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On Levi-degenerate homogeneous CR-manifolds

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Abstract. We give a survey on the main result of [9] where all homogeneous Levi degenerate CR-manifolds in dimension 5 have been classified up to local CR-equivalence. Furthermore, we discuss the only so far known examples of 3- and 4-nondegenerate locally homogeneous hypersurfaces by explicit equations and in greater detail than in [9].

1. Introduction and Preliminaries

At the beginning of every course on Complex Analysis the *Cauchy-Riemann* differential equations appear: For every domain $U \subset \mathbb{C}$ and every smooth function $f = u + iv : U \to \mathbb{C}$ complex differentiability holds if and only if at every point of U the real Jacobian $\partial(u,v)/\partial(x,y)$ in $\mathbb{R}^{2\times 2}$ is of the form $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$, or equivalently, induces a complex linear endomorphism of $\mathbb{C} \approx \mathbb{R}^2$. More generally, for every domain $U \subset \mathbb{C}^n$ a smooth mapping $f : U \to \mathbb{C}^m$ is holomorphic (i.e. locally representable by a convergent power series) if and only if at every $a \in U$ the real Jacobian in $\mathbb{R}^{2m\times 2n}$ induces a complex linear operator from $\mathbb{C}^n \approx \mathbb{R}^{2n}$ to $\mathbb{C}^m \approx \mathbb{R}^{2m}$.

Now consider instead of open subsets in $\mathbb{C}^n$ arbitrary connected smooth real submanifolds $M \subset \mathbb{C}^n$. At every $a \in M$ then the *tangent space* to M is an $\mathbb{R}$-linear subspace $T_aM \subset \mathbb{C}^n$ and for every smooth $f : M \to \mathbb{C}^m$ the *differential* at a is an $\mathbb{R}$-linear map $df_a : T_aM \to \mathbb{C}^m$. It is obvious that a necessary condition for f being locally the restriction of a holomorphic $\mathbb{C}^m$-valued map, defined in an open neighbourhood of $a \in$

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M with respect to $\mathbb{C}^n$, is that the restriction of df_a to the *holomorphic tangent space*

$$H_aM := T_aM \cap iT_aM \tag{1.1}$$

(that is the largest complex linear subspace of $\mathbb{C}^n$ contained in the real tangent space T_aM) is complex linear. M is called a *CR-submanifold* of $\mathbb{C}^n$ if the complex dimension of H_aM does not depend on $a \in M$. This dimension is called the *CR-dimension*, while the real dimension of T_aM/H_aM is called the *CR-codimension* of M. Then the collection of all H_aM gives the *holomorphic subbundle* HM of the tangent bundle TM, and multiplication with the imaginary unit i defines a bundle endomorphism J of HM with $J^2 = -\,\mathsf{id}$. The smooth sections in TM, the smooth *vector fields* on M, form a real Lie algebra and M satisfies the *integrability condition*, that is, for all smooth sections ξ, η in the subbundle HM also $[J\xi, \eta] + [\xi, J\eta]$ is a section in HM and

$$[J\xi, J\eta] - [\xi, \eta] \;=\; J\big([J\xi, \eta] + [\xi, J\eta]\big)\,. \tag{1.2}$$

The abstract version of a Cauchy-Riemann manifold that we intend to use here is as follows: A smooth CR-manifold is a triple (M, HM, J), where M is a connected smooth manifold, HM is a smooth subbundle of the tangent bundle TM and J is a smooth bundle endomorphism of HM such that $J^2 = -\,\mathsf{id}$ and the integrability condition (1.2) holds. J defines a complex vector space structure on every *holomorphic tangent space* H_aM. Instead of (M, HM, J) we simply write M if the corresponding HM and J are clear. The smooth CR-manifolds form a category in a natural way: A smooth mapping $\varphi : M \to M'$ between CR-manifolds is a CR-mapping, if for every $a \in M$ and $a' := \varphi(a)$ the differential $d\varphi_a : T_aM \to T_{a'}M'$ maps the corresponding holomorphic tangent space H_aM complex linearly to $H_{a'}M'$.

An important local invariant at every a of a smooth CR-manifold M is the (vector-valued) Levi from. We do not need here the full Levi form, only its *kernel* $K_aM \subset H_aM$. In case $M = \{z \in U : \rho(z) = 0\}$ for a domain $U \subset \mathbb{C}^n$ and a smooth submersion $\rho : U \to \mathbb{R}^d$ one way of defining the kernel is

$$K_aM := \Big\{ v \in H_aM : \sum_{j,k=1}^{n} v_j \overline{w}_k \frac{\partial^2 \rho\,(a)}{\partial z_j \partial \overline{z}_k} \;= 0 \;\in \mathbb{C}^d \quad \text{for all} \;\; w \in H_aM \Big\}\,,$$

which does not depend on the choice of the submersion ρ. In [3], compare also [16], a complete set of invariants has been given in the real-analytic setting that characterizes M near a up to CR-isomorphy provided that M is Levi nondegenerate at a (that is $KM = 0$) and in addition is of hypersurface type (that is, M has CR-codimension 1).

