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On local CR-transformations of Levi-degenerate group orbits in compact Hermitian symmetric spaces

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Abstract. We present a large class of homogeneous 2-nondegenerate CR-manifolds M , both of hypersurface type and of arbitrarily high CR-codimension, with the following property: Every CR-equivalence between domains U, V in M extends to a global real-analytic CR-automorphism of M . We show that this class contains G -orbits in Hermitian symmetric spaces Z of compact type, where G is a real form of the complex Lie group $\text{Aut}(Z)^0$ and G has an open orbit that is a bounded symmetric domain of tube type.

Keywords. Hermitian symmetric spaces, tube domains, Jordan algebras, CR-equivalence problems, 2-nondegenerate CR-manifolds

1. Introduction

Let S be a real-analytic hypersurface in \mathbb{C}^n or, more generally, a CR-submanifold in a complex manifold Z . This paper addresses the question when a local biholomorphic map between open sets in Z sending an open piece of S into S extends to a global biholomorphic self-map of Z preserving S . This question has been treated by various authors when S is a compact hypersurface and its Levi form is nondegenerate at least at some points [26], [1], [29], [21], [28], [5], [11], [24].

However, if S is not compact or is of higher codimension or its Levi form is everywhere degenerate, the question seems to be widely open, even for a basic example such as the tube

$$M := \left\{ z \in \mathbb{C}^3 : x_3 = \sqrt{x_1^2 + x_2^2} > 0 \right\}$$

over the 2-dimensional future light cone, where $z = (z_1, z_2, z_3)$ and $x_k = \text{Re}(z_k)$. Here M is the smooth boundary part of the associated tube domain (the interior of the convex hull of M)

$$H := \left\{ z \in \mathbb{C}^3 : x_3 > \sqrt{x_1^2 + x_2^2} \right\}$$

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over the corresponding future cone, whose holomorphic structure in connection with the Cauchy–Riemann structure of the boundary part M has been studied by various authors (also in higher dimensions, see e.g. [23]). M is the simplest known real hypersurface in \mathbb{C}^3 with everywhere degenerate Levi form that cannot be even locally biholomorphically straightened, i.e. it is not locally CR-equivalent to a direct product $S \times \mathbb{C}$ with S any real hypersurface in \mathbb{C}^2 (cf. [8], [6]). M is homogeneous as a CR-manifold since the group of all affine transformations of \mathbb{C}^3 fixing M acts transitively on M (and H). Actually, it can be seen that this group coincides with the group $\text{Aut}(M)$ of all real-analytic CR-automorphisms of M . By the homogeneity of M all local CR-equivalences (always understood to be real-analytic in the following) between domains in M are known as soon as for some (and hence every) $a \in M$ the automorphism group $\text{Aut}(M, a)$ of the CR-manifold germ (M, a) is known. Now, not every germ in $\text{Aut}(M, a)$ is affine. This is due to the fact that every transformation in the 10-dimensional biholomorphic automorphism group $\text{Aut}(H)$ of H extends to a birational (but not necessarily biholomorphic) transformation of \mathbb{C}^3 and hence induces local (but not necessarily global) CR-equivalences on M . Our main result, specialized to this example, states that actually all local CR-equivalences of M occur in this way and that M can be enlarged to a homogeneous CR-manifold S , containing M as a dense domain, such that all local CR-equivalences of M extend to global CR-automorphisms of S . In particular, every $\text{Aut}(M, a)$ turns out to be a solvable real Lie group of dimension 5 (compare the end of Section 6 for an explicit description).

In this paper we present a large class of further homogeneous Levi degenerate CR-manifolds M of arbitrary high codimension which have properties similar to the 5-dimensional hypersurface above: Let V be a real vector space of finite dimension with complexification $E := V \oplus iV$ and let $\Omega \subset V$ be an open convex cone such that the associated tube domain $H := \Omega + iV \subset E$ is symmetric (i.e. biholomorphically equivalent to a bounded symmetric domain). For simplicity and without essential loss of generality we always assume that the cone Ω is irreducible. The group $\text{GL}(\Omega)$ of all linear transformations in $\text{GL}(V)$ leaving Ω invariant has a finite number of orbits in V ; let $C \neq \{0\}$ be one of them (a typical example is the space $E = \mathbb{C}^{r \times r}$ of all complex $r \times r$ -matrices, $V \subset E$ the \mathbb{R} -linear subspace of all hermitian matrices, $\Omega \subset V$ the open cone of positive definite matrices and $C \subset V$ the cone of all hermitian matrices with p positive and q negative eigenvalues). The tube $M := C \oplus iV$ over the cone C is a locally closed Levi degenerate generic CR-submanifold of E , on which the affine group $\text{Aff}(H) := \{g \in \text{Aff}(E) : g(H) = H\}$ acts transitively. It turns out that in case $C \neq \pm\Omega$ the global CR-automorphism group $\text{Aut}(M)$ is just $\text{Aff}(H)$ (see Proposition 6.9; in case $C = -C$ the group $\text{Aff}(H)$ has to be extended by the transformation $z \mapsto -z$).

On the other hand, the group $\text{Aff}(H)$ is a subgroup of codimension $\dim(V)$ in $\text{Aut}(H)$, the biholomorphic automorphism group of the tube domain H . This group is a simple Lie group and is explicitly known in every case. Every $g \in \text{Aut}(H)$ extends to a birational transformation of \mathbb{C}^n and induces local CR-transformations on M . Actually, the following more precise statement is known from the theory of symmetric Hermitian spaces (see [9]): E can be compactified to a homogeneous rational complex manifold Z (the compact dual of H) in such a way that every $g \in \text{Aut}(H)$ extends to a biholomorphic transformation of Z . In fact, this way the simple real Lie group $\text{Aut}(H)$ is realized as a real form of

the simple complex Lie group $\text{Aut}(Z)$ (recall that we assumed Ω and hence also H to be irreducible). Now, there exists an $\text{Aut}(H)$ -orbit S in Z with $M = E \cap S$. This S is a non-compact locally closed CR-submanifold of Z that contains M as an open dense subset. Our main result, Theorem 4.7 together with Theorem 4.5, implies that, in case S is not open in Z , every CR-equivalence between domains in S extends to a biholomorphic transformation of Z respecting S . A consequence (Proposition 6.4) is that for every $a \in M$, the germ automorphism group $\text{Aut}(M, a)$ is canonically isomorphic to the isotropy subgroup $\text{Aut}(H)_a := \{g \in \text{Aut}(H) : g(a) = a\}$ (again, in case $C = -C$ the group $\text{Aut}(H)$ has to be extended by the transformation $z \mapsto -z$). An important step in the proof is that S , although Levi degenerate, is 2-nondegenerate and minimal as a CR-manifold.

We also consider arbitrary Hermitian symmetric spaces Z and orbits $S \subset Z$ with respect to arbitrary real forms of the connected identity component $\text{Aut}(Z)^0$. But, in contrast to the more special tube case discussed above, we have to assume $\dim \text{Aut}(S, a) < \infty$ for some $a \in S$ in order to obtain similar extension results for CR-equivalences between domains of S (see Theorem 4.5).

2. Preliminaries

Let X be a complex manifold and $M \subset X$ a connected (locally-closed) real-analytic submanifold. For every $a \in M$ the tangent space T_aM is an \mathbb{R} -linear subspace of the complex vector space T_aX . Recall that M is a (real-analytic) *CR-(sub)manifold* if the *holomorphic tangent space* $H_aM := T_aM \cap iT_aM \subset T_aX$ has the same complex dimension for all $a \in M$. The CR-manifold M is called *generic* in X if the tangent space T_aM spans T_aX over \mathbb{C} for every $a \in M$, that is, if $T_aX = T_aM + iT_aM$. In an abstract setting, a real-analytic *CR-manifold* is a real-analytic manifold with a real-analytic vector subbundle $HM \subset TM$ and a real-analytic bundle endomorphism $J : HM \rightarrow HM$ satisfying $J^2 = -\text{id}$ and the integrability condition $[\mathcal{H}^{0,1}, \mathcal{H}^{0,1}] \subset \mathcal{H}^{0,1}$ (see the Appendix). Given two CR-manifolds M and M' , a smooth map $f : M \rightarrow M'$ is called a *CR-map* if the differential $df : TM \rightarrow TM'$ maps HM into HM' and commutes with the corresponding complex structures J and J' on HM and HM' .

Denote by $\mathfrak{hol}(M)$ the real Lie algebra of all (globally defined) real-analytic vector fields on M whose local flows consist of CR-maps (these vector fields are also called *infinitesimal CR-transformations* of M). In particular, if M is a complex manifold, $\mathfrak{hol}(M)$ consists of all holomorphic vector fields on M . The value of the vector field $\xi \in \mathfrak{hol}(M)$ at the point $a \in M$ will be denoted by $\xi_a \in T_aM$. Furthermore, $\text{Aut}(M)$ is the group of all bi-analytic transformations of M that are CR in both directions.

For every $a \in M$ denote by $\text{Aut}(M, a)$ the group of all germs at a of real-analytic CR-isomorphisms $g : U \rightarrow V$ with $g(a) = a$, where U, V are arbitrary open neighbourhoods of a . For every $k \in \mathbb{N}$ let $\text{Aut}_k(M, a) \subset \text{Aut}(M, a)$ be the normal subgroup of all germs that have the same k -jet at a as the identity. Denote by $\mathfrak{hol}(M, a)$ the real Lie algebra of all germs at a of vector fields $\xi \in \mathfrak{hol}(U)$ with U being an arbitrary open neighbourhood of a . Furthermore, for every integer k , $\mathfrak{aut}_k(M, a) \subset \mathfrak{hol}(M, a)$ denotes the Lie subalgebra of all germs vanishing of order $> k$ at a , i.e. having zero k -jets at a . For shorter notation we also write $\mathfrak{aut}(M, a) := \mathfrak{aut}_0(M, a)$ for the Lie subalge-

bra of all germs in $\mathfrak{hol}(M, a)$ that vanish at a . There exists a canonical exponential map $\exp : \mathfrak{aut}(M, a) \rightarrow \text{Aut}(M, a)$ sending every $\mathfrak{aut}_k(M, a)$ into $\text{Aut}_k(M, a)$. In case the Lie algebra $\mathfrak{aut}(M, a)$ has finite dimension, there exists a unique Lie group structure on $\text{Aut}(M, a)$ such that the exponential map is locally bi-analytic in a neighbourhood of the origin in $\mathfrak{aut}(M, a)$. Throughout, the dependence (M) refers to global objects on M while (M, a) refers to germs at the point $a \in M$.

In case E is a complex vector space of finite dimension and $U \subset E$ is an open subset, we always identify for every $a \in U$ the tangent space $T_a U$ with E in the canonical way. In this sense every holomorphic vector field $\xi \in \mathfrak{hol}(U)$ is given by a holomorphic function $f : U \rightarrow E$ and vice versa. But since both objects have to be distinguished we write symbolically $\xi = f(z) \frac{\partial}{\partial z}$ (where z is meant as a *variable* in E). Actually, we consider ξ as a holomorphic differential operator acting on the space of holomorphic functions on U . More generally, for every complex vector space F of finite dimension and every holomorphic mapping $h : U \rightarrow F$, the F -valued holomorphic function ξh on U is defined by $z \mapsto h'(z)(f(z))$, where $h' : U \rightarrow \mathcal{L}(E, F)$ is the derivative of h and $\mathcal{L}(E, F)$ is the vector space of all linear operators $E \rightarrow F$. In particular, if $\iota : U \hookrightarrow E$ is the canonical embedding, then $\xi \iota = f$.

