading, there exists a matrix $A = (a_{jk}) \in \mathbb{F}^{n\times n}$ such that*

(i) $\xi p = p$ *for* $\xi := \sum_{j,k=1}^{n} a_{jk} x_k \, \partial/\partial x_j$.

(ii) *A is diagonalizable over $\mathbb{F}$.*

(iii) *Every eigen-value of A is a positive rational number, and for every eigen-value ε also $1-\varepsilon$ is an eigen-value with the same multiplicity.*

(iv) *The eigenvalues form the arithmetic progression $\frac{1}{\nu}, \frac{2}{\nu}, \ldots, \frac{\nu-1}{\nu}$ if $\mathcal{N}$ has a canonical grading.*

Proof. Let $\mathcal{N} = \bigoplus \mathcal{N}_k$ be a grading and put $d := \max\{k : \mathcal{N}_k \ne 0\}$. Then $\mathcal{N}_d$ is the annihilator of $\mathcal{N}$. Let $W := \bigoplus_{k<d} \mathcal{N}_k$ and choose a linear isomorphism $\varphi : \mathbb{F}^n \to W$ such that $\varphi(e_j) \in \mathcal{N}_k$ for all j and suitable $k = k(j)$, where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{F}^n$. Further let ω be a pointing on $\mathcal{N}$ with kernel W and put $\widetilde{p} := \omega \circ \exp_2 \circ \varphi$. Then (i) - (iii) hold for $\widetilde{p}$ and the diagonal matrix $\widetilde{A}$ with diagonal entries $\widetilde{a}_{jj} = k(j)/d$ for all j. In case the grading is canonical, $d = \nu$ and $\mathcal{N}_k \ne 0$ for $0 < k < \nu$ holds, implying (iv).

By Proposition 5.5 the nil-polynomials $p, \widetilde{p}$ are linearly equivalent. As a consequence of Proposition 5.4 there exists $C \in \mathrm{GL}(n, \mathbb{F})$ and $c \in \mathbb{F}^*$ with $c\,\widetilde{p}(x) = p(Cx)$ for all $x \in \mathbb{F}^n$. But then p satisfies (i) - (iv) with respect to $A = C\widetilde{A}C^{-1}$. □

The proof of 8.1 uses the non-trivial fact that every gradable $\mathcal{N}$ has property (AH) and hence that any two associated nil-polynomials are linearly equivalent, compare 3.2 and 4.10. Property (i) means that $p = \lambda_1 \partial p/\partial x_1 + \ldots + \lambda_n \partial p/\partial x_n$ for suitable linear forms λ_k on $\mathbb{F}^n$, and hence that p is in its Jacobi ideal (the ideal in $\mathbb{F}[x_1, \ldots, x_n]$ generated by all first partial derivatives of p). Part of 8.1 can also be reformulated in terms of quasi-homogeneous polynomials. By definition, $f \in \mathbb{F}[x_1, \ldots, x_n]$ is *quasi-homogeneous* if there exist positive integers $m, m_1, \ldots, m_n$ with $f(t^{m_1}x_1, \ldots, t^{m_n}x_n) \equiv t^m f(x_1, \ldots, x_n)$ for all $t \in \mathbb{F}$.

Remark. Let $\mathcal{N}$ be an admissible algebra having a grading. Then there exists a quasi-homogeneous nil-polynomial associated with $\mathcal{N}$.

Motivated by the main result of [11], see also [12], we organize our search for non-gradable admissible algebras as follows: For fixed indeterminates $T_1, \ldots, T_m$ denote by $\mathfrak{m}$ the maximal ideal in the localization R of $\mathbb{F}[T_1 \ldots, T_m]$ at the origin, that is, the ideal of all quotients P/Q with $P, Q \in \mathbb{F}[T_1 \ldots, T_m]$ satisfying $Q(0) \neq P(0) = 0$. For every $F \in \mathfrak{m}^2$ let $J(F)$ in R be the *Jacobi ideal* of F (the ideal generated by all first order derivatives of F). Then if $J(F) \supset \mathfrak{m}^k$ for some k, the Milnor algebra $R/J(F)$ has finite dimension and its maximal ideal

$$\mathcal{M}(F) := \mathfrak{m}\,/J(F) \tag{8.2}$$

is nilpotent. For our search we are looking for examples with $F \notin J(F)$.