For certain CR-manifolds M we define by induction on $k \in \mathbb{N}$ higher order Levi kernels $K^k_a M \subset H_a M$ and say that M has *constant degeneracy of order* k if the $K^k_a M$, $a \in M$, form a subbundle $K^k M$ of HM: To start the induction just put

$$K^0_a M := H_a M \quad \text{and} \quad K^{-1}_a M := T_a M \otimes \mathbb{C}/N_a$$

with $N_a := \{x \otimes 1 - Jx \otimes i : x \in H_a M\}$. Since N_a is a complex linear subspace, $K^{-1}_a M$ inherits a complex structure that we also denote by J. Embedding $T_a M$ into $K^{-1}_a M$ via $x \mapsto x \otimes 1 \bmod N_a$, the space $K^{-1}_a M$ may be thought of as the 'smallest' complex linear space containing $T_a M$ as real and $H_a M$ as complex linear subspace.
Now suppose as induction step that M has constant degeneracy of order $k \geq 0$ and that $K^k_a M \subset K^{k-1}_a M$ is already defined as complex linear subspace. Then it can be shown [13] that there is a unique mapping

$$\mathcal{L}^{k+1}_a : H_a M \times K^k_a M \to K^{k-1}_a M / K^k_a M$$

satisfying

$$\mathcal{L}^{k+1}_a(\xi_a, \eta_a) = [\xi, \eta]_a + J[\xi, J\eta]_a \mod K^k_a M$$

for all smooth sections ξ and η in HM and $K^k M$ respectively, and the next kernel is defined by

$$K^{k+1}_a M := \{v \in K^k_a M : \mathcal{L}^{k+1}_a(H_a M, v) = 0\}\,.$$

$\mathcal{L}^{k+1}_a$ is conjugate linear in the second and, as a consequence of the integrability condition (1.1), complex linear in the first variable. In particular, $K^{k+1}_a M$ is a complex linear subspace of $K^k_a M$. Also, $\mathcal{L}^1_a$ is (up to a non-zero constant factor) the usual Levi form at $a \in M$ and $K^1_a M = K_a M$.

We say that the CR-manifold M has constant degeneracy if it has constant degeneracy of any order in the above sense. For instance, M has this property, if for every pair of points $a, a' \in M$ there exist open neighbourhoods U, U' of a, a' together with a CR-diffeomorphism $\varphi : U \to U'$ satisfying $\varphi(a) = a'$. In this note we are mainly interested in manifolds of this type.

A CR-manifold M of constant degeneracy is called *finitely nondegenerate* if $K^k M = 0$ for some k and is called *k-nondegenerate* if $k \geq 1$ is minimal with respect to this property (for the definition of

k-nondegeneracy without the assumption of 'constant degeneracy' compare e.g. [1]). In case M is finitely non-degenerate there cannot exist a domain $N \subset M$ that is CR-isomorphic to a direct product $N' \times \mathbb{C}$ with a CR-manifold N'. It is clear that Levi nondegenerate is the same as 1-nondegenerate. In case $k \geq 2$ the minimal dimension for an everywhere k-nondegenerate CR-manifold is $2k+1$. A well studied example of a homogeneous 2-nondegenerate CR-manifold in the minimal possible dimension 5 is the tube

$$(1.3) \qquad \mathcal{M} := \{z \in \mathbb{C}^3 : (\mathrm{Re}\, z_1)^2 + (\mathrm{Re}\, z_2)^2 = (\mathrm{Re}\, z_3)^2,\ (\mathrm{Re}\, z_3) > 0\}$$

over the future light cone in 3-dimensional space time, compare [7], [9], [10], that will play a prominent role below.

The CR-manifold M is called *minimal* if every smooth submanifold $N \subset M$ with $H_aM \subset T_aN$ for all $a \in N$ is open in M. In case M is minimal in this sense there cannot exist a domain $N \subset M$ that is CR-isomorphic to a direct product $N' \times \mathbb{R}$ with a CR-submanifold N'.

2. The analytic category

From now on we will only consider *real-analytic* CR-manifolds (M, HM, J), that is, M is a real-analytic manifold, $HM \subset TM$ is a real-analytic subbundle and also J is real-analytic. Also these CR-manifolds form a category with respect to real-analytic CR-mappings. In particular, we now call two such manifolds M and M' CR-equivalent if there exists a CR-diffeomorphism $M \to M'$ that is real-analytic in both directions. It is well known that every (real-analytic) CR-manifold M can be realized locally as a real-analytic CR-submanifold of some $\mathbb{C}^n$.

By $\mathsf{Aut}(M)$ we denote the group of all (real-analytic) CR-automorphisms of M. In case $\mathsf{Aut}(M)$ acts transitively on M the CR-manifold M is called *homogeneous*. Vector fields on M are just the sections ξ in TM over M – for every $a \in M$ we write ξ_a instead of $\xi(a) \in T_aM$. A real-analytic vector field on M is called an *infinitesimal CR-transformation* if the corresponding local flow consists of local CR-isomorphisms. Denote by $\mathfrak{hol}(M)$ the space of all infinitesimal CR-transformations on M, which is a real Lie algebra with respect to the usual bracket. It is known, compare e.g. [1], that a vector field ξ on M is contained in $\mathfrak{hol}(M)$ if and only if every point of M has an open neighbourhood N that can be realized as a real-analytic submanifold of a domain U in some $\mathbb{C}^n$ in such a way that $\xi|_N$ extends to a holomorphic vector field on U.

