CR-manifolds and Jordan algebras

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1. CR-manifolds

Assume that E is a complex vector space of finite dimension and $M \subset E$ is a (locally-closed) connected real-analytic submanifold. Then for every $a \in M$ the tangent space T_aM is an $\mathbb{R}$-linear subspace of E and $H_aM := T_aM \cap iT_aM$ is the maximal complex linear subspace of E contained in T_aM. The submanifold M is called a CR-submanifold of E if the complex dimension of H_aM does not depend on $a \in M$. This dimension is called the CR-dimension of M and H_aM is called the holomorphic tangent space at a, compare [2] as general reference for CR-manifolds. Also, the real dimension of T_aM/H_aM is called the CR-codimension of M. For a further real-analytic CR-submanifold M' of a complex vector space E' a smooth mapping $\varphi : M \to M'$ is called CR if the differential $d\varphi_a : T_aM \to T_{\varphi a}M'$ maps the corresponding holomorphic tangent spaces in a complex linear way to each other. In terms of differential equations this just means that φ as E-valued mapping satisfies the Cauchy-Riemann partial differential equations at every point of M in the direction of the holomorphic tangent space. In particular, in case M is a complex analytic submanifold of E, then always $H_aM = T_aM$ and 'CR' just means 'holomorphic'.

The notion of a CR-function and hence of a CR-mapping can also be reformulated in terms of distributions and then also applies to not necessarily smooth mappings. In particular, due to the approximation theorem of Baouendi-Treves [1]), a continuous map $\varphi : M \to M'$ is CR if and only if it is locally the uniform limit of a sequence of smooth CR-mappings. It is a specialty of the Cauchy-Riemann differential equations that compositions of CR-mappings are also CR and, in particular, that the space $\mathcal{C}_{\text{CR}}(M)$ of all continuous CR-functions $f : M \to \mathbb{C}$ is a complex algebra with identity.

Typical questions in this context are for instance: When are two CR-manifolds M, M' CR-isomorphic, and if so, what is the space of all CR-isomorphisms $M \to M'$. Furthermore, what are the subsets $A \subset E$ with $M \subset A$ such that every continuous CR-function on M has a unique continuous extension to A that is 'holomorphic' in a suitable sense. Also, what is the spectrum (maximal ideal space) of the algebra $\mathcal{C}_{\text{CR}}(M)$?

The following is a well-known example: Let $E = \mathbb{C}^n$ and $(z|w) = \sum z_j \bar{w}_j$ the standard hermitian form on E. Then the unit sphere $S := \{z \in E : (z|z) = 1\}$ is a CR-submanifold on which the unitary group $\mathsf{U}(n)$ acts transitively by CR-transformations. For every $a \in S$ the holomorphic tangent space at a is $H_aS = \{z \in E : (z|a) = 0\}$ and $T_a = \mathbb{R}ia \oplus H_aS$.

2. Formally real Jordan algebras and associated CR-manifolds

A real vector space V together with a symmetric bilinear product $(x, y) \mapsto x \circ y$ is called a (real) Jordan algebra, if

$$x^2 \circ (x \circ y) = x \circ (x^2 \circ y)$$

holds for all $x, y \in V$, where $x^2 := x \circ x$. If we denote by L_x the multiplication operator $y \mapsto x \circ y$, then the above equation just means that L_x and L_{x^2} commute for every $x \in V$. Because of $2x \circ y = (x+y)^2 - x^2 - y^2$ the Jordan product is uniquely determined by the squaring map $x \mapsto x^2$ on V.

The real Jordan algebra $V \neq 0$ of finite dimension is called formally real [3] or euclidian [5] if $x^2 + y^2 = 0$ always implies $x = y = 0$, or equivalently, if the trace form $\langle x, y\rangle := \mathsf{tr}(L_x L_y)$ on V is positive definite. Assume in the following that V is formally real. Then V always has an identity e and there exists an integer $r \geq 1$ (called the rank of V) such that every $a \in V$ is an $\mathbb{R}$-linear combination

$$a = \lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_r e_r \,,$$

where the $e_1, \ldots, e_r$ form an orthogonal family of non-zero idempotents in V (i.e. $e_j \circ e_k = \delta_{jk} e_k$) with $e = e_1 + \ldots + e_r$. The coefficients λ_j are uniquely determined up to order and are called the eigenvalues of a.