Let us stress that we always use the notation $\mathcal{M}(F)$ in the following way: The indeterminates $T_1, \ldots, T_m$ giving the localization R above are precisely those occurring in F. As an example, for the $\mathcal{M}(F)$ occurring two lines below we understand $m = 2$ and $\{T_1, T_2\} = \{X, Y\}$.

Non-gradable algebras with Property (AH)

Let $\mathcal{N} := \mathcal{M}(X^5 + X^2Y^2 + Y^4)$. Then $\mathcal{N}$ has basis

$$x, x^2, x^3, x^4, y, xy, y^2, y^3, x^5, \tag{8.3}$$

where x, y are the residue classes of X, Y. We abbreviate this basis with $e_1, \ldots, e_9$. The annihilator $\mathcal{N}_{[1]}$ is spanned by $e_9 = x^5$ and the residue class of F is $-e_9/4 \in \mathcal{N}_{[1]}$. A nil-polynomial $\mathsf{p} \in \mathbb{F}[x_1, \ldots, x_8]$ obtained from the basis is[1]

$$\begin{aligned} &\tfrac{1}{120}x_1^5 + \left(\tfrac{1}{6}x_1^3x_2 + \tfrac{5}{96}x_5^4 - \tfrac{5}{8}x_1^2x_5^2\right) \\ &\quad + \left(\tfrac{1}{2}x_1^2x_3 + \tfrac{1}{2}x_1x_2^2 + \tfrac{5}{8}x_5^2x_7 - \tfrac{5}{4}x_1^2x_7 - \tfrac{5}{2}x_1x_5x_6 - \tfrac{5}{4}x_2x_5^2\right) \\ &\quad + \left(x_1x_4 + x_2x_3 + \tfrac{5}{4}x_5x_8 + \tfrac{5}{8}x_7^2 - \tfrac{5}{2}x_2x_7 - \tfrac{5}{4}x_6^2\right) . \end{aligned} \tag{8.4}$$

In particular, $\mathcal{N}$ is an admissible algebra of dimension 9 and nil-index 5. Also, $\mathcal{N}$ does not have a gradation but has Property (AH)[2].

Further examples of this type (but of higher dimension) are obtained by varying F. For instance $\mathcal{N} := \mathcal{M}(F)$ with $F = X^4 + X^2Y^3 + Y^5$ is an admissible algebra of dimension 11 and nil-index 5 without a grading but with Property (AH). As algebra $\mathcal{N}$ is isomorphic to the (unique) maximal ideal of $\mathbb{F}[X, Y]/I$, where $I := \left(\partial F/\partial X\,,\ \partial F/\partial Y\,,\ X^3Y\right)$. The latter algebra already appears as an example of a non-gradable algebra in Remark 3.3 in [1] (note that X^5 occurring there is already contained in I and hence is superfluous).

If we go to algebras with embedding dimension 3 we can get an example with nil-index 4 and dimension 8. Add an additional indeterminate Z and consider $\mathcal{M}(X^4 + XY^2 + Y^3 + XZ^2)$. Then a basis is given by $x, y, z, x^2, x^3, yz, z^2, x^4$ and

$$\begin{aligned} \mathsf{p} = &\tfrac{1}{24}x_1^4 + (\tfrac{1}{2}x_1^2x_4 + 3x_1^2x_2 - 2x_1x_2^2 + \tfrac{4}{9}x_2^3 - \tfrac{4}{3}x_2x_3^2) \\ &+ (x_1x_5 + \tfrac{1}{2}x_4^2 + 6x_2x_4 - \tfrac{8}{3}x_2x_7 - \tfrac{8}{3}x_3x_6) \end{aligned}$$

is the nil-polynomial derived from it. The Hilbert function is $\{1, 3, 3, 1, 1\}$. Also for this algebra there is no matrix $A \in \mathbb{F}^{7\times 7}$ satisfying (i) in Proposition 8.1. On the other hand, for suitable coefficients $\lambda_k \in \mathbb{F}[x_1, \ldots, x_7]$ there exists[1] a representation $\mathsf{p} = \lambda_1 \partial\mathsf{p}/\partial x_1 + \ldots + \lambda_7 \partial\mathsf{p}/\partial x_7$, that is, p lies in its Jacobi ideal.

[1] Computed with `Singular`, freely available at `http://www.singular.uni-kl.de/`

[2] Computed with `Maple`.

Failing Property (AH)

In this subsection we restrict to the special case $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. This allows us to use analytic arguments.