Our interest in the following is mainly in the local CR-structure at arbitrary points $a \in M$, that is, in the CR-manifold germs (M, a). Denote by $\mathfrak{hol}(M, a)$ the space of all germs of vector fields in $\mathfrak{hol}(N)$, where $N \subset M$ runs through all open neighbourhoods of a in M. Clearly, also $\mathfrak{hol}(M,a)$ is a real Lie algebra in an obvious way and there is a canonical embedding $\mathfrak{hol}(M) \hookrightarrow \mathfrak{hol}(M,a)$. In certain cases more can be said:

2.1 Proposition. *Suppose that the CR-manifold M is simply connected and that all Lie algebras $\mathfrak{hol}(M,a)$, $a \in M$, have the same finite dimension. Then for every $a \in M$ the canonical injection $\mathfrak{hol}(M) \to \mathfrak{hol}(M,a)$ is an isomorphism of Lie algebras.*

Proof. Denote by $\pi : \mathfrak{H} \to M$ the sheaf whose stalks $\pi^{-1}(a)$ are the Lie algebras $\mathfrak{hol}(M,a)$. For every domain $N \subset M$ then $\mathfrak{hol}(N)$ can be identified with the space of continuous sections over N in $\mathfrak{H}$. Every $a \in M$ has an open neighbourhood N in M such that $\mathfrak{hol}(N) \to \mathfrak{hol}(M,c)$ is an isomorphism for every $c \in N$. This implies that the connected components of $\mathfrak{H}$ are coverings of M and hence are single sheeted since M is simply connected. □

The CR-manifold M is called *locally homogeneous* if for every $a, a' \in M$ there exist open neighbourhoods N, N' in M together with a CR-isomorphism $N \to N'$ sending a to a'. By [17] this is equivalent to the condition: *To every $a \in M$ there exists a Lie subalgebra $\mathfrak{g} \subset \mathfrak{hol}(M,a)$ of finite dimension such that the canonical evaluation map $\mathfrak{g} \to T_aM$, $\xi \mapsto \xi_a$, is surjective.* Every locally homogeneous CR-manifold M has constant degeneracy in the sense of the preceding section. In particular, all complex subbundles $K^kM \subset TM$, $k \in \mathbb{N}$, are defined.

It is known that for every locally homogeneous CR-manifold M the condition $\dim \mathfrak{hol}(M,a) < \infty$ for some $a \in M$ (and hence for all $a \in M$) is equivalent to M being finitely nondegenerate and minimal (as defined in the preceding section).

For homogeneous Levi nondegenerate manifolds large classes of examples are known: One of the best studied examples is for every $n \geq 2$ the euclidian hypersphere

$$S := \{z \in \mathbb{C}^n : (z|z) = \sum z_k \overline{z}_k = 1\}\,, \tag{2.2}$$

the boundary of the euclidian ball $B := \{z \in \mathbb{C}^n : (z|z) < 1\}$. Every $g \in \mathsf{Aut}(S)$ extends to a biholomorphic automorphism of the ball B and thus gives a group isomorphism $\mathsf{Aut}(S) \cong \mathsf{Aut}(B) \cong \mathsf{PSU}(n,1)$. In particular, S is homogeneous (clearly, the subgroup $\mathsf{SU}(n) \subset \mathsf{GL}(n,\mathbb{C})$ is already transitive on S). For every $a \in S$ the holomorphic tangent space $H_aS = a^\perp$ is the (complex) orthogonal complement of the vector a. In particular, $a \mapsto H_aS$, defines an 'anti-CR map' from S to the Grassmannian of all complex hyperplanes in $\mathbb{C}^n$.

Further classes of examples can be found, for instance, in [12]. One of these is as follows: Fix arbitrary integers $p \geq q \geq 1$ and let

$E := \mathbb{C}^{p\times q}$ be the space of all complex $p\times q$-matrices. Then the compact group $K := \mathsf{SU}(p) \times \mathsf{SU}(q)$ acts on E by $z \mapsto uzv^*$, $(u, v) \in K$. For every $a \in E$ the orbit $M := K(a)$ is a Levi nondegenerate minimal homogeneous CR-submanifold of E. In case $M' = K'(a')$ for $a' \in \mathbb{C}^{p'\times q'}$ and $K' := \mathsf{SU}(p') \times \mathsf{SU}(q')$ with $p' \geq q' \geq 1$ is another orbit of this type, the manifolds M and M' are locally CR-equivalent if and only if $p' = p$, $q' = q$ and one of the following alternatives holds.
(1) *a invertible and hence $p = q$:* $M' = tM$ or $M' = tM^{-1}$ for some $t \in \mathbb{C}^*$ and $M^{-1} := \{z^{-1} : z \in M\} \subset \mathsf{GL}(p, \mathbb{C})$.
(2) *a not invertible:* $M' = tM$ for some $t > 0$.

3. Tube manifolds

In the following let E be a complex vector space of finite dimension. Fix a conjugation $z \mapsto \overline{z}$ on E and denote by $V := \{z \in E : z = \overline{z}\}$ the corresponding real form. Then $E = V \oplus iV$ and for every (connected immersed) real-analytic submanifold $F \subset V$ the corresponding *tube* $M := F + iV$ is a CR-submanifold of E. Indeed, the abelian group A of all translations $z \mapsto z + iv$, $v \in V$, satisfies $M = A(F)$ and for every $a \in F \subset M$ we have $H_aM = T_aF \oplus iT_aF$. More generally, since the conjugation leaves M invariant, all kernels K^k_aM (if defined) are also invariant under the conjugation and hence are of the form $K^k_aM = K^k_aF \oplus iK^k_aF$ for $K^k_aF := K^k_aM \cap T_aF$.

