A rationality criterion for biholomorphic mappings

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In [6], Theorem 10, the following result has been obtained:

Proposition. *Every biholomorphic mapping between Siegel domains in* $\mathbb{C}^n$ *is birational.*

Siegel domains generalize the classical Siegel's upper halfplanes which themselves are generalizations of the usual upper halfplane in $\mathbb{C}$. By definition a general Siegel domain is given by the following data:

(i) An open convex cone $\Omega \subset \mathbb{R}^p$ containing no affine real line.

(ii) An Ω-definite hermitian map $F : \mathbb{C}^q \times \mathbb{C}^q \to \mathbb{C}^p$, that is,

(1) F is conjugate linear in the first and complex linear in the second variable.

(2) $F(w, w) \in \Omega$ for all $w \in \mathbb{C}^q$ with $w \neq 0$.

With these data then the corresponding Siegel domain is defined as

$$D = D_{\Omega,F} := \big\{(v, w) \in \mathbb{C}^p \oplus \mathbb{C}^q : \mathrm{Im}(v) - F(w, w) \in \Omega\big\}\,.$$

Every Siegel domain D as above is biholomorphically equivalent to a bounded domain in $\mathbb{C}^n$, $n = p+q$. Also any two of them are biholomorphically equivalent if and only if they are affinely equivalent, see Theorem 11 in [6]. The main relevance of these domains, however, stems from the fact that every bounded *homogeneous* domain in $\mathbb{C}^n$ can be realized as a Siegel domain, see [8].

The easiest way to prove the above proposition seems to be the following: Let D in $\mathbb{C}^n$ be a Siegel domain. Since D is equivalent to a bounded domain its biholomorphic automorphism group $G := \mathrm{Aut}(D)$ is a real Lie group. The Lie algebra $\mathfrak{g}$ of G can be canonically identified with a Lie algebra of holomorphic vector fields on D. By a *holomorphic vector field* ξ *on* D we mean a holomorphic differential operator (acting on arbitrary holomorphic vector valued functions defined on D)

$$\xi = f(z)\,{\partial}/{\partial z} = f_1(z)\,{\partial}/{\partial z_1} + f_2(z)\,{\partial}/{\partial z_2} + \ldots + f_n(z)\,{\partial}/{\partial z_n}$$

with $f = (f_1, \ldots, f_n) : D \to \mathbb{C}^n$ holomorphic. The bracket of the two holomorphic vector fields $\xi = f\,{\partial}/{\partial z}$, $\eta = g\,{\partial}/{\partial z}$ on D is given by $[\xi, \eta] = (\xi g - \eta f)\,{\partial}/{\partial z}$.

Assume that $D = D_{\Omega,F}$ is given as above with $p + q = n$. As short hand let us write $E := \mathbb{C}^n$, $V := \mathbb{C}^p$ and $W := \mathbb{C}^q$ so that $E = V \oplus W$ and every $z \in E$ is of the form $z = (v, w)$ with $v \in V$, $w \in W$. We also write $\xi = f\,{\partial}/{\partial z} = g\,{\partial}/{\partial v} + h\,{\partial}/{\partial w}$, where $g : D \to V$ and $h : D \to W$ are the partial maps of f.

Since G contains all translations $(v, w) \mapsto (v + \alpha, w)$, $\alpha \in V$, the Lie algebra $\mathfrak{g}$ contains all constant vector fields $\alpha\,{\partial}/{\partial v}$ with $\alpha \in V$. Also it is easy to see that $\mathfrak{g}$ contains all affine vector fields $2iF(\beta, w)\,{\partial}/{\partial v} + \beta\,{\partial}/{\partial w}$ with $\beta \in W$, as well as the linear vector fields $2v\,{\partial}/{\partial v} + w\,{\partial}/{\partial w}$ and $iw\,{\partial}/{\partial w}$.