It can be seen[1] that $\mathcal{M}(X^6 + X^2Y^3 + Y^5)$ is an admissible algebra of dimension 17 and nil-index 7. In the same way $\mathcal{M}(X^7 + X^2Y^3 + X^3Y^2 + Y^4)$ is an admissible algebra of dimension 15 and nil-index 8. Also $\mathcal{M}(X^5 + X^2Y^2 + Y^4 + XZ^2)$ for fixed indeterminates X, Y, Z is an admissible algebra of dimension 15 with nil-index 6. All three algebras do not have Property (AH). The automorphism groups have dimension $25, 20, 23$ respectively. Instead of giving further details for these examples we do this for another one (of even lower nil-index but higher dimension).

For indeterminates X, Y, Z, U consider $\mathcal{N} := \mathcal{M}(X^3 + X^2Y^2 + Y^4 + XZ^2 + ZU^2)$. Then $\mathcal{N}$ is an admissible algebra of dimension 23 and nil-index 5, in fact, $\mathcal{N}$ has symmetric Hilbert function $\{1, 4, 7, 7, 4, 1\}$. A basis for $\mathcal{N}$ is[1]

$$x, y, z, u, xy, y^2, xy^2, yz, y^2z, z^2, yz^2, y^2z^2, xu, yu, xyu, y^2u, xy^2u, u^2, xu^2, yu^2, xyu^2, y^2u^2, xy^2u^2$$

with the last vector spanning $\operatorname{Ann}(\mathcal{N})$. Also, the nil-polynomial given by the above basis is[1]

$$
\begin{aligned}
(8.5)\quad \mathsf{p} = {} & \left(\tfrac{1}{4}x_1x_4^2 - \tfrac{1}{8}x_1^2x_3 + \tfrac{1}{96}x_2^2x_3 + \tfrac{1}{8}x_3^3\right)x_2^2 \\
& + \Big(\tfrac{1}{18}x_1^3x_3 - \tfrac{1}{4}x_1^2x_2x_8 - \tfrac{1}{4}x_1^2x_6x_3 - \tfrac{1}{6}x_1^2x_4^2 + \tfrac{1}{2}x_1x_2^2x_{18} - \tfrac{1}{2}x_1x_2x_5x_3 + \tfrac{1}{4}x_6x_3^3 + x_1x_2x_4x_{14} \\
& \quad + \tfrac{1}{2}x_1x_6x_4^2 + \tfrac{1}{24}x_2^3x_8 + \tfrac{1}{8}x_2^2x_6x_3 + \tfrac{3}{4}x_2^2x_3x_{10} + \tfrac{1}{2}x_2^2x_4x_{13} + \tfrac{1}{2}x_2x_5x_4^2 + \tfrac{3}{4}x_2x_3^2x_8\Big) \\
& + \Big(x_1x_2x_{20} - \tfrac{1}{4}x_1^2x_9 - \tfrac{1}{3}x_1^2x_{18} - \tfrac{1}{2}x_1x_5x_8 + x_1x_6x_{18} - \tfrac{1}{2}x_1x_7x_3 - \tfrac{2}{3}x_1x_4x_{13} + x_1x_4x_{16} \\
& \quad + x_2x_{13}x_{14} + \tfrac{1}{8}x_2^2x_9 + \tfrac{1}{2}x_2^2x_{19} + x_2x_5x_{18} + \tfrac{1}{4}x_2x_6x_8 + \tfrac{3}{2}x_2x_3x_{11} + \tfrac{3}{2}x_2x_8x_{10} + x_2x_4x_{15} \\
& \quad + \tfrac{1}{2}x_1x_{14}^2 - \tfrac{1}{4}x_5^2x_3 + x_5x_4x_{14} + \tfrac{1}{8}x_6^2x_3 + \tfrac{3}{2}x_6x_3x_{10} + x_6x_4x_{13} + \tfrac{1}{2}x_7x_4^2 + \tfrac{3}{4}x_3^2x_9 + \tfrac{3}{4}x_3x_8^2\Big) \\
& + \Big(x_1x_{22} - \tfrac{2}{3}x_1x_{19} + x_2x_{21} + x_5x_{20} + \tfrac{1}{4}x_6x_9 + x_6x_{19} + x_7x_{18} + \tfrac{3}{2}x_3x_{12} \\
& \quad + \tfrac{3}{2}x_8x_{11} + \tfrac{3}{2}x_9x_{10} + x_4x_{17} - \tfrac{1}{3}x_{13}^2 + x_{13}x_{16} + x_{14}x_{15}\Big)\,.
\end{aligned}
$$