For a certain class of tube manifolds $M = F + iV$ a simple method for the actual computation of the kernels K^k_aF has been given in [10]: Let $\mathfrak{A}$ be a linear space of affine mappings $\xi : V \to V$ such that

(i) $\xi_x := \xi(x) \in T_xF$ for all $\xi \in \mathfrak{A}$ and all $x \in F$,

(ii) the mapping $\mathfrak{A} \to T_aF$, $\xi \mapsto \xi_a$, is a linear isomorphism.

Then the CR-submanifold $M = F + iV$ is locally homogeneous and the kernels K^k_aF are recursively given by

$$\begin{aligned} K^0_aF &= T_aF \quad \text{and} \\ K^{k+1}_aF &= \{v \in K^k_aF : \xi^{\mathrm{lin}}(v) \in K^k_aF \quad \text{for all} \quad \xi \in \mathfrak{A}\}\,, \end{aligned}$$

where $\xi^{\mathrm{lin}} = \xi - \xi_0$ is the linear part of ξ.

Tubes form a very special class of CR-manifolds: Indeed, for every tube M and every $a \in M$ the Lie algebra $\mathfrak{hol}(M, a)$ must contain an abelian subalgebra of dimension

$$(*) \qquad n := \text{CR-dim}M + \text{CR-codim}M.$$

As an example consider for fixed $p \geq q \geq 1$ and $a \in E := \mathbb{C}^{p\times q}$ noninvertible the submanifold $M := \{uav : u \in \mathsf{SU}(p), v \in \mathsf{SU}(q)\}$ of E, that is, the case (2) at the end of Section 2. Then M is homogeneous, Levi nondegenerate, minimal and $n = r(p+q-r)$ for the number n defined by the formula $(*)$, where r is the rank of the matrix a (compare [12] for details). Furthermore, M is of hypersurface type if and only if $r = 1$. In M there exists a unique (rectangular) diagonal matrix d with non-negative real diagonal entries $d_{11} \geq d_{22} \geq \cdots \geq d_{qq}$. In case $q > 1$ and $d_{11} \neq d_{qq}$ the Lie algebra $\mathfrak{hol}(M,a)$ is isomorphic to $\mathfrak{u}(p) \times \mathfrak{su}(q)$ and hence does not contain any abelian Lie subalgebra of dimension $p+q$. Therefore, in this case the germ (M,a) has no local tube realization. On the other hand, it can be shown that (M,a) always has a tube realization if $r = 1$. Notice that among these cases the sphere S from (2.2) occurs for $p = n$, $q = 1$, $d_{11} = 1$ and that S is locally CR-equivalent, for instance, to the tube with base

$$F := \{x \in \mathbb{R}^n : \sum e^{2x_k} = 1\}\,.$$

Indeed, the locally biholomorphic map $z \mapsto (e^{z_1}, e^{z_2}, \ldots, e^{z_n})$ realizes $F + i\mathbb{R}^n$ as universal cover of $S \cap (\mathbb{C}^*)^n$.

In [4] all *closed* tube hypersurfaces in $\mathbb{C}^n$ have been classified up to affine equivalence that are locally CR-equivalent to the sphere (2.2). In [11] the same has been done for the pseudo-sphere

$$\{z \in \mathbb{C}^n : \sum_{k<n} z_k \overline{z}_k = 1 + z_n \overline{z}_n\}$$

of signature $(n-1, 1)$. In both cases there is only a finite number of equivalence classes. These were obtained by solving a certain system of second order differential equations coming from [3]. In particular, this method does not extend to the Levi degenerate case nor to higher CR-codimensions.

4. Examples of 2- 3- 4-nondegenerate manifolds

In the following we discuss some of the examples presented in section 5 of [10]: For fixed integers $k \in \{2,3,4\}$ and $c \geq 1$ let $V \subset \mathbb{R}[u,v]$ be the subspace of all homogeneous polynomials of degree $m := k+c-1$. We consider every $p \in V$ as polynomial function on $\mathbb{R}^2$ and also on $\mathbb{C}^2$ if convenient. By $G \subset \mathsf{GL}(V)$ we denote the subgroup of all transformations $p \mapsto \pm p \circ g$ with $g \in \mathsf{GL}(2,\mathbb{R})$. Then G acts irreducibly on V and, as a consequence, for every non-zero G-orbit F in V the corresponding tube manifold $M = F + iV$ is a minimal (not necessarily connected) CR-submanifold of $E := V \oplus iV$. Note that G always has two connected components and is isomorphic to $\mathsf{GL}(2,\mathbb{R})$ if m is odd. The connected identity component G^0 of G consists of all transformations $p \mapsto t{\cdot}(p \circ g)$ with $t > 0$ and $g \in \mathsf{SL}(2,\mathbb{R})$.