In case $E = \mathbb{C}^n$ the vector field $\xi = f(z) \frac{\partial}{\partial z} \in \mathfrak{hol}(U)$ can be written as

$$\xi = f_1(z) \frac{\partial}{\partial z_1} + \cdots + f_n(z) \frac{\partial}{\partial z_n}, \quad f = (f_1, \dots, f_n),$$

where $\frac{\partial}{\partial z}$ is interpreted as the column $(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n})^t$.

3. Reductive Lie algebras of holomorphic vector fields

Recall that a real or complex Lie algebra \mathfrak{l} is called *reductive* if its radical coincides with its center, or equivalently, if \mathfrak{l} is the direct sum of an abelian Lie algebra with a semisimple one (cf. [12]). Every (finite-dimensional) linear representation of a semisimple Lie algebra is completely reducible by Weyl's theorem ([12, p. 28] or [17, p. 382]), i.e. every invariant subspace in a representation space has an invariant complement. This property is crucial in the proof of the next proposition.

We also recall the notion of a nonresonant vector field (cf. e.g. [2, p. 177]): A finite subset $\Lambda \subset \mathbb{C}$ is called *nonresonant* if $\sum_{\lambda \in \Lambda} m_\lambda \cdot \lambda \notin \Lambda$ for every family of integers $m_\lambda \geq 0$ with $\sum_{\lambda \in \Lambda} m_\lambda \geq 2$. For given $\delta \in \mathfrak{aut}(\mathbb{C}^n, 0)$ consider its linear part as an endomorphism of \mathbb{C}^n . Then δ is called *nonresonant* if the spectrum of this endomorphism (i.e. the set of eigenvalues) is nonresonant.

3.1. Proposition. *Let $\mathfrak{l} \subset \mathfrak{hol}(\mathbb{C}^n, 0)$ be a complex Lie subalgebra of finite dimension such that*

- (i) \mathfrak{l} is reductive,
- (ii) \mathfrak{l} spans the full tangent space to \mathbb{C}^n at 0, that is, $\mathbb{C}^n = \{\xi_0 : \xi \in \mathfrak{l}\}$,
- (iii) \mathfrak{l} contains a nonresonant $\delta \in \mathfrak{aut}(\mathbb{C}^n, 0)$.

Then \mathfrak{l} is semisimple and contains all finite-dimensional \mathfrak{l} -submodules of $\mathfrak{hol}(\mathbb{C}^n, 0)$.

Proof. Let $zA \frac{\partial}{\partial z}$ be the linear part of δ where $z = (z_1, \dots, z_n)$ and A is a complex $n \times n$ -matrix. After a linear change of coordinates we may assume that A is upper triangular and has $\lambda_1, \dots, \lambda_n$ as diagonal entries. Clearly, $\Lambda := \{\lambda_1, \dots, \lambda_n\}$ is the spectrum of A .

Denote by Ξ the set of all monomial vector fields $\alpha = z_1^{m_1} \dots z_n^{m_n} \frac{\partial}{\partial z_j}$ in $\mathfrak{hol}(\mathbb{C}^n, 0)$. Then, by restricting the lexicographic order on \mathbb{N}^{n+2} to $\Xi \hookrightarrow \mathbb{N}^{n+2}$ embedded via

$$\alpha \mapsto (m_1 + \dots + m_n, m_1, \dots, m_n, j),$$

Ξ becomes a well ordered set with minimal element $\frac{\partial}{\partial z_1}$. Every $\xi \in \mathfrak{hol}(\mathbb{C}^n, 0)$ has a unique power series expansion $\xi = \sum_{\beta \in \Xi} c_\beta \beta$ with complex coefficients c_β . For every $\alpha \in \Xi$ denote by $F_\alpha \subset \mathfrak{hol}(\mathbb{C}^n, 0)$ the linear subspace of all those ξ such that $c_\beta = 0$ for all $\beta \leq \alpha$ in the above expansion. It is easily verified that $\text{ad}(\delta)$ (defined as $\xi \mapsto [\delta, \xi]$) leaves F_α invariant and that

$$[\delta, \alpha] \equiv (m_1 \lambda_1 + \dots + m_n \lambda_n - \lambda_j) \alpha \pmod{F_\alpha} \tag{3.2}$$

if $\alpha = z_1^{m_1} \dots z_n^{m_n} \frac{\partial}{\partial z_j}$.

Now let $\mathfrak{h} \subset \mathfrak{hol}(\mathbb{C}^n, 0)$ be an arbitrary finite-dimensional \mathfrak{l} -submodule, i.e. $[\mathfrak{l}, \mathfrak{h}] \subset \mathfrak{h}$. Denote by Θ the restriction of $\text{ad}(\delta)$ to \mathfrak{h} and consider the direct sum decomposition

$$\mathfrak{h} = \bigoplus_{\lambda \in \mathbb{C}} \mathfrak{h}^\lambda, \tag{3.3}$$

where every \mathfrak{h}^λ is the largest Θ -invariant linear subspace on which $\Theta - \lambda \text{id}$ is nilpotent (the generalized λ -eigenspace of Θ in case $\mathfrak{h}^\lambda \neq 0$). An immediate consequence of (3.2) is that $\mathfrak{h}^\lambda \subset \text{aut}(\mathbb{C}^n, 0)$ for every $\lambda \notin -\Lambda$. Assume on the other hand that there exists a vector field $\xi = \sum_{\beta} c_\beta \beta \neq 0$ in

$$\mathfrak{h}^- := \bigoplus_{\lambda \in \Lambda} \mathfrak{h}^{-\lambda}$$

with $\xi_0 = 0$. Choose $\alpha = z_1^{m_1} \dots z_n^{m_n} \frac{\partial}{\partial z_j} \in \Xi$ minimal with respect to the property $c_\alpha \neq 0$, say $c_\alpha = 1$ without loss of generality. Clearly, α has degree $d = m_1 + \dots + m_n \geq 1$ because $\xi_0 = 0$. Since $\prod_{\lambda \in \Lambda} (\Theta + \lambda \text{id})$ is nilpotent on \mathfrak{h}^- we deduce from (3.2) that $-\lambda_k = \sum_i m_i \lambda_i - \lambda_j$ for some k , a contradiction to the nonresonance of Λ . Therefore the evaluation map $\xi \mapsto \xi_0$ defines a linear injection $\varepsilon_0 : \mathfrak{h}^- \hookrightarrow T_0 \mathbb{C}^n = \mathbb{C}^n$.

We first discuss the special case where \mathfrak{l} is semisimple and assume $\mathfrak{h} \not\subset \mathfrak{l}$ contrary to the claim. To get a contradiction we may assume $\mathfrak{l} \cap \mathfrak{h} = 0$ without loss of generality, since by Weyl's theorem, \mathfrak{l} has an $\text{ad}(\mathfrak{l})$ -invariant complement in the \mathfrak{l} -module $\mathfrak{l} + \mathfrak{h}$. But then $(\mathfrak{l} \oplus \mathfrak{h})^- = \mathfrak{l}^- \oplus \mathfrak{h}^-$. Since the evaluation map ε_0 is an injection on $\mathfrak{l}^- \oplus \mathfrak{h}^-$ as mentioned above and $\varepsilon_0(\mathfrak{l}^-) = \mathbb{C}^n$ by assumption (ii), we conclude that all vector fields in \mathfrak{h} vanish at 0. On the other hand, if $\xi \in \mathfrak{h}$ is a nontrivial vector field, taking subsequent Lie brackets with suitable vector fields from \mathfrak{l} and using (ii) we obtain a vector field $\eta \in \mathfrak{h}$ with $\eta_0 \neq 0$, a contradiction.

In the general case, if \mathfrak{l} is arbitrary reductive, let \mathfrak{h} be the center of \mathfrak{l} . From $[\delta, \mathfrak{h}] = 0$ we get $\mathfrak{h} \subset \text{aut}(\mathbb{C}^n, 0)$ since $0 \notin \Lambda$. But then, since \mathfrak{h} is an \mathfrak{l} -module, the above argument implies $\mathfrak{h} = 0$, that is, \mathfrak{l} is semisimple. \square

Simple examples show that none of the conditions (i)–(iii) in Proposition 3.1 can be omitted. For condition (iii) we present the following

3.4. Example. Let $n = 2m - 1$ be an arbitrary odd integer ≥ 3 and consider \mathbb{C}^n in the usual way as an open dense subset of the complex projective space \mathbb{P}_n . The standard action of the complex Lie group $SL(2m, \mathbb{C})$ on \mathbb{P}_n induces a complex Lie algebra of holomorphic vector fields on \mathbb{P}_n whose germs at $0 \in \mathbb{C}^n$ form a simple complex Lie subalgebra $\mathfrak{h} \subset \mathfrak{ho}(\mathbb{C}^n, 0)$ isomorphic to $\mathfrak{sl}(2m, \mathbb{C})$. It is easily verified that \mathfrak{h} contains (the germ of) the Euler field $z \frac{\partial}{\partial z}$, which is nonresonant since $\Lambda = \{1\}$ in this case. Now, the symplectic group $Sp(m, \mathbb{C}) \subset SL(2m, \mathbb{C})$ also acts transitively on \mathbb{P}_n and induces a proper simple Lie subalgebra $\mathfrak{l} \subset \mathfrak{h}$ isomorphic to $\mathfrak{sp}(m, \mathbb{C})$. Therefore the conclusion of Proposition 3.1 does not hold for this \mathfrak{l} . It is not difficult to see that \mathfrak{l} contains a linear vector field $\delta \in \mathfrak{aut}(\mathbb{C}^n, 0)$ with spectrum $\Lambda = \{1, 2\}$, where the eigenvalue 1 has multiplicity $n - 1$.

3.5. Remark. Since $\text{ad}(\delta)$ is a derivation, for the special case $\mathfrak{h} = \mathfrak{l}$ in Proposition 3.1 the decomposition (3.3) actually gives the \mathbb{C} -grading

$$\mathfrak{l} = \bigoplus_{\lambda \in \mathbb{C}} \mathfrak{l}^\lambda \quad \text{with} \quad [\mathfrak{l}^\lambda, \mathfrak{l}^\mu] \subset \mathfrak{l}^{\lambda+\mu} \tag{3.6}$$

for all $\lambda, \mu \in \mathbb{C}$ and $\delta \in \mathfrak{h}^0$. Furthermore, due to condition (ii) the linear subspace $\mathfrak{l}^- = \bigoplus_{\lambda \in \Lambda} \mathfrak{l}^{-\lambda}$ is isomorphic to \mathbb{C}^n via the evaluation map ε_0 , and the action of $\text{ad}(\delta)$ on \mathfrak{h}^- is equivalent to the endomorphism of \mathbb{C}^n given by the linear part of δ . Denote by $\delta' \in \mathfrak{l}$ the semisimple part of δ (see [12, p. 29] for basic properties of this concept). Then the linear part of δ' is the semisimple part of the linear part of δ and hence also has Λ as spectrum. In particular, with δ also δ' is nonresonant. Furthermore, $\text{ad}(\delta')$ is diagonalizable on \mathfrak{l} , that is, $\mathfrak{l}^\lambda = \ker(\text{ad}(\delta') - \lambda \text{id})$ for all $\lambda \in \mathbb{C}$.