It is convenient to consider $\mathfrak{l} := \mathfrak{g} + i\,\mathfrak{g} \subset \mathfrak{hol}(D)$, where $\mathfrak{hol}(D)$ is the complex Lie algebra of all holomorphic vector fields on D. Then it is clear that the complex Lie algebra $\mathfrak{l}$ contains the affine Lie algebra

$$\mathfrak{s} := \{(tz + \alpha)\,{\partial}/{\partial z} : t \in \mathbb{C}, \alpha \in E\} \;\subset\; \mathfrak{hol}(D)$$

and this forces $\mathfrak{g} \subset \mathfrak{P}$, where $\mathfrak{P}$ is the complex Lie algebra of all polynomial holomorphic vector fields on E. Indeed, fix a point $a \in D$ and consider for $\theta := (z-a)\,\partial/\partial z \in \mathfrak{s} \subset \mathfrak{l}$ the endomorphism $\Theta := \mathrm{ad}(\theta) \in \mathrm{End}(\mathfrak{l})$, $\Theta(\xi) = [\theta, \xi]$. For given $\xi = f(z)\,\partial/\partial z \in \mathfrak{l}$ then expand f into a power series $f(z) = \sum_{k \ge 0} f_k(z-a)$ about a, where every $f_k : E \to E$ is a homogeneous polynomial map of degree k. Elementary calculus then gives $\Theta(\xi) = \sum_{k \ge 0}(k-1) f_k(z-a) \in \mathfrak{l}$ near $a \in D$. Since $\mathfrak{l}$ has finite dimension, the spectrum of Θ must be finite as well, that is, $f_k = 0$ for k big enough and thus $\xi \in \mathfrak{P}$.

With these preparations the proof of the above proposition follows with the following criterion that seems to date back to Koecher [7], compare also [5], p. 511 and [3]:

Rationality criterion. *Let $g : D_1 \to D_2$ be a biholomorphic map between domains in E and $g^* : \mathfrak{hol}(D_2) \to \mathfrak{hol}(D_1)$ the induced Lie algebra isomorphism. Then g extends to a rational map on E if the Lie subalgebra $g^*(\mathfrak{s}) \subset \mathfrak{hol}(D_1)$ contains only polynomial vector fields.*

Proof of the criterion. For every $\xi = f(z)\,\partial/\partial z \in \mathfrak{hol}(D_2)$ we have by definition

$$(*) \qquad g^*(\xi) = \big(g'(z)^{-1} f(g(z))\big)\,\partial/\partial z \,,$$

where $g'(z) \in \mathrm{GL}(E)$ for every $z \in D_1$ is the derivative of g at z. Because of $g^*(\mathfrak{s}) \subset \mathfrak{P}$ we can define polynomial maps

$$p : E \to E \quad \text{and} \quad q : E \to \mathrm{End}(E) \quad \text{by}$$

$$g^*(z\,\partial/\partial z) = p(z)\,\partial/\partial z \quad \text{and} \quad g^*(\alpha\,\partial/\partial z) = (q(\alpha)z)\,\partial/\partial z \quad \text{for all } \alpha \in E\,.$$

Together with $(*)$ this implies $p(z) = g'(z)^{-1} g(z)$ and $q(z) = g'(z)^{-1}$ and finally $g(z) = q(z)^{-1} p(z)$ for all $z \in D_1$. But the matrix fractional transformation $q^{-1}p$ is rational on E.

Proof of the proposition. Let $g : D_1 \to D_2$ be a biholomorphic mapping between Siegel domains in $E = \mathbb{C}^n$. Let furthermore $\mathfrak{g}_j$ be the Lie algebra of $\mathrm{Aut}(D_j)$ and $\mathfrak{l}_j := \mathfrak{g}_j + i\,\mathfrak{g}_j \subset \mathfrak{hol}(D_j)$ for $j = 1, 2$. Then $\mathfrak{l}_j$ consists of polynomial vector fields and contains the subalgebra $\mathfrak{s}$. In particular, $g^*(\mathfrak{s}) \subset \mathfrak{l}_1$ implies that g is rational. But g^{-1} is rational by the same argument giving that g is birational on E. □

The rationality criterion can be extended. For this denote by $\mathfrak{R}$ the complex Lie algebra of all rational vector fields on E. Clearly, $\mathfrak{R}$ contains $\mathfrak{P}$ as a Lie subalgebra.