By definition, the function $p = p(u,v) \in V$ vanishes of order $\geq d \in \mathbb{N}$ at $(u_0 : v_0) \in \mathbb{P}_1(\mathbb{C})$ if all partial derivatives up to degree $d-1$ vanish at $(u_0, v_0) \in \mathbb{C}^2$. Counted with multiplicities, every $p \neq 0$ has exactly m zeroes. The group $\mathsf{GL}(2, \mathbb{R})$ acts by linear fractional transformations on $\mathbb{P}_1(\mathbb{C})$ with two orbits, $\mathbb{P}_1(\mathbb{R})$ and its complement. For every $g \in \mathsf{GL}(2, \mathbb{R})$ the zeroes of p and $\pm\, p \circ g$ differ by an application of g on $\mathbb{P}_1(\mathbb{C})$.

Now consider the set P of all $p \neq 0$ in V with the following property: All zeroes of p are in $\mathbb{P}_1(\mathbb{R})$, one of these has order $m - (k-2)$ while the remaining $(k-2)$ zeroes have order 1. From the above it is clear that P for our particular choices of k is a G-orbit. Denote by F the connected component of P that contains the polynomial

$$p := \begin{cases} v^m & k = 2 \\ uv^{m-1} & k = 3 \\ (u^2 - v^2)v^{m-2} & k = 4\,. \end{cases}$$

Then the connected identity component G^0 of G acts transitively on F. With the criterion in Section 3 it is easily seen that

$$K_p^r F = \sum_{j=0}^{k-1-r} \mathbb{R}\, u^j v^{m-j} \quad \text{for all} \quad r \geq 0\,.$$

In particular, the tube manifold $M := F + iV$ is a k-nondegenerate homogeneous CR-submanifold of $E = V \oplus iV$ with CR-dimension k and CR-codimension c. For better distinction we also write $F^{k,c} := F$ and $M^{k,c} := M$ in the following.

The CR-manifolds $\mathcal{M}^{k,c}$ from Section 5 in [10] coincide with our $M^{k,c}$ for $k = 2, 3$. The 4-nondegenerate homogeneous CR-submanifold $\mathcal{M}^{4,c}$ is the tube over the G^0-orbit $\mathcal{F}^{4,c}$ defined as follows: It is a connected component of the set Q of all $q \neq 0$ in V having a zero of order $(m-2)$ in $\mathbb{P}_1(\mathbb{R})$ and 2 zeros outside $\mathbb{P}_1(\mathbb{R})$. A polynomial in Q is, for instance, $(u^2 + v^2)v^{m-2}$.

Let us identify $\mathbb{R}^{m+1}$ and V via

$$(x_0, x_1, \ldots, x_m) \;\cong\; \sum_{j=0}^{m} x_j \binom{m}{j} u^j v^{m-j}\,. \tag{4.1}$$

Then $F^{2,c}$, $F^{3,c}$, $F^{4,c}$, $\mathcal{F}^{4,c}$ are respectively the G^0-orbits of the points

$(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, $(-1, 0, 1, 0, \ldots, 0)$, $(1, 0, 1, 0, \ldots, 0)$.

Notice that $F^{2,c} \cup -F^{2,c}$ is the unique G-orbit that is closed in $V \setminus \{0\}$. It is also the unique 2-dimensional G-orbit in V and consists of all $p \in V$ that are a power of a non-zero linear form on V. The $F^{2,c}$ will occur again in Section 5.

Denote for every $x = (x_0, \ldots, x_m)$ by $D(x)$ the resultant of the two polynomials $m^{-1}\partial f/\partial u$ and $m^{-1}\partial f/\partial v$, where $f \in V$ corresponds to x by (4.1). Then D is a homogeneous polynomial of degree $2m-2$ in x and $D(x) = 0$ if and only if the corresponding polynomial f has a multiple zero in $\mathbb{P}_1(\mathbb{C})$. In particular, every F of the form $F^{k,c}$ or $\mathcal{F}^{4,c}$ is contained in the hypersurface $S := \{x \in \mathbb{R}^{m+1} : D(x) = 0\}$. In case $c = 1$ the orbit F is open in the nonsingular part of S and we have the following explicit formula for D:

Case $k = 2$: Here $D(x) = x_0x_2 - x_1^2$. As a consequence,

$$F^{2,1} = \{(x_0, x_1, x_2) \in \mathbb{R}^3 : D(x) = 0 < x_0 + x_2\}$$

and thus $M^{2,1}$ coincides with the future light cone tube $\mathcal{M}$ in (1.3) up to a complex linear isomorphism.

Case $k = 3$: Here $D(x) = x_0^2x_3^2 - 3x_1^2x_2^2 - 6x_0x_1x_2x_3 + 4x_1^3x_3 + 4x_0x_2^3$ (the formula presented in Example 1.22 of [15] is not correct). Furthermore, $D(x) = 0$ if and only if the orbit $G(x)$ has dimension < 4, see [10]. Therefore $F^{3,1}$ consists of all $x \in \mathbb{R}^4$ for which the matrix

$$\begin{pmatrix} 0 & x_1 & 2x_2 & 3x_3 \\ 3x_1 & 2x_2 & x_3 & 0 \\ 0 & x_0 & 2x_1 & 3x_2 \\ 3x_0 & 2x_1 & x_2 & 0 \end{pmatrix}$$

has rank 3. In particular, $F^{3,1}$ is the nonsingular part of S.