Now assume that the linear part of δ is the Euler field $z \frac{\partial}{\partial z}$, that is, $\delta = \delta'$ and $\Lambda = \{1\}$ (this case will be of special interest in the next sections). Then (3.6) reduces to $\mathfrak{l} = \bigoplus_{k=-1}^\infty \mathfrak{l}^k$ with $\mathfrak{l}^k \subset \mathfrak{aut}_k(\mathbb{C}^n, 0)$ for every integer $k \geq -1$. But then with standard arguments for semisimple Lie algebras it follows that actually

$$\mathfrak{l} = \mathfrak{l}^{-1} \oplus \mathfrak{l}^0 \oplus \mathfrak{l}^1 \tag{3.7}$$

with abelian Lie algebras $\mathfrak{l}^{\pm 1}$ of dimension n and $\mathfrak{l}^0 = [\mathfrak{l}^{-1}, \mathfrak{l}^1]$. Indeed, for every $\eta \in \mathfrak{l}^k$ with $k > 1$ the endomorphism $\text{ad}(\xi) \text{ad}(\eta)$ is nilpotent for every $\xi \in \mathfrak{l}$, and hence η is orthogonal to \mathfrak{l} with respect to the Killing form of \mathfrak{l} , that is, η is in the radical of \mathfrak{l} , proving (3.7). That $\mathfrak{l}^{-1}, \mathfrak{l}^1$ have the same dimension follows from $\text{tr}(\text{ad}(\delta)) = 0$ (cf. [12, p. 28]). Finally, $\mathfrak{m} := \mathfrak{l}^{-1} \oplus [\mathfrak{l}^{-1}, \mathfrak{l}^1] \oplus \mathfrak{l}^1$ and $\mathfrak{n} := \mathbb{C}\delta + \mathfrak{m}$ are ideals in \mathfrak{l} and hence are semisimple themselves. Therefore $\mathfrak{n} = \mathfrak{m} \oplus \mathfrak{c}$ for some ideal \mathfrak{c} of dimension ≤ 1 in \mathfrak{n} . Since also \mathfrak{c} is semisimple, only $\mathfrak{c} = 0$ is possible, that is, $\mathfrak{m} = \mathfrak{n}$. Finally, Proposition 3.1 applied to \mathfrak{m} in place of \mathfrak{l} shows $\mathfrak{l} = \mathfrak{m}$ and thus the claim. We mention that the vector field δ is actually linearizable, that is, after a suitable biholomorphic change of coordinates becomes the Euler vector field. Such a change of coordinates can be obtained in the following way: There exists an open neighbourhood U of $0 \in \mathbb{C}^n$ such that every $\xi \in \mathfrak{l}^{-1}$ can be represented by a vector field in $\mathfrak{ho}(\mathfrak{l}(U))$. For a suitable open neighbourhood

V of $0 \in \Gamma^{-1}$ the assignment $\xi \mapsto \exp(\xi)(0)$ defines a local biholomorphic transformation $V \rightarrow U$ doing the job.

There is also a real version of Proposition 3.1. Let $M \subset \mathbb{C}^n$ be a (locally closed) generic real-analytic CR-submanifold containing the origin 0 . We consider $\mathfrak{hol}(M, a)$ as a real Lie subalgebra of the complex Lie algebra $\mathfrak{hol}(\mathbb{C}^n, a)$ in the obvious way and call M *holomorphically nondegenerate* at $a \in M$ if $\mathfrak{hol}(M, a)$ is totally real in $\mathfrak{hol}(\mathbb{C}^n, a)$, that is, $\mathfrak{hol}(M, a) \cap i\mathfrak{hol}(M, a) = 0$. This definition is equivalent to the usual one ([3, p. 322]). Recall that M is called *minimal* (in the sense of [27]) if every real submanifold $N \subset M$ with $H_a M \subset T_a N$ for all $a \in N$ is necessarily open in M . In case M is a real hypersurface of \mathbb{C}^n minimality already follows from holomorphic nondegeneracy.

3.8. Proposition. *Suppose that $M \subset \mathbb{C}^n$ is holomorphically nondegenerate at $0 \in M$ and that $\mathfrak{s} \subset \mathfrak{hol}(M, 0) \subset \mathfrak{hol}(\mathbb{C}^n, 0)$ is a real Lie subalgebra of finite dimension such that*

- (i) \mathfrak{s} is reductive,
- (ii) \mathfrak{s} spans the full tangent space of M at 0 ,
- (iii) $(\mathfrak{s} + i\mathfrak{s}) \cap \mathfrak{aut}(\mathbb{C}^n, 0)$ contains a nonresonant vector field.

Then \mathfrak{s} is semisimple and contains every finite-dimensional \mathfrak{s} -submodule of $\mathfrak{hol}(M, 0)$. If, in addition, M is minimal at 0 then $\mathfrak{hol}(M, 0) = \mathfrak{s}$.

Proof. Let $\mathfrak{h} \subset \mathfrak{hol}(M, 0)$ be any \mathfrak{s} -submodule of finite dimension. Since \mathfrak{s} is totally real in $\mathfrak{hol}(\mathbb{C}^n, 0)$, the sum $\mathfrak{l} := \mathfrak{s} + i\mathfrak{s} \subset \mathfrak{hol}(\mathbb{C}^n, 0)$ is direct and hence a complex reductive Lie subalgebra. Since M is generic in \mathbb{C}^n , (ii) implies that \mathfrak{l} spans the full tangent space of \mathbb{C}^n at 0 . Therefore, by Proposition 3.1, \mathfrak{l} is semisimple and the finite-dimensional \mathfrak{l} -module $\mathfrak{h} + i\mathfrak{h}$ is contained in \mathfrak{l} . It follows that \mathfrak{s} is also semisimple and \mathfrak{h} is contained in $\mathfrak{l} \cap \mathfrak{hol}(M, 0)$. But $\mathfrak{l} \cap \mathfrak{hol}(M, 0) = \mathfrak{s}$ since M is holomorphically nondegenerate at 0 . The last claim now follows for $\mathfrak{h} = \mathfrak{hol}(M, 0)$ since $\dim \mathfrak{hol}(M, 0) < \infty$ for any minimal holomorphically nondegenerate germ $(M, 0)$ of a real-analytic generic submanifold in \mathbb{C}^n (see e.g. (12.5.16) in [3]). \square

In the following we consider a connected complex Lie group L acting holomorphically on a complex manifold Z . We always assume that L acts *almost effectively* on Z , that is, the subgroup $\bigcap_{a \in Z} L_a$ is discrete in L , where $L_a := \{g \in L : g(a) = a\}$ is the *isotropy subgroup* of L at $a \in Z$. Then the Lie algebra \mathfrak{l} of L can be considered in a natural way to be a subalgebra of $\mathfrak{hol}(Z)$, which in turn can be considered to be a Lie subalgebra of $\mathfrak{hol}(Z, a)$ for every $a \in Z$. We denote by $\mathfrak{l}_a := \{\xi \in \mathfrak{l} : \xi_a = 0\}$ the *isotropy subalgebra* of \mathfrak{l} at $a \in Z$.

Recall that a *real form* of L is any closed connected real Lie subgroup $S \subset L$ with $\mathfrak{l} = \mathfrak{s} \oplus i\mathfrak{s}$ for their Lie algebras. Then every S -orbit $S \subset Z$ may be viewed as an immersed real-analytic CR-submanifold of Z . In case L acts transitively on Z , every such orbit is generic in Z . The next result together with Proposition 3.8 will be the key for our first main result, Theorem 4.5.

3.9. Proposition. *Let L and L' be connected complex Lie groups acting holomorphically, transitively and almost effectively on simply connected complex manifolds Z and Z' respectively. Let furthermore $S \subset Z$, $S' \subset Z'$ be orbits with respect to real forms S , S' of L , L' and assume $\mathfrak{hol}(S, a) = \mathfrak{s}$ and $\mathfrak{hol}(S', a') = \mathfrak{s}'$ for some (and hence all) $a \in S$, $a' \in S'$, where $\mathfrak{s} \subset \mathfrak{l}$ and $\mathfrak{s}' \subset \mathfrak{l}'$ are the Lie algebras of the real forms S and S' . Then every real-analytic CR-equivalence $\varphi : U \rightarrow U'$ between domains $U \subset S$ and $U' \subset S'$ extends to a (unique) biholomorphic map $Z \rightarrow Z'$ sending S onto S' .*

Proof. Fix a point $a \in U$ and write $Z = L/L_a$ as well as $Z' = L'/L'_{a'}$ for $a' := \varphi(a)$. The CR-equivalence φ extends to a biholomorphic map between suitable open neighbourhoods of a and a' in Z and Z' respectively (see e.g. Corollary 1.7.13 in [3]). Therefore φ induces a Lie algebra isomorphism from $\mathfrak{hol}(S, a) = \mathfrak{s}$ onto $\mathfrak{hol}(S', a') = \mathfrak{s}'$ and hence, by complexification, a complex Lie algebra isomorphism $\psi : \mathfrak{l} \rightarrow \mathfrak{l}'$ with $\psi(\mathfrak{l}_a) = \mathfrak{l}'_{a'}$ for the corresponding isotropy Lie subalgebras. Without loss of generality we assume that L and L' are simply connected (otherwise pass to the universal coverings). Since Z , Z' are simply connected by assumption, the isotropy subgroups $L_a, L'_{a'}$ are connected and hence ψ induces a biholomorphic group isomorphism $\Psi : L \rightarrow L'$ with $\Psi(L_a) = L'_{a'}$ and $\Psi(S) = S'$. The induced biholomorphic map $Z \rightarrow Z'$ extends φ and maps S onto S' , as desired. \square

4. Real form orbits in Hermitian symmetric spaces

In the following let E be a complex vector space of finite dimension and $D \subset E$ a bounded symmetric domain. Without loss of generality we assume that D is convex and circular (cf. [9] and [19]). The group $\text{Aut}(D)$ of all biholomorphic automorphisms of D is a semisimple real Lie group acting analytically and transitively on D . The linear group

$$\text{GL}(D) := \{g \in \text{GL}(E) : g(D) = D\}$$

is the isotropy subgroup of $\text{Aut}(D)$ at the origin and acts transitively on the Shilov boundary $\partial_s D$ of D , which in this case coincides with the set of all extremal points of the compact convex body \overline{D} . The Shilov boundary $\partial_s D$ is a connected CR-submanifold of E , and D is called *of tube type* if $\partial_s D$ is totally real in E . This is equivalent to D being biholomorphically equivalent to a domain $\Omega \oplus iV \subset V \oplus iV$ for some real vector space V and some open cone $\Omega \subset V$.

We denote by Z the *compact dual* of D in the sense of Hermitian symmetric spaces (cf. e.g. [9]). Z is a simply connected compact homogeneous complex manifold that contains E in a canonical way as a Zariski-open subset such that every biholomorphic automorphism of D extends to an automorphism of Z , i.e.

$$\text{Aut}(D) \cong \{g \in \text{Aut}(Z) : g(D) = D\}. \tag{4.1}$$

The connected component $L := \text{Aut}(Z)^0$ of identity is a semisimple complex Lie group acting transitively and holomorphically on Z whereas $G := \text{Aut}(D)^0$ is a noncompact real form of L . The corresponding Lie algebras \mathfrak{l} and \mathfrak{g} with $\mathfrak{l} = \mathfrak{g} \oplus i\mathfrak{g}$ are realized as

Lie algebras of holomorphic vector fields on Z , in fact, \mathfrak{l} coincides with the Lie algebra $\mathfrak{hol}(Z)$ of all holomorphic vector fields on Z and we have canonical inclusions $\mathfrak{hol}(Z) \subset \mathfrak{hol}(E) \subset \mathfrak{hol}(D)$ by restriction. In particular, every $\xi \in \mathfrak{l}$ is of the form $\xi = f(z) \frac{\partial}{\partial z}$ for a certain holomorphic mapping $f : E \rightarrow E$ (see Section 2 for this notation).