For all integers $p \geq q \geq 0$ with $0 < p + q \leq r$ we denote by $C_{p,q}$ the cone of all $a \in V$ having precisely p positive and q negative eigenvalues. Then $C_{p,q}$ is open in V if and only if $p + q = r$. Of a particular interest is the positive cone $\Omega := C_{r,0}$, which is open, convex and coincides with the interior of the closed cone $\{x^2 : x \in V\}$. The linear group

$$G = \mathsf{GL}(\Omega) := \{g \in \mathsf{GL}(V) : g(\Omega) = \Omega\}$$

is a reductive Lie group and its connected identity component G^0 acts transitively on every connected component of $C_{p,q}$. In particular, every cone $C_{p,q}$ is a locally linearly homogeneous real-analytic submanifold of V. Furthermore, the isotropy subgroup

$$\{g \in G : g(e) = e\} = \{g \in \mathsf{GL}(V) : g(x \circ y) = g(x) \circ g(y) \ \text{ for all } \ x, y \in V\} =: \mathsf{Aut}(V)$$

at the identity e is a maximal compact subgroup of G.

Now denote by $E := V \oplus iV$ the complexification of V. The complex bilinear extension of the Jordan product from V makes E to a complex Jordan algebra also with identity e. Denote by

$$D := \Omega \times iV \ \subset \ E$$

the tube domain over Ω (up to a factor i a generalized upper halfplane). Then the group $G \ltimes V$ acts transitively by affine transformations on D and $z \mapsto (z - e) \circ (z + e)^{-1}$ defines a biholomorphic mapping from D to a bounded circular convex domain in E. Therefore D is a symmetric tube domain (a bounded symmetric domain of tube type), compare [7], [11] for details. On the other hand, every symmetric tube domain arises from a uniquely determined formally real Jordan algebra in the above way.

V is the direct sum of all its minimal ideals, we may therefore assume without loss of generality in the following that the formally real Jordan algebra V is simple. Then $C_{p,q}$ is connected and hence an orbit of the group $G = \mathsf{GL}(\Omega)$. Now let

$$M_{p,q} := C_{p,q} \times iV \ \subset \ E$$

be the tube manifold over $C_{p,q}$. Then the group $G \ltimes V$ acts transitively on $M_{p,q}$ by affine transformations and $M_{p,q}$ is a homogeneous CR-submanifold of E. Some of the results obtained in [10] can be stated as follows: Suppose that V' is another simple formally real Jordan algebra of rank r' and $p' \geq q' \geq 0$ are integers with $0 < p' + q' \leq r'$. Then the tube manifolds $M_{p,q}$ and $M'_{p',q'}$ are CR-isomorphic if and only if the Jordan algebras V, V' are isomorphic and $p = p'$ as well as $q = q'$ holds. In case $p + q < r$ the tube manifolds $M_{p,q}$ and $M'_{p',q'}$ are already locally CR-equivalent if and only if they are globally

CR-equivalent. Also in case $p + q < r$ every continuous CR-function on $M_{p,q}$ has a unique continuous extension to the convex hull

$$\hat{M}_{p,q} = \begin{cases} E & q > 0 \\ \bigcup_{m \geq p} M_{m,0} & q = 0 \end{cases}$$

of $M_{p,q}$ that is holomorphic on the interior (compare also Proposition 5.2 in [6] for the last statement).