Case $k = 4$: Here, compare [15] p. 29, $D(x) = g_2(x)^3 - 27g_3(x)^2$ with

$$\begin{aligned} g_2(x) &:= x_0x_4 - 4x_1x_3 + 3x_2^2 \quad \text{and} \\ g_3(x) &:= x_0x_2x_4 - x_0x_3^2 - x_1^2x_4 + 2x_1x_2x_3 - x_2^3 \,. \end{aligned}$$

The polynomials g_2, g_3 are invariant under the action of the group $\mathsf{SL}(2,\mathbb{R})$ on $V = \mathbb{R}^5$ but not under G^0. This allows to describe the orbit structure of G^0 from the CR-viewpoint in more detail: For $0 \le j \le 4$ let $S^{[j]}$ be the union of all G^0-orbits in S of dimension j, that is, the set of all $x \in S$ for which the matrix

$$\begin{pmatrix} 0 & x_1 & 2x_2 & 3x_3 & 4x_4 \\ 4x_1 & 3x_2 & 2x_3 & x_4 & 0 \\ 0 & x_0 & 2x_1 & 3x_2 & 4x_3 \\ 4x_0 & 3x_1 & 2x_2 & x_3 & 0 \end{pmatrix}$$

has rank j (for every $x \notin S$ the orbit $G^0(x)$ has dimension 4). A further decomposition of S is given by the three G^0-invariant subsets

$$S^{\pm} := \{x \in S : \pm g_3(x) > 0\}\,, \quad S^0 := \{x \in S : g_3(x) = 0\}$$

satisfying $S^- = -S^+$. In a total, S consists of 13 G^0-orbits, more precisely:

(i) $S^+ \setminus S^{[3]} \ = \ S^+ \cap S^{[4]} = F^{4,1} \cup \mathcal{F}^{4,1}$

(ii) $S^+ \cap S^{[3]} \ = \ G^0(0,0,1,0,0) \ \cup \ -G^0(1,0,2,0,1)$

(iii) $S^0 \cap S^{[4]} = S^0 \cap S^{[1]} = \emptyset$

(iv) $S^0 \cap S^{[3]} = F^{3,2} \cup -F^{3,2}$

(v) $S^0 \cap S^{[2]} \ = \ F^{2,3} \cup -F^{2,3}$

with 2-nondegenerate orbits in (ii). The nonsingular part of S is

$$\pm\big(F^{4,1} \ \cup \ \mathcal{F}^{4,1} \ \cup \ -G^0(1,0,2,0,1)\big)$$

and $\overline{F^{4,1}} \cap \overline{\mathcal{F}^{4,1}} \ = \ G^0(0,0,1,0,0) \ \cup \ F^{2,3} \ \cup \ \{0\}$.

The case $k = 3$, $c = 1$ gives some affinely homogeneous tube domains in $\mathbb{C}^4$: As before, $(x_0, x_1, x_2, x_3) = x_0v^3 + 3x_1uv^2 + 3x_2u^2v + x_3u^3$ identifies $\mathbb{R}^4$ and V. The subgroup $G \subset \mathsf{GL}(V) = \mathsf{GL}(4, \mathbb{R})$ is isomorphic to $\mathsf{GL}(2, \mathbb{R})$ and has precisely two open orbits in $\mathbb{R}^4 = V$, namely

$$\Omega^\pm := \{x \in \mathbb{R}^4 : \pm D(x) > 0\}\,.$$

As subset of V then Ω^- consists of all polynomials $p \neq 0$ having three distinct zeroes in $\mathbb{P}_1(\mathbb{R})$ while Ω^+ consists of all $p \neq 0$ with two zeroes outside $\mathbb{P}_1(\mathbb{R})$. This shows that every $\Omega^\pm$ is connected and $\mathcal{D}^\pm := \Omega^\pm + i\mathbb{R}^4$ is an affinely homogeneous tube domain in $\mathbb{C}^4$. From $\Omega^\pm = -\Omega^\pm$ we see that the convex hull of $\Omega^\pm$ contains the origin and hence coincides with $\mathbb{R}^4$. This implies that every holomorphic function on $\mathcal{D}^\pm$ has a holomorphic extension to $\mathbb{C}^4$. In particular, every bounded holomorphic function on $\mathcal{D}^\pm$ is constant. Since $D(x)$ is invariant under the action of $\mathsf{SL}(2, \mathbb{R})$ on $V = \mathbb{R}^4$ the $\mathsf{SL}(2, \mathbb{R})$-orbits in $\Omega^\pm$ are the hypersurfaces $S^\alpha := \{x \in \mathbb{R}^4 : D(x) = \alpha\}$ with $\alpha \in \mathbb{R}^\pm$. With the criterion in Section 3 it is easily seen that all S^α, $\alpha \neq 0$, are Levi nondegenerate.

So far we do not have any example of a locally homogeneous CR-manifold that is k-nondegenerate with $k \geq 5$.

5. Conical manifolds of CR-dimension 2

In this section we discuss linear homogeneous cones F of dimension 2 in a real vector space V of dimension $n \geq 3$. For every such F the corresponding tube $M := F + iV$ is a Levi degenerate homogeneous CR-submanifold of $E := V \oplus iV$ with CR-dimension 2 and CR-codimension $(n-2)$. In case F is contained in a hyperplane $W \subset V$ the manifold M is CR-equivalent to the direct product $(F + iW) \times \mathbb{R}$. We therefore always assume in the following that F is not contained in any hyperplane of V. Then M is automatically minimal and 2-nondegenerate as CR-manifold, compare [10]. In particular, the Lie algebra $\mathfrak{hol}(M, a)$ then has finite dimension for every $a \in M$.