Since D is circular we have $i\delta \in \mathfrak{g}$ for $\delta := z \frac{\partial}{\partial z} \in \mathfrak{l}$. It is clear from the definition that δ is nonresonant and thus Proposition 3.8 can be applied to G -orbits. In the decomposition (3.7), \mathfrak{l}^k is the space of all homogeneous vector fields of degree $k+1$ in \mathfrak{l} for $k = -1, 0, 1$. In particular, $\mathfrak{l}^{-1} = \{\alpha \frac{\partial}{\partial z} : \alpha \in E\}$ is the space of all constant holomorphic vector fields on E (when restricted to $E \subset Z$) and $\mathfrak{l}^0 \oplus \mathfrak{l}^1 = \mathfrak{l}_0$ is the isotropy subalgebra of \mathfrak{l} at 0 . The isotropy subalgebras $\mathfrak{l}_a = \{\xi \in \mathfrak{l} : \xi_a = 0\}$ of \mathfrak{l} separate points of Z in the sense that $\mathfrak{l}_a \neq \mathfrak{l}_b$ for all $a \neq b$ in Z . Indeed, in case $a, b \in E$, the vector field $(z - a) \frac{\partial}{\partial z}$ is in \mathfrak{l}_a but not in \mathfrak{l}_b . The general case is reduced to that of $a, b \in E$ as a consequence of the known fact that, for any two points in Z , there exists a transformation in L mapping them into E .

For the Lie algebra \mathfrak{g} of the group $G = \text{Aut}(D)^0$ consider the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into the range and kernel of the projection $\text{id} + (\text{ad } i\delta)^2$. Clearly $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{l}^0$ and $\mathfrak{p} = \mathfrak{g} \cap (\mathfrak{l}^{-1} \oplus \mathfrak{l}^1)$. As a consequence of Cartan's uniqueness theorem, every $\xi \in \mathfrak{g}$ is uniquely determined by its 1-jet at $0 \in D$ (cf. for instance [15]). Hence \mathfrak{k} is the isotropy subalgebra of \mathfrak{g} at 0 , and $\xi \mapsto \xi_0$ defines an \mathbb{R} -linear isomorphism $\mathfrak{p} \rightarrow T_0E = E$. As a consequence, there exists a unique mapping

$$E \times E \times E \rightarrow E, \quad (x, y, z) \mapsto \{xyz\}, \tag{4.2}$$

that is symmetric complex bilinear in the outer variables (x, z) such that

$$\mathfrak{p} = \{(\alpha - \{z\alpha z\}) \frac{\partial}{\partial z} : \alpha \in E\}.$$

Since both $(\alpha - \{z\alpha z\}) \frac{\partial}{\partial z}$ and $\xi := (i\alpha - \{z(i\alpha)z\}) \frac{\partial}{\partial z}$ are in \mathfrak{p} , one has

$$\eta := [(\alpha - \{z\alpha z\}) \frac{\partial}{\partial z}, i z \frac{\partial}{\partial z}] = (i\alpha + i\{z\alpha z\}) \frac{\partial}{\partial z} \in \mathfrak{p}.$$

Now ξ and η have the same 1-jet at 0 and the above unique determination implies that $\{xyz\}$ is conjugate linear in the inner variable y . Consequently,

$$\mathfrak{l}^1 = \{\{z\alpha z\} \frac{\partial}{\partial z} : \alpha \in E\} \quad \text{and} \quad \mathfrak{l}^{-1} = \{\alpha \frac{\partial}{\partial z} : \alpha \in E\}.$$

In addition, the triple product $\{xyz\}$ satisfies certain algebraic identities as well as a positivity condition. It is called the *Jordan triple product* on E given by the bounded symmetric domain D (see e.g. [19] and [16] for details).

Let S be a *real form* of the complex Lie group L , that is, a closed connected real subgroup whose Lie algebra \mathfrak{s} satisfies $\mathfrak{l} = \mathfrak{s} \oplus i\mathfrak{s}$ (for instance, $\text{Aut}(D)^0$ is such a real form). Let S be an S -orbit in the compact dual Z . Then S is a locally closed connected real-analytic submanifold of Z and hence a homogeneous CR-manifold. Since the complex Lie group L acts transitively on Z , the CR-manifold S is generic in Z , i.e. $T_a S + iT_a S = T_a Z$ for the tangent spaces at every $a \in S$. The Lie algebra \mathfrak{s} of S can be considered as a real Lie subalgebra of $\mathfrak{hol}(S)$ and hence for every $a \in S$ also of $\mathfrak{hol}(S, a)$ in a natural way. Note that we have $iz \frac{\partial}{\partial z}, ia \frac{\partial}{\partial z} \in \mathfrak{s}$ and hence $\delta := (z - a) \frac{\partial}{\partial z} \in \mathfrak{s} + i\mathfrak{s}$. As a consequence of Proposition 3.8 we state

4.3. Proposition. *Suppose that $\mathfrak{hol}(S, a)$ is of finite dimension for some a in the S -orbit S (for instance, if S is holomorphically nondegenerate and minimal as a CR-manifold). Then $\mathfrak{hol}(S, a) = \mathfrak{s}$ for every $a \in S$.*

For the formulation of our first main result we introduce the following notation:

4.4. Definition. *Denote by \mathfrak{C} the class of all pairs (S, Z) , where Z is an arbitrary Hermitian symmetric space of compact type and $S \subset Z$ is an S -orbit with $\dim \mathfrak{hol}(S, a) < \infty$ for some $a \in S$ and some real form S of $\text{Aut}(Z)^0$.*

4.5. Theorem. *Let (S, Z) and (S', Z') be arbitrary pairs in the class \mathfrak{C} and assume that $\varphi : U \rightarrow U'$ is a real-analytic CR-equivalence where $U \subset S$ and $U' \subset S'$ are arbitrary domains. Then φ has a unique extension to a biholomorphic transformation $Z \rightarrow Z'$ mapping S onto S' . In particular, there are canonical isomorphisms*

$$\begin{aligned}\text{Aut}(S) &\cong \{g \in \text{Aut}(Z) : g(S) = S\}, \\ \text{Aut}(S, a) &\cong \{g \in \text{Aut}(S) : g(a) = a\}\end{aligned}$$

for every $a \in S$. Every $g \in \text{Aut}(S, a)$ is uniquely determined by its 2-jet at a .

Proof. We show that the assumptions of Proposition 3.9 are satisfied. By [9, p. 305], Z is simply connected, and the semisimple complex Lie group $L = \text{Aut}(Z)^0$ acts transitively on Z . By the definition of \mathfrak{C} there is a real form S of L with $S = S(a)$ and $\dim \mathfrak{hol}(S, a) < \infty$ for some $a \in S$. Proposition 4.3 gives $\mathfrak{hol}(S, a) = \mathfrak{s}$ for the Lie algebra \mathfrak{s} of S . Since the same properties hold for (S', Z') Proposition 3.9 gives the continuation statement. The last statement about the jet determination follows from the known fact that elements of $\text{Aut}(Z)$ are uniquely determined by their 2-jets at any given point $a \in Z$. \square

4.6. Corollary. *Given any $(S, Z) \in \mathfrak{C}$, the group $\text{Aut}(S)$ of all real-analytic CR-automorphisms is a Lie group with finitely many connected components and S as the identity component. More precisely, $\text{Aut}(S)$ is canonically isomorphic to an open subgroup of $\text{Aut}(\mathfrak{s})$, where \mathfrak{s} is the Lie algebra of S .*

Proof. The group $\text{Aut}(S)$ acts on the real Lie algebra $\mathfrak{hol}(S) = \mathfrak{s}$ and hence induces an injective Lie homomorphism $\varphi : \text{Aut}(S) \rightarrow \text{Aut}(\mathfrak{s}) \subset \text{Aut}(\mathfrak{l})$ (the injectivity follows from the fact that the isotropy subalgebras of \mathfrak{l} separate points of Z , i.e. are different at different points). Since $\text{Aut}(S)$ contains the semisimple subgroup S it follows that φ is open. Furthermore, $\text{Aut}(\mathfrak{s})$ is an algebraic subgroup of $\text{GL}(\mathfrak{s})$ and hence has only finitely many connected components. \square

The irreducible Hermitian symmetric spaces of compact type come in four series and two exceptional spaces (see for instance [9]). As an example let us briefly recall the first series. It consists of all spaces Z for which the automorphism group $L = \text{Aut}(Z)^0$ is of the form $\text{PSL}(p, \mathbb{C}) := \text{SL}(p, \mathbb{C})/\text{center}$ for some $p \geq 2$: Fix integers $m \geq n \geq 1$ with $m + n = p$ and denote by $Z := \mathbb{G}_{n,m}$ the Grassmannian of all linear subspaces of dimension n in \mathbb{C}^p . Then $\mathbb{G}_{n,m}$ is a connected compact complex manifold of dimension nm on which the

complex Lie group $\mathrm{SL}(p, \mathbb{C})$ acts transitively as a holomorphic transformation group and its center is the kernel of ineffectivity. Up to a positive factor, there exists a unique $\mathrm{SU}(p)$ -invariant Hermitian metric on Z making it a Hermitian symmetric space of rank n with $\mathrm{L} = \mathrm{Aut}(Z)^0 = \mathrm{PSL}(p, \mathbb{C})$. The real forms of $\mathrm{SL}(p, \mathbb{C})$ are $\mathrm{SL}(p, \mathbb{R})$ and all $\mathrm{SU}(j, k)$ with arbitrary integers $j \geq k \geq 0$ satisfying $j + k = p$. Fix such a pair (j, k) with $k > 0$ and an $\mathrm{SU}(j, k)$ -invariant Hermitian form Φ on \mathbb{C}^p of type (j, k) (i.e. Φ has j positive and k negative eigenvalues). The orbits of the real form $\mathrm{S} := \mathrm{PSU}(j, k)$ in Z can be indexed as

$$Z_{p,q} = \{V \in \mathbb{G}_{n,m} : \Phi \text{ has type } (p, q) \text{ on } V\},$$

where $p, q \geq 0$ are certain integers satisfying $p + q \leq n$, $p \leq j$, $q \leq k$ and $\max(p, q) \geq n - k$. The simplest case occurs for rank $n = 1$, that is, for $Z = \mathbb{G}_{1,m} = \mathbb{P}_m$ the complex projective space of dimension m . Then S has exactly three orbits: $Z_{1,0}$, $Z_{0,1}$ are open in \mathbb{P}_m and $Z_{0,0}$ is a closed Levi-nondegenerate real hypersurface. Tanaka [26] has shown that in case $m \geq 2$ every CR-equivalence between connected open subsets $U, V \subset Z_{0,0}$ extends to a biholomorphic transformation of \mathbb{P}_m leaving $Z_{0,0}$ invariant. In particular, (for every choice of $j \geq k > 0$) the pair $(Z_{0,0}, \mathbb{P}_m)$ belongs to the class \mathcal{C} and Theorem 4.5 may be considered as an extension of Tanaka's result to more general situations.