2.1 Example. Let H be a real Hilbert space of finite dimension $n \geq 2$, that is, a real vector space together with a positive definite symmetric bilinear form $(x, y) \mapsto \langle x|y\rangle$ and corresponding norm $\|x\| = \sqrt{\langle x|x\rangle}$. Then $V := \mathbb{R} \oplus H$ is a simple formally real Jordan algebra of rank 2 with respect to $(s, x) \circ (t, y) := (st + \langle x|y\rangle, sy + tx)$. The positive cone is $\Omega = \{(s, x) \in V : s > \|x\|\}$, the future cone in $n+1$-dimensional space time. Its smooth boundary part is the future light cone $C_{1,0} = \{(s, x) \in V : s = \|x\| > 0\}$.

2.2 Example. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$ and $\mathcal{H}_r(\mathbb{K}) := \{x \in \mathbb{K}^{r\times r} : x = x^*\}$ the $\mathbb{R}$-linear space of all hermitian $r \times r$-matrices over $\mathbb{K}$. Then $V := \mathcal{H}_r(\mathbb{K})$ is a simple formally real Jordan algebra of rank r with respect to $x \circ y := (xy + yx)/2$. The notion of eigenvalue and inverse coincides with the usual one for matrices. In particular, Ω is the open cone of positive definite matrices.

Example 2.2 can be extended by taking $\mathcal{H}_r(\mathbb{H})$ for the skew-field $\mathbb{H}$ of quaternions and also by $\mathcal{H}_3(\mathbb{O})$ for the division algebra $\mathbb{O}$ of octonions. Together with these extensions the above examples give a complete list of all simple formally real Jodan algebras.

3. Hermitian Jordan triple systems

For the next bunch of CR-manifolds let us recall the classical singular value decomposition of rectangular matrices: Fix integers $n \geq r \geq 1$ and let $E := \mathbb{C}^{r\times n}$ be the space of all complex $r \times n$-matrices. Then to every $a \in E$ there exist unitary matrices $u \in \mathsf{U}(r)$ and $v \in \mathsf{U}(n)$ such that $d := uav \in E$ is a real diagonal matrix with diagonal entries $d_{11} \geq d_{22} \geq \ldots \geq d_{rr} \geq 0$, called the singular values of a. In particular, for every $a \neq 0$ the subset $S \subset E$ of all matrices having the same singular values as a is a generalization of the euclidian spheres in $\mathbb{C}^n$ (which occur for $r = 1$).

A singular value decomposition exists in every positive hermitian Jordan triple system. This is a complex vector space E of finite dimension together with a map (called Jordan triple product)

$$E \times E \times E \to E, \qquad (x, y, z) \mapsto \{xyz\},$$

that satisfies the following properties, compare [11]:

(i) $\{xyz\}$ is symmetric complex bilinear in the outer variables x, z and conjugate linear in the inner variable y,

(ii) $\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$ for all $a, b, x, y, z \in E$,

(iii) $\{xxx\} = \lambda x$ implies $x = 0$ or $\lambda \in e^{\mathbb{R}}$ for all $x \in E$ and $\lambda \in \mathbb{C}$.

Examples are for instance every $E = \mathbb{C}^{r\times n}$ with $\{xyz\} = (xy^*z + zy^*x)/2$, but also the subspaces $\{z \in \mathbb{C}^{n\times n} : z' = z\}$ of all symmetric, as well as $\{z \in \mathbb{C}^{n\times n} : z' = z\}$ of all skew-symmetric $n \times n$-matrices with the triple product restricted from $\mathbb{C}^{n\times n}$. Also, for every formally real Jordan algebra V the complexification $E = V \oplus iV$ becomes a positive hermitian Jordan triple system with respect to the triple product $\{xyz\} = (x \circ \bar{y}) \circ z + (z \circ \bar{y}) \circ x - x^2 \circ \bar{y}$, where $y \mapsto \bar{y}$ is the conjugation of E with respect to the real form V of E.