Generalized rationality criterion. Let $g : D_1 \to D_2$ be a locally biholomorphic map between domains in E and denote by $g^* : \mathfrak{hol}(D_2) \to \mathfrak{hol}(D_1)$ the injective Lie algebra homomorphism defined by $(*)$ above. Assume that $\mathfrak{l} \subset \mathfrak{hol}(D_2)$ is a linear subspace with the following properties:

(iii) $\mathfrak{l}$ contains every constant vector field $\alpha\,\partial/\partial z$, $\alpha \in E$.

(iv) $\mathfrak{l}$ contains a vector field $h(z)\,\partial/\partial z$, where h is birational on E.

Then g is rational if $g^*(\mathfrak{l})$ is contained in $\mathfrak{R}$.

Proof. We define the rational maps

$$p : E \to E \quad \text{and} \quad q : E \to \mathrm{End}(E)$$

as above with the only modification $g^*(h(z)\,\partial/\partial z) = p(z)\,\partial/\partial z$. This implies as before $h(g(z)) = q(z)^{-1} p(z)$ and hence the rationality of g. □

Notice that condition (iv) is already satisfied for every $h \in \mathrm{GL}(E)$. On the other hand the above reasoning for the Euler field $\eta = z\,{}^{\partial}\!/_{\partial z}$ can be extended.

Lemma. *Let* $D \subset E$ *be a domain and* $\mathfrak{l} \subset \mathfrak{hol}(D)$ *a complex linear subspace satisfying* (iii), (iv) *with a linear operator* $h \in \mathrm{GL}(E)$ *such that* $[\eta, \mathfrak{l}] \subset \mathfrak{l}$ *for* $\eta := h(z)\,{}^{\partial}\!/_{\partial z}$ *. Then, if the endomorphism* h *is semi-simple on* E *and* $\mathrm{Re}(\lambda) > 0$ *for every eigenvalue* λ *of* h*, every vector field in* $\mathfrak{l}$ *is polynomial, that is* $\mathfrak{l} \subset \mathfrak{P} \subset \mathfrak{R}$*.*

Proof. By assumption we may assume that $\eta = \sum_{k=1}^{n} \lambda_k z_k\,{}^{\partial}\!/_{\partial z_k}$ with $\mathrm{Re}(\lambda_k) > 0$ for all k. Then the mapping

$$\psi : \mathbb{N}^n \times \{1, \ldots, n\} \to \mathbb{C}\,, \qquad (\nu, j) \longmapsto \Big(\sum_{k=1}^{n} \nu_k \lambda_k\Big) - \lambda_j$$

has finite fibers, and for every monomial vector field $\xi = z^\nu\,{}^{\partial}\!/_{\partial z_j}$ we have $[\eta, \xi] = \psi(\nu, j)\xi$, that is, ξ is an eigenvector of $\Theta := \mathrm{ad}(\eta)$ to the eigenvalue $\psi(\nu, j)$.
Because of (iii) we may assume that D contains the origin $0 \in E = \mathbb{C}^n$. Expanding every vector field in $\mathfrak{l}$ about 0 into a series of monomial vector fields we see that Θ splits $\mathfrak{l}$ into a finite direct sum of Θ-eigenspaces and all these consist of polynomial vector fields. □

The extended rationality criterion and the lemma show that for the proof of the proposition it is not necessary to use $iw\,{}^{\partial}\!/_{\partial w} \in \mathfrak{g}$ (as we did by assuring $z\,{}^{\partial}\!/_{\partial z} \in \mathfrak{l} = \mathfrak{g} + i\mathfrak{g}$), it is enough to know $(2v\,{}^{\partial}\!/_{\partial v} + w\,{}^{\partial}\!/_{\partial w}) \in \mathfrak{g}$.

The case of CR-submanifolds of E. Suppose that $M, M' \subset E = \mathbb{C}^n$ are real submanifolds and $a \in M$, $a' \in M'$ are given points. A classical problem is the following: When can we find domains $D, D' \subset E$ together with a biholomorphic map $g : D \to D'$ such that $a \in D$, $a' = g(a)$ and $D' \cap M' = g(D \cap M)$? If such a g exists it is uniquely determined by the restriction $g_{|D\cap M}$, provided M is *generically embedded at* a, that is, the real tangent space $T_aM \subset E$ satisfies $T_aM + i\,T_aM = E$. For simplicity we assume for the following that this property is always satisfied and also that the M, M' are real-analytic submanifolds of E. In addition we assume that M (and also M') is a CR-submanifold of E, that is, that the complex dimension of $T_xM \cap i\,T_xM$ does not depend on $x \in M$. As general reference for CR-manifolds we refer to [1].