Now the question arises for which real form orbits S in a Hermitian symmetric space Z of compact type the pair (S, Z) belongs to the class \mathcal{C} and hence has the properties stated in Theorem 4.5. Since the class \mathcal{C} is closed under taking direct products (that is, together with (S_k, Z_k) in \mathcal{C} for $k = 1, 2$ also $(S_1 \times S_2, Z_1 \times Z_2)$ is in \mathcal{C}), for the above question we only have to consider situations $S \subset Z$ where S is an orbit with respect to a simple real form S of the complex Lie group $\mathrm{L} = \mathrm{Aut}(Z)^0$, that is, where one of the two following cases holds:

- (i) Z is irreducible, or equivalently, the complex Lie group L is simple.
- (ii) $Z = Z_1 \times Z_2$ is the direct product of two irreducible Hermitian symmetric spaces Z_k and $\mathrm{S} = \{(g, \tau g) : g \in \mathrm{L}_1\}$ is the graph of an antiholomorphic group isomorphism $\tau : \mathrm{L}_1 \rightarrow \mathrm{L}_2$ with $\mathrm{L}_k := \mathrm{Aut}(Z_k)^0$ for $k = 1, 2$.

Our second main result gives a complete answer for the tube case in (i), more precisely:

4.7. Theorem. *Let Z be an irreducible Hermitian symmetric space of compact type and let S be a real form of $\mathrm{L} := \mathrm{Aut}(Z)^0$ that has an open orbit $D \subset Z$ which is biholomorphically equivalent to a bounded symmetric domain of tube type. Let furthermore $S \subset Z$ be an S -orbit that is neither open nor totally real in Z . Then S is a minimal 2-nondegenerate CR-manifold and hence (S, Z) belongs to the class \mathcal{C} .*

The proof will be given at the end of the next section (see the Appendix for the definition of 2-nondegeneracy). Locally, every S in Theorem 4.7 will be realized as a tube manifold over a suitable cone in some real vector space. As an example, for the Grassmannian $Z = \mathbb{G}_{n,m}$ the real form $\mathrm{G} = \mathrm{PSU}(n, m)$ of $\mathrm{L} = \mathrm{PSL}(n+m, \mathbb{C})$ has a bounded symmetric domain as an orbit, but this domain is of tube type only if $n = m$ and then the cone is the set of all positive definite Hermitian $n \times n$ -matrices.

5. Tube manifolds

Some bounded symmetric domains can be realized as generalized half planes (tube domains). Besides the Lie theoretic approach (cf. [18]), there is also a Jordan algebraic one that we shall use in the following. It will turn out that all necessary computations become relatively easy in the Jordan context.

Let V be a *real Jordan algebra* of finite dimension, that is, a real vector space with a commutative bilinear product $(x, y) \mapsto x \circ y$ satisfying $[L(a), L(a^2)] = 0$ for every $a \in V$, $a^2 := a \circ a$ and $L(a)$ the multiplication operator $x \mapsto a \circ x$ on V (see [4] for this and the following). Assume in addition that V is *formally real* in the sense that always $x^2 + y^2 = 0$ implies $x = 0$. Then the algebra V automatically has a unit e and the open subset of all invertible elements of V decomposes into a finite number of connected components. Denote by Ω the component containing e . Then Ω is an open convex cone (i.e. $t\Omega = \Omega$ for all $t > 0$) in V and the corresponding *tube domain* $H := \Omega \oplus iV$ in the complexified Jordan algebra $E := V \oplus iV$ is biholomorphically equivalent via the Cayley transformation $z \mapsto (z - e) \circ (z + e)^{-1}$ to a (circular) bounded symmetric domain $D \subset E$ whose (totally real) Shilov boundary is the generalized unit circle $\exp(iV)$. (Here $\exp(z) = \sum_{k \geq 0} z^k / k!$ with powers z^k defined with respect to the Jordan product.) On the other hand, every bounded symmetric domain with totally real Shilov boundary occurs this way and hence is said to be of *tube type*. The example of lowest possible dimension occurs for $V = \mathbb{R}$ with the usual product, and then H is the right halfplane in $E = \mathbb{C}$.

The linear group

$$\mathrm{GL}(\Omega) := \{g \in \mathrm{GL}(V) : g(\Omega) = \Omega\}$$

is a reductive Lie group (cf. also Lemma 5.2) acting transitively on Ω . The Jordan algebra automorphism group $\mathrm{Aut}(V)$ is a maximal compact subgroup of $\mathrm{GL}(\Omega)$ and coincides with the isotropy subgroup $\{g \in \mathrm{GL}(\Omega) : g(e) = e\}$ at the identity. The *trace form* $(x|y) := \mathrm{tr}(L(x)L(y))$ yields an $\mathrm{Aut}(V)$ -invariant positive definite inner product on V such that all operators $L(a)$, $a \in V$, are self-adjoint. The cone Ω is self-dual in the sense

$$\Omega = \{x \in V : (x|y) > 0 \text{ for all } y \in \Omega\}.$$

For all $x, y \in V$ define the linear operators

$$\begin{aligned} P(x, y) &:= L(x)L(y) + L(y)L(x) - L(x \circ y), \\ P(x) &:= P(x, x) = 2L(x)^2 - L(x^2). \end{aligned}$$

Then $P(a)$ is contained in $\mathrm{GL}(\Omega)$ and maps e to $a^2 \in \Omega$ for every invertible $a \in V$ (see [4, p. 325]). Actually, $(g, a) \mapsto P(a) \circ g$ defines a homeomorphism $\mathrm{Aut}(V) \times \Omega \rightarrow \mathrm{GL}(\Omega)$ and gives a Cartan decomposition

$$\mathfrak{gl}(\Omega) = \mathfrak{der}(V) \oplus L(V) \tag{5.1}$$

with $L(V) := \{L(a) : a \in V\}$, $\mathfrak{der}(V)$ the derivation algebra of the Jordan algebra V and $\mathfrak{gl}(\Omega) \subset \mathfrak{gl}(V)$ the Lie algebra of $\mathrm{GL}(\Omega) \subset \mathrm{GL}(V)$. Furthermore for $H = \Omega \oplus iV$ as above,

$$\mathrm{Aff}(H) := \{z \mapsto g(z) + iv : g \in \mathrm{GL}(\Omega), v \in V\} \subset \mathrm{Aut}(H)$$

is the group of all affine holomorphic transformations of H , where we consider $\mathrm{GL}(V)$ in the canonical way as a subgroup of $\mathrm{GL}(E)$. Since $\mathrm{GL}(\Omega)$ acts transitively on Ω , $\mathrm{Aff}(H)$ acts transitively on H .

The Lie algebra $\mathfrak{gl}(D)$ of the compact group $\mathrm{GL}(D)$ (for the corresponding bounded symmetric domain $D \subset E$) is canonically isomorphic to \mathfrak{k} (as defined in Section 4) and hence has complexification isomorphic to \mathfrak{l}^0 (see (3.7)). But also $\mathfrak{gl}(\Omega)$ has complexification isomorphic to \mathfrak{l}^0 (see (6.1)). The centers of $\mathfrak{gl}(\Omega)$ and $\mathfrak{gl}(D)$ have dimension equal to the number of irreducible factors of the bounded symmetric domain D . In case D has an irreducible factor of dimension > 1 , $\mathfrak{d}\mathrm{er}(V)$ is a proper maximal compact subalgebra of the semisimple part of $\mathfrak{gl}(\Omega)$. Therefore we can state

5.2. Lemma. *The real Lie algebras $\mathfrak{gl}(\Omega)$ and $\mathfrak{gl}(D)$ have isomorphic complexifications. In case D is not biholomorphically equivalent to a polydisk, the Lie algebras $\mathfrak{gl}(\Omega)$ and $\mathfrak{gl}(D)$ are not isomorphic.*

For the rest of the section we assume that the formally real Jordan algebra V is *simple*, that is, the symmetric tube domain H is irreducible, or equivalently, $\mathrm{GL}(\Omega)$ has 1-dimensional center $\{x \mapsto tx : t > 0\}$. Then

$$\mathrm{SL}(\Omega) := \mathrm{GL}(\Omega) \cap \mathrm{SL}(V)$$

is the semisimple part of $\mathrm{GL}(\Omega)$ and has codimension 1 in $\mathrm{GL}(\Omega)$. Every $\mathrm{GL}(\Omega)$ -orbit C in V is a connected locally closed cone and the associated tube manifold $C \oplus iV$ is a CR-submanifold of E , on which $\mathrm{Aff}(H)$ acts transitively. Clearly, the $\mathrm{GL}(\Omega)$ -orbits in V and the $\mathrm{Aff}(H)$ -orbits in E are in 1-1 correspondence to each other.

There exists a uniquely determined integer $r \geq 1$, the *rank* of V , such that every $a \in V$ has a representation

$$a = \lambda_1 e_1 + \cdots + \lambda_r e_r, \quad (5.3)$$

where e_1, \dots, e_r is a *frame* in V (i.e. a sequence of mutually orthogonal minimal idempotents in V with $e = e_1 + \cdots + e_r$) and the coefficients $\lambda_k \in \mathbb{R}$ (called the *eigenvalues* of a) are uniquely determined up to a permutation. For all integers $p, q \geq 0$ with $p + q \leq r$ denote by $C_{p,q}$ the set of all elements in V having p positive and q negative eigenvalues (multiplicities counted). Then $\Omega = C_{r,0}$ and $C_{q,p} = -C_{p,q}$ for all p, q . It is well known that the group $\mathrm{Aut}(V)$ acts transitively on the space of all frames in V . Furthermore, the element a with representation (5.3) is mapped by $P(c) \in \mathrm{GL}(\Omega)$ to $t_1^2 \lambda_1 e_1 + \cdots + t_r^2 \lambda_r e_r$ for every $c = t_1 e_1 + \cdots + t_r e_r \in \Omega$. This implies that every $C_{p,q}$ is contained in a $\mathrm{GL}(\Omega)$ -orbit. In case $p + q = r$ actually it is easy to see that $C_{p,q}$ is an open $\mathrm{GL}(\Omega)$ -orbit in V . But then for arbitrary p, q the closure $\overline{C}_{p,q} = \bigcup_{p' \leq p, q' \leq q} C_{p',q'}$ is $\mathrm{GL}(\Omega)$ -invariant as follows inductively from the formula $\overline{C}_{p,q} = \overline{C}_{p+1,q} \cap \overline{C}_{p,q+1}$ for $p + q < r$. The next statement now follows from the fact that $C_{p,q}$ is the complement in $\overline{C}_{p,q}$ of $(\bigcup_{p' < p} \overline{C}_{p',q}) \cup (\bigcup_{q' < q} \overline{C}_{p,q'})$.

5.4. Lemma. *There are precisely $\binom{r+2}{2}$ $\mathrm{GL}(\Omega)$ -orbits in V . These are the cones $C_{p,q}$.*

On V there exists a unique homogeneous real polynomial N of degree r (called the *generic norm* of V) with $N(e) = 1$ and $N^{-1}(0) = \{a \in V : a \text{ not invertible}\}$. The value $N(a)$ is the product of all eigenvalues of a , therefore N may be considered a generalization of the determinant for matrices. The characteristic polynomial

$$N(Te - x) = \sum_{k=0}^r N_{r-k}(x)T^k$$

determines homogeneous polynomials N_j of degree j for $0 \leq j \leq r$ on V that give local equations for every cone $C_{p,q}$, more precisely,

$$U \cap C_{p,q} = \{x \in U : N_j(x) = 0 \text{ for all } j > p + q\}$$

for every $a \in C_{p,q}$ and a suitable neighbourhood U of a in V .