In [10] all M of the above type have been classified up to local CR-equivalence and all $\mathfrak{hol}(M, a)$ have been explicitly determined. For a description we recall some elementary facts: $a \in V$ is called a *cyclic vector* of the endomorphism $\varphi \in \mathsf{End}(V)$ if the powers $\varphi^k(a)$, $k \in \mathbb{N}$, span the vector space V. By $\mathsf{Cyc}(V) \subset \mathsf{End}(V)$ we denote the subset of all *cyclic* endomorphisms, that is, of all φ having a cyclic vector. Every cyclic φ is uniquely determined up to conjugation in $\mathsf{Cyc}(V)$ by the (unordered) sequence $\alpha_1, \ldots, \alpha_n$ of all its characteristic roots in $\mathbb{C}$ (more precisely the roots of the characteristic polynomial of φ with mulitplicities counted). We say that $\alpha_1, \ldots, \alpha_n$ form an *arithmetic progression* if, after a suitable permutation, the differences $(\alpha_{k+1} - \alpha_k)$, $1 \leq k < n$, do not depend on k. In case φ has trace 0 this is equivalent to $\varphi \in \rho(\mathfrak{sl}(2, \mathbb{R}))$ where $\rho : \mathfrak{sl}(2, \mathbb{R}) \to \mathsf{End}(V)$ is the (essentially) unique irreducible Lie algebra representation.

To every $\varphi \in \mathsf{Cyc}(V)$ we associate a linearly homogeneous surface $F = F^\varphi$ in V as follows: For the abelian Lie subalgebra $\mathfrak{h} := \mathfrak{h}^\varphi := \mathbb{R}\,\mathsf{id} \oplus \mathbb{R}\varphi \subset \mathsf{End}(V)$ consider the subgroup $H := \exp(\mathfrak{h}) \subset \mathsf{GL}(V)$. Then $a \in V$ is cyclic for φ if and only if the orbit $H(a)$ is not contained in a hyperplane of V and any two cyclic orbits $H(a)$, $H(a')$ differ by some $g \in \mathsf{GL}(V)$. Therefore, if we fix a cyclic vector $a \in V$ and put $F^\varphi := H(a)$, the CR-structure on the tube $M^\varphi := F^\varphi + iV$ does not depend on the choice of the cyclic vector $a \in V$. Also, this structure only depends on the characteristic roots of φ counted with multiplicities.

The case where $\alpha_1, \ldots, \alpha_n$ form an arithmetic progression: Then there exists a faithful irreducible representation $\rho : \mathfrak{gl}(2, \mathbb{R}) \to \mathsf{End}(V)$ with $\varphi \in \rho(\mathfrak{gl}(2, \mathbb{R}))$ such that every $\psi \in \rho(\mathfrak{gl}(2, \mathbb{R}))$, considered as vector field on V, is tangent to $F^\varphi \subset V$. This means that F^φ is locally linearly equivalent to $F^{2,n-2}$ as considered in Section 4. It can be shown that for every $a \in M^\varphi$

$$\mathfrak{hol}(M^\varphi, a) \cong \begin{cases} \mathfrak{so}(2,3) & n = 3 \\ \mathfrak{gl}(2, \mathbb{R}) \ltimes_\rho V & \text{otherwise.} \end{cases}$$

The case where $\alpha_1, \ldots, \alpha_n$ do not form an arithmetic progression: Without loss of generality we may always assume for M^φ that the cyclic endomorphism φ is tracefree, that is $\alpha_1 + \ldots + \alpha_n = 0$. Then, if φ' is another tracefree cyclic endomorphism with eigenvalues $\alpha'_1, \ldots, \alpha'_n$, the following conditions are equivalent: (compare Prop. 7.6 in [10])

(i) M^φ and $M^{\varphi'}$ are locally (globally) CR-equivalent.

(ii) F^φ and $F^{\varphi'}$ are locally (globally) linearly equivalent.

(iii) Modulo a suitable permutation of the indices in one of the eigenvalue strings, $\alpha'_k = t\alpha_k$ holds for some $t \in \mathbb{R}^*$ and all $1 \le k \le n$.

(iv) $\mathfrak{hol}(M^\varphi)$ and $\mathfrak{hol}(M^{\varphi'})$ are isomorphic as Lie algebras.

Furthermore, $\mathfrak{hol}(M^\varphi, a) \cong \mathfrak{hol}(M^\varphi) \cong \mathfrak{h}^\varphi \ltimes V$ holds for all $a \in M^\varphi$.

Notice that conversely every linearly homogeneous connected cone $F \subset V$ not contained in a hyperplane of V is locally linearly equivalent to some M^φ with $\varphi \in \mathsf{Cyc}(V)$ tracefree.