Under the above assumptions we denote by $\mathfrak{hol}(M, a)$ the real Lie algebra of all germs of holomorphic vector fields $\xi = f(z)\,{}^{\partial}\!/_{\partial z}$ that are defined in an arbitrary open neighbourhood $U \subset E$ of a and are tangent to $U \cap M$. Since M is assumed to be generically embedded at a we can consider $\mathfrak{hol}(M, a)$ in a canonical way as real Lie subalgebra of the complex Lie algebra $\mathfrak{hol}(E, a)$. It is well understood when $\mathfrak{hol}(E, a)$ is of finite dimension, see [1] for details. We give some examples that fit into our discussion above:

For the Siegel domain $D = D_{\Omega,F}$ the boundary part

$$S := \big\{(v, w) \in \mathbb{C}^p \oplus \mathbb{C}^q : \mathrm{Im}(v) - F(w, w) = 0\big\}$$

(the *Shilov boundary* of the unbounded domain D) is a CR-submanifold with $\mathfrak{hol}(S, a) \cong \mathfrak{g}$ for every $a \in S$, where $\mathfrak{g}$ is the Lie algebra of the biholomorphic automorphism group $G = \mathrm{Aut}(D)$. Usually, S is defined by the above formula in more generality – instead of (2) in (ii) only the following condition is required

(2') $F(w_1, w_2) = 0$ for all w_1 implies $w_2 = 0$ and $\{F(w, w) : w \in \mathbb{C}^q\}$ spans $\mathbb{C}^p$.

Then S is called a *standard* quadric in $E = \mathbb{C}^n$ of CR-codimension p, and a *non-degenerate hyperquadric* if $p = 1$. For every $a \in S$ and $\mathfrak{g} := \mathfrak{hol}(S, a)$ the complex Lie algebra

$\mathfrak{l} = \mathfrak{g} + i\mathfrak{g} \subset \mathfrak{hol}(E,a)$ contains the affine Lie algebra $\mathfrak{s}$ implying $\mathfrak{g} \subset \mathfrak{P}$. In fact, it is known that all vector fields in $\mathfrak{g}$ are even polynomials of degree ≤ 2, see [2]. The rationality criterion therefore gives the following statement, that is already contained in [9]:
Let $D, D' \subset E$ be domains and let $S, S' \subset E$ be standard quadrics with $D \cap S \neq \emptyset$. Then every biholomorphic mapping $g : D \to D'$ with $g(D \cap S) = D' \cap S'$ is rational.

We discuss a second type of CR-submanifolds M of $E = \mathbb{C}^n$ such that for $\mathfrak{l} := \mathfrak{hol}(M,a) + i\,\mathfrak{hol}(M,a)$ condition (iii) is automatically satisfied: Let $N \subset \mathbb{R}^n$ be a real-analytic submanifold. Then $M := \mathbb{R}^n + iN \subset \mathbb{C}^n$ is a generically embedded CR-submanifold, called the *tube manifold with base* N.
Now assume that the vector field $\eta = \lambda_1 \partial/\partial x_1 + \ldots + \lambda_n \partial/\partial x_n$ is tangent to $N \subset \mathbb{R}^n$, where $\lambda_1, \ldots, \lambda_n > 0$ are real coefficients. Then, if $\mathfrak{g} = \mathfrak{hol}(M,a)$ has finite dimension for the tube $M = \mathbb{R}^n + iN$, every vector field in $\mathfrak{g}$ is polynomial and every biholomorphic map $g : D_1 \to D_2$ between domains of E with $D_1 \cap M \neq \emptyset$ and $g(D_1 \cap M) \subset M$ is rational. An example for this situation is the *twisted n-ic* $N := \{(t, t^2, \ldots, t^n) : t \in \mathbb{R}\}$. Here the vector field $\eta = \sum_{k=1}^n k z_k \partial/\partial x_k$ is tangent to $N \subset \mathbb{R}^n$.

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