In the following fix a $GL(\Omega)$ -orbit $C = C_{p,q}$ in V and let $M := M_{p,q} = C \oplus iV$ be the corresponding tube manifold in E . We denote by $\rho := p + q$ the common rank of all elements $a \in C$, that is, the number of all nonzero eigenvalues of a . Obviously, $T_aM = T_aC \oplus iV$ for the tangent spaces at every $a \in C \subset M$, and also $H_aM = T_aC \oplus iT_aC$ for the holomorphic tangent space at a . Therefore, every smooth vector field on C has a unique extension to a smooth vector field in $\Gamma(M, HM)$ that is invariant under all translations $z \mapsto z + iv, v \in V$.

For fixed $a \in C$ choose a representation (5.3) and denote by $c := \sum_{\lambda_k \neq 0} e_k$ the *support idempotent* of a , which does not depend on the chosen frame in (5.3). Consider the corresponding *Peirce decompositions* (see for instance [4, p. 155]) with respect to c ,

$$V = V_1 \oplus V_{1/2} \oplus V_0 \quad \text{and} \quad E = E_1 \oplus E_{1/2} \oplus E_0, \tag{5.5}$$

where V_k and $E_k = V_k \oplus iV_k$ are the k -eigenspaces of $L(c)$ in V and E . Then V_1 and V_0 are Jordan subalgebras with $V_1 \circ V_0 = 0$ and with identity elements c and $c' := e - c$ respectively. The operators $L(e_j)$ commute and induce a *joint Peirce decomposition*

$$V = \bigoplus_{1 \leq j \leq k \leq r} V_{jk} \tag{5.6}$$

into pairwise orthogonal (with respect to the trace form) Peirce spaces

$$V_{jk} = \{x \in V : 2L(e_l)x = (\delta_{jl} + \delta_{lk})x \text{ for all } l\}$$

satisfying

$$L(a) = \sum_{1 \leq j \leq k \leq r} \frac{\lambda_j + \lambda_k}{2} \pi_{jk}, \quad P(a) = \sum_{1 \leq j \leq k \leq r} \lambda_j \lambda_k \pi_{jk}, \tag{5.7}$$

where π_{jk} is the orthogonal projection on V with range V_{jk} . On the other hand,

$$V_1 = \sum_{\lambda_j \neq 0 \neq \lambda_k} V_{jk}, \quad V_{1/2} = \sum_{\lambda_j \neq 0 = \lambda_k} V_{jk}, \quad V_0 = \sum_{\lambda_j = 0 = \lambda_k} V_{jk}.$$

We have $V_{jj} = \mathbb{R}e_j$ for every j , and all V_{jk} with $j \neq k$ have the same dimension, which in case $r \geq 3$ can only be one of the numbers 1, 2, 4, 8 (see the classification in the next section). Furthermore, V_1 is the range of $P(a)$ and $V_{1/2} \subset L(a)V \subset V_1 \oplus V_{1/2}$. The same decompositions and spectral resolutions for $L(a)$ and $P(a)$ also occur for E in place of V . For every $z = x + iy \in E$ with $x, y \in V$ let $z^* := x - iy$ (we prefer z^* to \bar{z} as notation here since the conjugation bar serves a different purpose later, in Section 6). Then $z \mapsto z^*$ is a conjugate linear algebra involution of the complex Jordan algebra E that leaves all Peirce spaces E_k invariant. By setting $P(z, w) = L(z)L(w) + L(w)L(z) - L(z \circ w)$ and $P(z) := P(z, z)$ for $z, w \in E$ we extend our previous definition and get complex linear operators on E satisfying $(P(z)w)^* = P(z^*)w^*$.

5.8. Lemma. $T_a C = V_1 \oplus V_{1/2}$ and hence $H_a M = E_1 \oplus E_{1/2}$ for the corresponding tangent spaces at $a \in C$. In particular, $L(z + z^*)E \subset H_z M$ for all $z \in M$. Furthermore, $L(a)E = H_a M$ provided

$$\lambda_j + \lambda_k = 0 \quad \text{implies} \quad \lambda_j = \lambda_k = 0. \tag{*}$$

Proof. For every given $\lambda \in \partial \text{er}(V)$ denote by $v_0 \in V_0$ the component of $\lambda(a)$ with respect to the Peirce decomposition (5.5). Then $a \circ c' = 0$ implies $\lambda(a) \circ c' = -a \circ \lambda(c')$ and hence $v_0 = v_0 \circ c' \in L(a)V \subset V_1 \oplus V_{1/2}$, that is, $v_0 = 0$ and thus $\lambda(a) \in V_1 \oplus V_{1/2}$. Therefore (5.1) and $L(V)a = L(a)V$ imply

$$L(a)V \subset T_a C = \mathfrak{gl}(\Omega)a \subset V_1 \oplus V_{1/2}. \tag{**}$$

In case a satisfies (*) the spectral resolution for $L(a)$ in (5.7) implies $L(a)V = V_1 \oplus V_{1/2}$ and hence $T_a C = V_1 \oplus V_{1/2}$ by (**). Since $\dim(V_1 \oplus V_{1/2})$ does not depend on the choice of $a \in C$ and since on the other hand an $a \in C$ can always be chosen that satisfies (*) we conclude that $\dim T_a C = \dim(V_1 \oplus V_{1/2})$ and hence $T_a C = V_1 \oplus V_{1/2}$ by (**) for every choice of $a \in C$. Finally, for every $v \in V$ and $w := a + iv$ we have $L(a)E \subset H_a M = H_w M$, where the latter identity is obvious from the fact that $z \mapsto z + iv$ is a CR-automorphism of M . □

To simplify our arguments we assume without loss of generality in the following that $a \in C$ always satisfies the condition (*) above. Then the restriction of $L(a)$ to $H_a M = E_1 \oplus E_{1/2}$ is invertible and E_0 is the kernel of $L(a)$ in E . Also we assume for the rank $\rho = p + q$ of a that $\rho > 0$ (i.e. M is not totally real in E) and, in addition, that $\rho < r$ (i.e. M is not open in E). Furthermore we identify $E/H_a M$ in the canonical way with E_0 .

At this point it is convenient to compare the Jordan algebra product $z \circ w$ on E with the Jordan triple product $\{xyz\}$ associated with the bounded symmetric domain $D \subset E$ that is the image of $H \subset E$ under the Cayley transformation $z \mapsto (z - e) \circ (z + e)^{-1}$ (see the first part of this section). The following identities are well known:

$$\{z w z\} = P(z)w^*, \quad z \circ w = \{z e w\}, \quad z^* = \{e z e\} \quad \text{for all } z, w \in E.$$

For every Peirce space $V_{jk} = V_{kj}$ in (5.6) the inclusion $\{V_{jm} V_{mn} V_{nk}\} \subset V_{jk}$ holds for all index pairs, and all triple products of Peirce spaces that cannot be written this way vanish (after transposing indices in some pairs if necessary).

An important CR-invariant for every $a \in M$ is the (vector-valued) *Levi form*

$$\Lambda_a : H_a M \times H_a M \rightarrow E/H_a M,$$

which we define in the following way: For every $x, y \in H_a M$ choose smooth sections ξ, η in HM over M with $\xi_a = x, \eta_a = y$ and put

$$\Lambda_a(x, y) := ([\xi, \eta] + i[i\xi, \eta])_a \text{ mod } H_a M.$$

Since $[\xi, \eta] - [i\xi, i\eta], [\xi, i\eta] + [i\xi, \eta] \in HM$ in view of the integrability condition, it follows that $\Lambda_a(x, y)$ is conjugate linear in x , complex linear in y and satisfies $\Lambda_a(v, v) \in iT_a M/H_a M \subset E/H_a M$ for all $v \in H_a M$.

For every $v \in H_a M$ define the smooth vector field ξ^v on E by $\xi_z^v = \frac{1}{2}(z + z^*) \circ v \in E \cong T_z E$ for all $z \in E$. Then $\xi_a^v = a \circ v$ and $\xi_z^v \in H_z M$ for all $z \in M$ by Lemma 5.8. A simple computation shows

$$\Lambda_a(\xi_a^v, \xi_a^w) = (a \circ v)^* \circ w \text{ mod } H_a M.$$

Since the operator $L(a)$ is bijective on $H_a M$ we thus get

$$\Lambda_a(v, w) = v^* \circ L(a)^{-1} w \text{ mod } H_a M$$

for all $v, w \in H_a M$. In particular,

$$K_a M := \{w \in H_a M : \Lambda_a(v, w) = 0 \text{ for all } v \in H_a M\} = E_1 \tag{5.9}$$

for the *Levi kernel* at a . Indeed, $E_1 \subset K_a M$ follows from the fact that every Peirce space E_k is invariant under $L(z)$ for every $z \in E_1$. On the other hand, for every $w \in E_{1/2}$ the E_0 -component of $w^* \circ w$ is $\{w w c'\}$, which vanishes only for $w = 0$ (see [19, p. 10.6]). This proves the opposite inclusion $K_a M \subset E_1$.

The Levi kernel $K_a M = E_1$ is the image of E under the operator $P(a)$ and its restriction to this space is invertible. For every $w \in K_a M$ define the vector field η^w on E by $\eta_z^w = \frac{1}{4}P(z + z^*)w$. Then $\eta_a^w = P(a)w$ and, by Lemma 5.8 and (5.9), $\eta_z^w \in K_z M$ for all $z \in M$, where $K_z M$ is the Levi kernel at z . A simple calculation shows

$$[\xi^v, \eta^w]_a = P(a, a \circ (v + v^*))w - \frac{1}{2}(P(a)(w + w^*)) \circ v \in E_{1/2} \tag{5.10}$$

for all $v \in E_{1/2}, w \in E_1$. The part

$$\beta(\xi_a^v, \eta_a^w) := P(a, a \circ v^*)w$$

of (5.10) that is antilinear in v and linear in w is the sesquilinear map

$$\beta : E_{1/2} \times E_1 \rightarrow E_{1/2} \quad \text{given by} \quad \beta(v, P(a)w) = P(a, v^*)w \tag{5.11}$$

for $v \in E_{1/2} \cong H_a M/K_a M$ and $w \in E_1 = K_a M$.

5.12. Lemma. *$R = 0$ for the right β -kernel*

$$R := \{w \in E_1 : \beta(v, w) = 0 \text{ for all } v \in E_{1/2}\}.$$

Proof. Assume on the contrary $R \neq 0$. Since R is invariant under the involution $w \mapsto w^*$ and since $P(a)$ is bijective on V_1 there exists a vector $w \neq 0$ in V_1 with $P(a)w \in R$. Therefore $P(a, v)w = 0$ for all $v \in V_{1/2}$, or in triple product notation, $\{awv\} = 0$ for all $v \in V_{1/2}$. Furthermore $r \geq 2$ since $0 < \rho < r$ for the rank ρ of a .