We claim that the graph S of p in $\mathbb{F}^{23}$ cannot be affinely homogeneous. Define f on $\mathbb{F}^{23}$ by $\mathsf{f}(x_1, \dots, x_{23}) := \mathsf{p}(x_1, \dots, x_{22}) - x_{23}$. Since $\operatorname{Aff}(S)$ is a Lie group over $\mathbb{F}$, affine homogeneity of S would imply for every fixed $1 \le \ell \le 22$ the existence of an affine vector field

$$\xi = \partial/\partial x_\ell + \sum_{j,k=1}^{23} a_{jk} x_k \partial/\partial x_j$$

with coefficients $a_{jk} \in \mathbb{F}$ such that $\xi \mathsf{f} = \rho \mathsf{f}$ for some $\rho \in \mathbb{F}$. Since the polynomial f has rational coefficients the vector field ξ can be chosen in such a way that all a_{jk} and ρ are rational. But $\xi \mathsf{f} = \rho \mathsf{f}$ is equivalent to a system of linear equations in a_{jk}, ρ. It can be seen[2] that this system has a rational solution only in case $\ell \neq 3$. Therefore the orbit of $0 \in S$ under the group $\operatorname{Aff}(S)$ has dimension 21. In particular, S is not even locally affinely homogeneous at the origin. Furthermore[2], the group $\operatorname{Aff}(S) \cong \operatorname{Aut}(\mathcal{N})$ is a Lie group of dimension 42 over $\mathbb{F}$.

Remark. Since computer software may contain errors we carried out all computations in a highly redundant manner: Different machines were used, the same computations were repeated with different routines, bases of the admissible algebras $\mathcal{N}$ under consideration were changed resulting in totally different (but affinely equivalent) nil-polynomials and finally, the linear system $\xi \mathsf{f} = \rho \mathsf{f}$ with ξ from above was replaced by a totally different one (namely Lemma 8.6 for the special case $r = 22$ and $n = 23$). Notice that $\xi \mathsf{f} - \rho \mathsf{f} \in \mathbb{F}[x_1, \dots, x_{22}]$ is a polynomial of degree 5 having a_{jk} and ρ as parameters. An alternative condition for ξ being tangent to $S \subset \mathcal{N}$ clearly is that $\xi \mathsf{f}$ vanishes at every point of S. This leads to a linear system in terms of a polynomial of degree bigger than 5. This works also for arbitrary codimension:

For fixed integers $r, s \ge 1$ and $n := r + s$ consider $\mathbb{F}^r$, $\mathbb{F}^s$ with coordinates $(x_1, \dots, x_r)$, $(x_{r+1}, \dots, x_n)$ respectively. Let $\mathsf{p} : \mathbb{F}^r \to \mathbb{F}^s$ be a polynomial map. Then $\mathsf{p} = (\mathsf{p}_{r+1}, \dots, \mathsf{p}_n)$

for scalar valued polynomials p_k on $\mathbb{F}^r$. Denote by S the graph of p in $\mathbb{F}^n$, which is a smooth variety of codimension s. The proof of the following statement is an easy exercise in differentiation and will be omitted.

8.6 Lemma. *Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. Then $S \subset \mathbb{F}^n$ is locally affinely homogeneous at the origin if and only if for every $1 \leq \ell \leq r$ the following linear system in the unknowns a_{jk}, $1 \leq j, k \leq n$, has a solution over $\mathbb{F}$*

$$(\dagger) \qquad P_m = \sum_{j=1}^{r} (P_j + \delta_{j\ell}) \partial p_m / \partial x_j \quad \text{for} \quad m = r+1, \dots, n\,, \ \textit{where}$$

$$P_j := \sum_{k=1}^{r} a_{jk} x_k + \sum_{k=r+1}^{n} a_{jk} p_k \quad \text{for} \quad 1 \leq j \leq n$$

and δ is the Kronecker delta.

The equations in Lemma 8.6 can also be used to compute the Lie algebra $\mathfrak{aff}(S)$ of $\mathrm{Aff}(S)$ numerically. Indeed, add to the a_{jk} further unknowns $c_1, \dots, c_r$ and replace in $(\dagger)$ the term $\delta_{j\ell}$ by c_j. Then the solution space for this altered linear system is canonically isomorphic to $\mathfrak{aff}(S)$.

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