Now fix an arbitrary positive hermitian Jordan triple system $E \neq 0$. The element $e \in E$ is called a tripotent if $\{eee\} = e$ holds. Two tripotents $e, c \in E$ are called orthogonal if $\{eec\} = 0$, or equivalently, if $\{cce\} = 0$ holds. There exists an integer $r \geq 1$, called the rank of E, such that every $a \in E$ has a representation

$$a = \sigma_1 e_1 + \sigma_2 e_2 + \ldots , \sigma_r e_r +$$

with pairwise orthogonal non-zero tripotents $e_1, e_2, \ldots , e_r$ and uniquely determined real coefficients

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq 0 \,.$$

For every k with $1 \leq k \leq r$ the number $\sigma_k(a) := \sigma_k$ is called the k^{th} singular value of a. It is known that for every k the sum $\alpha_k := \sigma_1 + \sigma_2 + \ldots + \sigma_r : E \to \mathbb{R}$ defines a norm on E. In particular, the open unit ball $D := \{z \in E : \sigma_1(z) < 1\}$ with respect to the norm $\sigma_1 = \alpha_1$ is homogeneous under biholomorphic automorphisms and hence a bounded symmetric domain, compare [7] and [11] for details. On the other hand, every bounded symmetric domain occurs as the open σ_1-unit ball of a suitable positive hermitian Jordan triple system E.

The triple automorphism group

$$\mathsf{Aut}(E) := \big\{g \in \mathsf{GL}(E) : g\{xyz\} = \{(gx)(gy)(gz)\} \text{ for all } x, y, z\big\}$$

is compact and coincides with the group of all σ_1-isometries of E. Clearly, $\mathsf{Aut}(E)$ leaves invariant the mapping $\sigma := (\sigma_1, \ldots , \sigma_r) : E \to \mathbb{R}^r$. Denote by K the connected identity component of $\mathsf{Aut}(E)$. In case of $E = \mathbb{C}^{r\times n}$, for instance, K is the group of all transformations $z \mapsto uzv$ with $u \in \mathsf{U}(r)$ and $v \in \mathsf{U}(n)$.

Since E is the direct sum of its minimal ideals (the linear subspace $I \subset E$ is called an ideal, if $\{IEE\} + \{EIE\} \subset E$) we assume in the following that E is simple, or equivalently, that K acts irreducibly on E. For every s in

$$\Delta := \{s \in \mathbb{R}^r : 1 = s_1 \geq s_2 \geq \ldots \geq s_r \geq 0\}$$

the set

$$M_s := \{z \in E : \sigma(z) = s\}$$

is a K-orbit and hence a homogeneous CR-submanifold of E. On the other hand, every non-zero K-orbit obviously is of the form $t{\cdot}M_s$ for uniquely determined $t > 0$ and $s \in \Delta$ and hence is CR-equivalent to M_s. The following statements can be found in [9] and [8].

For every $a \in M_s$ The linear convex hull of M_s is given by

$$\text{ch}(M_s) = \{z \in E : \alpha_k(z) \leq \alpha_k(a) \text{ for all } k\} \,.$$

If the multiplicative analogon to the norms α_k is defined as the product $\mu_k := \sigma_1\sigma_2 \cdots \sigma_k$ then the polynomial convex hull of M_s can be written as

$$\text{pch}(M_s) = \{z \in E : \mu_k(z) \leq \mu_k(a) \text{ for all } k\} \,.$$

For the formualtion of further results on has to distinguish the two cases $a \in E$ invertible and $a \in E$ non-invertible. By definition, a is invertible in E if the conjugate linear operator $z \mapsto \{aza\}$ is invertible on E (in case $E = \mathbb{C}^{r\times n}$ invertibility of a is just the usual notion for matrices, i.e. $r = n$ and $\det(a) \neq 0$).

For simplicity let us assume for the rest of the section that $a \in M_s$ and hence every element of M_s is non-invertible. Then M_s is a minimal Levi-nondegenerate CR-manifold and is CR-isomorphic to M_t, $t \in \Delta$, if and only $t = s$. Also, every continuous CR-function f on M_s has a unique continuous extension to the polynomial convex hull pch(M_s) which is holomorphic in a certain sense (in particular is holomorphic in the usual sense on the interior of pch(M_s) $\subset E$ if not empty). In fact, pch(M_s) can be canonically be identified with the spectrum of the Banach algebra $\mathcal{C}_{\text{CR}}(M_s)$.

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