For every $x \in V$ denote by $x_{jk} := \pi_{jk}(x) \in V_{jk}$ the corresponding component with respect to the decomposition (5.6). As $w \neq 0$ there exist j, k with $w_{jk} \neq 0$. In particular, $\lambda_j \lambda_k \neq 0$ and there exists an index n with $\lambda_n = 0$, that is, $0 \neq V_{kn} \subset V_{1/2}$. This forces

$$0 = \lambda_j^{-1} \{awv_{kn}\}_{jn} = \{e_j w_{jk} v_{kn}\} = 0 \quad \text{for all } v \in V.$$

From $V_{kk} = \mathbb{R}e_k$ and $2\{e_k e_k v_{kn}\} = v_{kn}$ we derive $j \neq k$ and hence $r \geq 3$. As a consequence, $V = \mathcal{H}_r(\mathbb{K})$ for \mathbb{K} one of the division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and \mathbb{O} (see the next section for the notation). If we realize $a \in \mathcal{H}_r(\mathbb{H})$ as the diagonal matrix $[\lambda_1, \dots, \lambda_r]$ and let $v_{kn} \in V_{kn}$ be the matrix that has 1 $\in \mathbb{K}$ at positions $(k, n), (n, k)$ and zeros elsewhere we get $w_{jk} = 0$, a contradiction. \square

The bilinear map β in (5.11) corresponds to the mapping β^2 in (7.2) evaluated at a . In particular, the right β -kernel R can be identified with \mathcal{H}^2 in Lemma 7.3. Thus M is 2-nondegenerate by Lemma 5.12 (recall that by (5.9), M is Levi degenerate), and we have all ingredients for the postponed

Proof of Theorem 4.7. We may assume that there exists in $E \subset Z$ a symmetric tube domain $H \subset E$ with $S = \text{Aut}(H)^0$. Since the S -orbit S is generic in Z the intersection $M := S \cap E$ is not empty. Clearly, M is invariant under the subgroup $\text{Aff}(H) \subset S$, and we claim that actually M is an $\text{Aff}(H)$ -orbit in E . This follows from the well known fact that in the irreducible Hermitian symmetric space Z of rank r the number of S -orbits is $\binom{r+2}{2}$ (see e.g. [13]), which by Lemma 5.4 is also the number of $\text{Aff}(H)$ -orbits in E . By the above discussion M is a 2-nondegenerate CR-manifold, and by homogeneity this is therefore also true for S . Finally, minimality of S follows from Theorem 3.6 in [13]. \square

6. Examples and applications

We begin by presenting briefly the classification of all formally real Jordan algebras in the notation of [16]. From $2x \circ y = (x + y)^2 - x^2 - y^2$ it is clear that the Jordan product is uniquely determined by the square mapping. For every integer $n \geq 1$ let \mathbb{K}_n be the vector space \mathbb{R}^n with the following additional structure: $(x|y) = \sum x_i y_i$ is the usual scalar product and $\bar{x} := (x_1, -x_2, \dots, -x_n)$ for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. The field \mathbb{R} is identified with $\{x \in \mathbb{K}_n : \bar{x} = x\}$ via $t \mapsto te$, where $e := (1, 0, \dots, 0)$. In addition, define the product of x and \bar{x} formally as $x\bar{x} := (x|x) \in \mathbb{R} \subset \mathbb{K}_n$. For every integer $r \geq 1$ denote by $\mathcal{H}_r(\mathbb{K}_n) \subset (\mathbb{K}_n)^{r \times r}$ the linear subspace of all Hermitian $r \times r$ -matrices (x^{ij}) over \mathbb{K}_n , that is, $x^{ij} \in \mathbb{K}_n$ and $\bar{x}^{ij} = x^{ji}$ for all $1 \leq i, j \leq r$. Obviously, $\mathcal{H}_r(\mathbb{K}_n)$ has real dimension $r + \binom{r}{2}n$.

Our conventions so far suffice to define all squares x^2 for $x \in \mathcal{H}_2(\mathbb{K}_n)$ (just formally as matrix squares). For $r > 2$ we need an additional structure on some \mathbb{K}_n : Identify \mathbb{K}_2

with the field \mathbb{C} , \mathbb{K}_4 with the (skew) field \mathbb{H} of quaternions and \mathbb{K}_8 with the real division algebra \mathbb{O} of octonions in such a way that $x \mapsto \bar{x}$ is the standard conjugation of these structures. With these identifications also squares are defined in $\mathcal{H}_r(\mathbb{K}_n)$ for all r and $n = 1, 2, 4, 8$ (again in terms of the usual matrix product). Now the simple formally real Jordan algebras are precisely the following, where r denotes the rank:

- $r = 1 : \mathbb{R}$,
- $r = 2 : \mathcal{H}_2(\mathbb{K}_n), n \geq 1$,
- $r = 3 : \mathcal{H}_3(\mathbb{R}), \mathcal{H}_3(\mathbb{C}), \mathcal{H}_3(\mathbb{H}), \mathcal{H}_3(\mathbb{O})$,
- $r > 3 : \mathcal{H}_r(\mathbb{R}), \mathcal{H}_r(\mathbb{C}), \mathcal{H}_r(\mathbb{H})$.

In $\mathcal{H}_2(\mathbb{K}_n)$ the generic norm is given by $N\begin{pmatrix} \alpha & x \\ \bar{x} & \beta \end{pmatrix} = \alpha\beta - x\bar{x}$, and

$$C_{1,0} = \left\{ \begin{pmatrix} \alpha & x \\ \bar{x} & \beta \end{pmatrix} \in \mathcal{H}_2(\mathbb{K}_n) : \alpha + \beta > 0, \alpha\beta = x\bar{x} \right\}$$

is the future light cone, which can be written in a more familiar form as

$$\{(t, x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+2} : t > 0, t^2 = x_0^2 + \dots + x_n^2\}$$

via $\alpha = t + x_0, \beta = t - x_0$. In $V = \mathcal{H}_r(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ the cone Ω is the subset of all positive definite matrices. The group of all transformations $x \mapsto gxg^*$ with $g \in \text{GL}(r, \mathbb{K})$ is an open subgroup of $\text{GL}(\Omega)$, in particular $P(a)$ is then the operator $x \mapsto axa$ for every $a \in V$. The kernel of ineffectivity for the action of $\text{GL}(r, \mathbb{K})$ on Ω is the group of all λ in the center of \mathbb{K} with $\lambda\bar{\lambda} = 1$ (that is, $\{\pm 1\}$ in the cases \mathbb{R} and \mathbb{H}). The complexified Jordan algebra E is the matrix algebra $\mathbb{C}^{r \times r}$ in case $\mathbb{K} = \mathbb{C}$ and is the Jordan subalgebra of all symmetric matrices in case $\mathbb{K} = \mathbb{R}$. The realization of \mathbb{H} as a matrix algebra,

$$\mathbb{H} = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C} \right\},$$

gives a canonical embedding $\mathcal{H}_r(\mathbb{H}) \subset \mathcal{H}_{2r}(\mathbb{C})$ as a Jordan subalgebra. The usual determinant function on $\mathcal{H}_{2r}(\mathbb{C})$ restricted to $V = \mathcal{H}_r(\mathbb{H})$ is the square of the generic norm of V . In case $\mathbb{K} = \mathbb{R}, \mathbb{C}$ the generic norm on $V = \mathcal{H}_r(\mathbb{K})$ coincides with the determinant. The subgroup $\text{SL}(r, \mathbb{H})$ has real codimension 1 in $\text{GL}(r, \mathbb{H})$ and Lie algebra $\mathfrak{sl}(r, \mathbb{H}) = \{x \in \mathfrak{gl}(r, \mathbb{H}) : \text{tr}(x) = 0\}$, where tr is the reduced (center-valued) trace on $\mathfrak{gl}(r, \mathbb{H})$ (see [22, p. 267] or [17] for details).

Now fix a simple formally real Jordan algebra $V = \mathcal{H}_r(\mathbb{K}_n)$ in the following and denote as before by $\Omega = \exp(V)$ ($= C_{r,0}$) the positive cone in V . There exists a unique $\text{GL}(\Omega)$ -invariant Riemannian metric on Ω that coincides at $e \in \Omega$ with the $\text{Aut}(V)$ -invariant inner product $(x|y) = \text{tr}(L(x)L(y))$ on $V = T_e\Omega$. Since $x \mapsto x^{-1}$ is an isometry of Ω with unique fixed point e in Ω , the positive cone is actually an irreducible Riemannian symmetric space of noncompact type.

As before let $E = V \oplus iV$ be the complexification of V . The tube domain $H = \Omega \oplus iV$ in E is homogeneous under the affine group $\text{Aff}(H)$ and it is well known that the full automorphism group $\text{Aut}(H)$ is generated by the subgroup $\text{Aff}(H)$ and the involutory transformation $z \mapsto z^{-1}$ whose unique fixed point in H is e . As already mentioned

before, H is biholomorphically equivalent to a bounded symmetric domain $D \subset E$ via the Cayley transformation $\gamma(z) = (z - e) \circ (z + e)^{-1}$. In fact, D is the interior of the convex hull of $\exp(iV)$ in E , and also $\exp(iV)$ is a Riemannian symmetric space (the compact dual of $\exp(V) = \Omega$).

Let again Z be the compact dual of D and $L := \text{Aut}(Z)^0$ with Lie algebra $\mathfrak{l} = \mathfrak{hol}(Z) \subset \mathfrak{hol}(E)$. The Cayley transformation γ is contained in L and has order 4. Therefore the Lie algebra \mathfrak{h} of $\text{Aut}(H)$ is also a real form of \mathfrak{l} . Since $z \frac{\partial}{\partial z} \in \mathfrak{h}$ the Lie algebra \mathfrak{h} has a \mathbb{Z} -grading (cf. also (3.7))

$$\mathfrak{h} = \mathfrak{h}^{-1} \oplus \mathfrak{h}^0 \oplus \mathfrak{h}^1$$

with $\mathfrak{h}^k = \mathfrak{h} \cap \mathfrak{l}^k$ a real form of the complex Lie algebra \mathfrak{l}^k , more precisely

$$\mathfrak{h}^{-1} = \{iv \frac{\partial}{\partial z} : v \in V\}, \quad \mathfrak{h}^0 = \mathfrak{gl}(\Omega) = [\mathfrak{h}^{-1}, \mathfrak{h}^1], \quad \mathfrak{h}^1 = \{i\{z v z\} \frac{\partial}{\partial z} : v \in V\}, \quad (6.1)$$

where $\{z v z\} = P(z)v$ is the corresponding Jordan triple product (cf. e.g. [14]). The affine subalgebra $\mathfrak{a} := \mathfrak{h}^{-1} \oplus \mathfrak{h}^0$ is the Lie algebra of $\text{Aff}(H)$. With (5.1) and the above we see that the codimension of every Lie algebra from the chain $\partial \text{er}(V) \subset \mathfrak{h}^0 \subset \mathfrak{a} \subset \mathfrak{h}$ in its successor is $\dim V = r + \binom{r}{2}n$.

The Lie algebra $\mathfrak{h} = \text{aut}(H)$ is explicitly known in all cases; actually the table below can be verified (cf. e.g. [7]). In the table, $\mathfrak{sl}(D)$ is the Lie algebra of the compact group $\text{SL}(D) := \text{GL}(D) \cap \text{SL}(E)$ with $\text{GL}(D)$ being isomorphic to the isotropy subgroup $\text{Aut}(H)_e$ at e . The notation used is as in [9, p. 354]. In particular, every exceptional simple real Lie algebra in the last line is uniquely identified by its character (in parentheses), which by definition is $\text{codim} - \text{dim}$ for a maximal compact subalgebra.