6. A nonconical manifold of CR-dimension 2

As in Section 4 let $V \subset \mathbb{R}[u,v]$ be the subspace of all homogeneous polynomials of degree $m \ge 3$ and denote by $\Gamma \subset \mathsf{GL}(2,\mathbb{R})$ the subgroup of all transformations $(u,v) \mapsto (ru, tu+v)$ with $t \in \mathbb{R}$ and $r > 0$. Then Γ acts on V by $p \mapsto p \circ g^{-1}$ and leaves the affine hyperplane

$$W := v^m + \sum_{k=1}^{m} \mathbb{R}\, v^k v^{m-k}$$

invariant. Setting $(x_1, x_2, \ldots, x_m) = v^m + \sum_{k=1}^m x_k \binom{m}{k} v^k v^{m-k}$ identifies $\mathbb{R}^m$ with W and Γ becomes a group Σ of affine transformations on $\mathbb{R}^m$. The orbit

$$C := \Sigma(0) = \{(t, t^2, \ldots, t^m) \in \mathbb{R}^m : t \in \mathbb{R}\}$$

is the *twisted m-ic*. Its development $S := \bigcup_{c \in C}(c + T_cC)$ is divided by C into the two linearly equivalent orbits $\Sigma(\pm a)$, where $a := (1, 0, \ldots, 0)$. Then with $\gamma(t) := (t, t^2, \ldots, t^m)$

$$F := \Sigma(a) = \{\gamma(t) + r\gamma'(t) : t \in \mathbb{R},\ r > 0\}$$

is an affinely homogeneous surface in $\mathbb{R}^m$ (compare [6], p. 45 for the special case $m = 3$). The corresponding tube $M := F + i\mathbb{R}^m$ is a homogeneous minimal 2-nondegenerate CR-manifold of CR-dimension 2 not locally CR-equivalent to any of the examples in the previous section. For every $a \in M$ the Lie algebra $\mathfrak{hol}(M, a)$ has dimension $m + 2$.

7. The classification in dimension 5

Our examples so far give the following affinely homogeneous 2-nondegenerate tube manifolds $M := F + i\mathbb{R}^3 \subset \mathbb{C}^3$ of dimension 5, where r, t run over all real numbers with $r > 0$:

(i) $F = \{\, r\,(\cos t, \sin t, 1) \in \mathbb{R}^3\}$ the future light cone,

(ii) $F = \{\, r\,(\cos t, \sin t, e^{\omega t}) \in \mathbb{R}^3\}$ with $\omega > 0$ arbitrary,

(iii) $F = \{\, r\,(1, t, e^t) \in \mathbb{R}^3\}$,

(iv) $F = \{\, r\,(1, e^t, e^{\theta t}) \in \mathbb{R}^3\}$ with $\theta > 2$ arbitrary,

(v) $F = \{\, \gamma(t) + r\gamma'(t) \in \mathbb{R}^3\}$, where $\gamma(t) := (t, t^2, t^3)$ parameterizes the twisted cubic and $\gamma'(t) = (1, 2t, 3t^2)$.

In case (i) $M \cong \mathcal{M}$, see (1.3), and $\mathfrak{hol}(M) \cong \mathfrak{hol}(M, a) \cong \mathfrak{so}(2, 3)$ is semisimple with dimension 10 for all $a \in M$. In all other cases M is simply connected and $\mathfrak{hol}(M) \cong \mathfrak{hol}(M, a)$ is a solvable Lie algebra of dimension 5.

In [6], compare also [5], all affinely homogeneous surfaces in $\mathbb{R}^3$ have been classified up to local affine equivalence. Among the equivalence classes the surfaces F in (i) - (v) represent precisely those classes that can be given in a neighbourhood U of the origin in $\mathbb{R}^3$ as

$$\{(x, y, z) \in U : z = f(x, y)\}$$

with f a convergent power series of the form

$$f(x, y) = x^2 + x^2 y + \text{higher order terms.}$$

The surfaces F above are pairwise locally affinely inequivalent and even give pairwise locally CR-inequivalent CR-manifolds (since the Lie algebras $\mathfrak{hol}(M, a)$ are pairwise non-isomorphic). Note that this property is not evident a priori: In [11] two CR-equivalent affinely homogeneous tube manifolds in $\mathbb{C}^3$ are given that are locally affinely inequivalent. These examples are Levi nondegenerate and hence not 2-nondegenerate as in our situation. The main result of [10] now states:

7.1 Theorem. *Every locally homogeneous* 2*-nondegenerate CR-manifold of dimension* 5 *is locally CR-equivalent to a tube* $M = F + i\mathbb{R}^3$ *with* F *a unique surface from the list in (i) – (v) above.*

Since every locally homogeneous Levi degenerate CR-manifold M of dimension 5 is either 2-nondegenerate or locally of the form $M' \times \mathbb{C}$, Cartan's classification [2] in dimension 3 together with Theorem 7.1 also gives a local classification of all locally homogeneous Levi degenerate CR-manifolds in dimension 5. For a local classification of a certain class of Levi nondegenerate hypersurfaces in $\mathbb{C}^3$ compare [14].

The tube manifolds $M = F + i\mathbb{R}^3$ with F from (i) - (v) are all non-closed hypersurfaces in $\mathbb{C}^3$ and we may ask: *Do there exist closed hypersurfaces* $N \subset \mathbb{C}^3$ *that are locally homogeneous and* 2*-nondegenerate?* At least tube submanifolds N of this type cannot exist that are locally CR-equivalent to one of the manifolds $M = F + i\mathbb{R}^3$ with F from (ii) - (v). This follows from the fact that for those M in $\mathfrak{hol}(M, a)$ there is a unique abelian subalgebra of dimension 3.

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