V	$\partial \text{er}(V)$	$\mathfrak{sl}(\Omega)$	$\text{aut}(H)$	$\mathfrak{sl}(D)$
\mathbb{R}	0	0	$\mathfrak{sl}(2, \mathbb{R})$	0
$\mathcal{H}_2(\mathbb{K}_n)$	$\mathfrak{so}(n+1)$	$\mathfrak{so}(1, n+1)$	$\mathfrak{so}(2, n+2)$	$\mathfrak{so}(n+2)$
$\mathcal{H}_r(\mathbb{R})$	$\mathfrak{so}(r)$	$\mathfrak{sl}(r, \mathbb{R})$	$\mathfrak{sp}(r, \mathbb{R})$	$\mathfrak{su}(r)$
$\mathcal{H}_r(\mathbb{C})$	$\mathfrak{su}(r)$	$\mathfrak{sl}(r, \mathbb{C})$	$\mathfrak{su}(r, r)$	$\mathfrak{su}(r) \times \mathfrak{su}(r)$
$\mathcal{H}_r(\mathbb{H})$	$\mathfrak{sp}(r)$	$\mathfrak{sl}(r, \mathbb{H})$	$\mathfrak{so}^*(4r)$	$\mathfrak{su}(2r)$
$\mathcal{H}_3(\mathbb{O})$	$\mathfrak{f}_4(-52)$	$\mathfrak{e}_6(-26)$	$\mathfrak{e}_7(-25)$	$\mathfrak{e}_6(-78)$

The semisimple Lie algebras $\mathfrak{sl}(\Omega)$ and $\mathfrak{sl}(D)$ have isomorphic complexifications (see Lemma 5.2) and in particular have the same dimensions. These are easily read off from the table as

$$\dim \mathfrak{sl}(\Omega) = \dim \mathfrak{sl}(D) = \begin{cases} 78, & V = \mathcal{H}_3(\mathbb{O}), \\ n(r^2 - 2) + \binom{n}{2} + 1, & \text{otherwise.} \end{cases} \quad (6.2)$$

Denote by $s \in \text{Aut}(D) \subset \text{Aut}(Z)$ the symmetry $s(z) \equiv -z$ of D . Then $g := \text{Ad}(s)$ satisfies $g(\xi) = (-1)^k \xi$ for all $\xi \in \mathfrak{l}^k$ and hence also leaves $\mathfrak{h} \subset \mathfrak{l}$ invariant. It is obvious that $\pm \text{Aut}(H) := \text{Aut}(H) \cup (s \circ \text{Aut}(H))$ is a group containing $\text{Aut}(H)$ as a subgroup of index 2. In the same way we also define the subgroups $\pm \text{GL}(\Omega) \subset \pm \text{Aff}(H) \subset \pm \text{Aut}(H) \subset \text{Aut}(Z)$. As an improvement of Corollary 4.6 we state:

6.3. Proposition. *The group $\pm \text{Aut}(H)$ is isomorphic to $\text{Aut}(\mathfrak{h})$ via Ad. With this identification*

$$\begin{aligned} \pm \text{GL}(\Omega) &= \{g \in \text{Aut}(\mathfrak{h}) : g(\delta) = \delta\}, \\ \pm \text{Aff}(H) &= \{g \in \text{Aut}(\mathfrak{h}) : g(\mathfrak{a}) = \mathfrak{a}\} \quad \text{for } \mathfrak{a} = \mathfrak{h}^{-1} \oplus \mathfrak{h}^0. \end{aligned}$$

Proof. The antiholomorphic transformation $\tau(z) = z^*$ of H induces the same Lie algebra automorphism of \mathfrak{h} as s . Therefore the first claim follows from Proposition 4.5 in [13] (stated for the biholomorphically equivalent domain D). Suppose $g(\delta) = \delta$. Then g leaves the $\text{ad}(\delta)$ -eigenspace \mathfrak{h}^{-1} invariant, that is, $g \in \pm \text{Aut}(H)$ is linear and hence in $\pm \text{GL}(\Omega)$. Next, assume $g(\mathfrak{a}) = \mathfrak{a}$. Since $g(iV) \cap iV \neq \emptyset$ there exist translations $t_1, t_2 \in \exp(\mathfrak{h}^{-1}) \subset \text{Aff}(H)$ such that $h(0) = 0$ for $h := t_1 g t_2$. But h leaves \mathfrak{a} as well as \mathfrak{h}^0 invariant and hence induces an invertible endomorphism of $\mathfrak{a}/\mathfrak{h}^0 \cong \mathfrak{h}^{-1}$. Therefore $[g(\delta), g(\alpha)] = -g(\alpha)$ for all $\alpha \in \mathfrak{h}^{-1}$ implies $h(\delta) = \delta$ and hence $h \in \pm \text{GL}(\Omega)$, that is, $g \in \pm \text{Aff}(H)$. \square

For the rest of the section fix a $\text{GL}(\Omega)$ -orbit $C := C_{p,q}$ in V together with a point $a \in C$ and denote by $M := M_{p,q} = C + iV$ the corresponding tube manifold. As before, $\rho := p + q \leq r$ is the rank of a . For convenience we call $\rho' := r - \rho$ the *corank* of a . The affine group $\text{Aff}(H)$ acts transitively on M , in case $p = q$ also the bigger group $\pm \text{Aff}(H)$ acts on M (since then $C = -C$). From Lemma 5.8 it is easily derived that M has CR-dimension $\rho + \binom{\rho}{2}n + \rho\rho'n$ and CR-codimension $\rho' + \binom{\rho'}{2}n$. In particular, M is of hypersurface type if and only if $\rho' = 1$. Furthermore, by (5.9) the complex dimension of the Levi kernel at every point of M is $\rho + \binom{\rho}{2}n$.

The isotropy subgroup

$$(\pm \text{Aut}(H))_a := \{g \in \pm \text{Aut}(H) : g(a) = a\} \subset \text{Aut}(Z)$$

can be canonically identified with a subgroup of $\text{Aut}(M, a)$ and clearly coincides with the isotropy subgroup $\text{Aut}(M)_a$ in case $p \neq q$.

6.4. Proposition. *In case M is neither totally real nor open in E ,*

$$\text{Aut}(M, a) = (\pm \text{Aut}(H))_a$$

for every $a \in M$. In particular,

$$\dim \text{Aut}(M, a) = \begin{cases} 72 + 8\rho', & V = \mathcal{H}_3(\mathbb{O}), \\ n(r^2 + \binom{\rho'}{2}) - 2 + \binom{n}{2} + \rho' + 2, & \text{otherwise,} \end{cases}$$

where $\rho' = r - \text{rank}(a)$ is the corank of a in V .

Proof. Let S be the $\text{Aut}(H)$ -orbit of a in Z . Then M is an open subset of S and the pair (S, Z) belongs to the class \mathfrak{C} . By Theorem 4.5 every germ in $\text{Aut}(M, a)$ extends to a transformation $g \in \text{Aut}(Z)$ with $g(S) = S$. Therefore $g \in \text{Aut}(\mathfrak{h}) \cong \pm \text{Aut}(H)$ as a consequence of Proposition 6.3. The dimension formula follows from (6.2), $\dim \text{Aut}(M, a) = \dim \text{Aut}(M) - \dim M = \dim \mathfrak{gl}(\Omega) + \text{codim}_{\text{CR}}$ and the explicit expression for the last summand above. \square

As an example, if $r = 2$, $\rho' = 1$ and $d := n + 2 \geq 3$, that is, $M \subset \mathbb{C}^d$ is the tube over the future light cone $\{x \in \mathbb{R}^d : x_1 = \sqrt{x_2^2 + \dots + x_d^2} > 0\}$ in d -dimensional space time we have $\dim \text{Aut}(M, a) = \binom{d}{2} + 2$ for every $a \in M$.

We proceed with the above fixed cone $C = C_{p,q}$. Let $f(z) \frac{\partial}{\partial z} \in \mathfrak{h} = \text{aut}(H)$ be an arbitrary vector field. By (6.1), f has the form $f(z) = \lambda(z) + i(\{zvz\} - w)$ for suitable $\lambda \in \mathfrak{gl}(\Omega)$ and $v, w \in V$. For every $a \in C$ we then have

- 6.5. Lemma.** (i) $f(a) = 0 \Leftrightarrow \lambda(a) = 0$ and $w = \{ava\}$.
 (ii) $f'(a) = 0 \Leftrightarrow \lambda = 0$ and $v \in V_0$, where V_0 is the Peirce space according to (5.5).

In particular, $\{i\{zvz\} \frac{\partial}{\partial z} : v \in V_0\}$ is the space of all vector fields in $\text{aut}(H)$ with vanishing 1-jet at a . The dimension of this space coincides with the CR-codimension of M .

Proof. (i) follows from $\lambda(a) \in V$ and $i(\{ava\} - w) \in iV$. Obviously, $f'(a)(z) = \lambda(z) + 2i\{avz\}$ for all $z \in E$ and in particular for all $z \in V$. Therefore $f'(a) = 0$ is equivalent to $\lambda = 0$ and $\{avz\} = 0$ for all $z \in V$. But the latter condition is equivalent to $v \in V_0$. The last claim follows from the fact that V_0 is isomorphic to the normal space at a to M in E . □

6.6. Corollary. *The following conditions are equivalent:*

- (i) Every $\xi \in \text{aut}(H)$ is uniquely determined by its 1-jet at $a \in M$.
 (ii) M is open in E .

Proof. Both conditions are equivalent to $V_0 = \{0\}$. □

Recall that $\text{aut}_1(M, a)$ is the space of all germs of vector fields in $\mathfrak{hol}(M, a)$ that vanish of order ≥ 2 at a , that is, which have vanishing 1-jet at a . Lemma 6.5 also immediately implies

6.7. Corollary. $\text{aut}_1(M, a) = \{i\{zvz\} \frac{\partial}{\partial z} : v \in V_0\}$ if M is neither totally real nor open in E .

Denote by $\text{aut}(M) \subset \mathfrak{hol}(M)$ the subset of all vector fields that are *complete on M* , that is, generate global flows on M .

6.8. Lemma. $\mathfrak{h} \cap \text{aut}(M) = \mathfrak{h}^{-1} \oplus \mathfrak{h}^0 (= \mathfrak{a})$ if M is not open in E .

Proof. The linear span of $\mathfrak{b} := \mathfrak{h} \cap \text{aut}(M)$ in $\mathfrak{hol}(M)$ has finite dimension by [20], therefore $\mathfrak{b} \subset \mathfrak{h}$ is a Lie subalgebra with $\mathfrak{a} \subset \mathfrak{b}$. Assume there exists a vector field $\xi \in \mathfrak{b} \setminus \mathfrak{a}$. Without loss of generality we may assume $\xi = i\{zvz\} \frac{\partial}{\partial z} \in \mathfrak{h}^1$ for some $v \in V$. There exist minimal orthogonal idempotents e_1, \dots, e_r in E with $v = v_1e_1 + \dots + v_re_r$, and we may assume $v_1 = 1$. Since M is not open in E there exists a point $c = c_1e_1 + \dots + c_re_r$ in M with $c_1 = i$. The vector field ξ is tangent to the linear subspace $\sum_j \mathbb{C}e_j$ of E . As a consequence, $g(t) := \exp(t\xi)(c)$ has the form $g(t) = \sum_j g_j(t)e_j$ with certain real-analytic functions $g_j : \mathbb{R} \rightarrow \mathbb{C}$. It is easily verified that $g_1(t) = i(1+t)^{-1}$, which has a singularity at $t = -1$ and thus gives a contradiction. □

It is easily seen that $M = M_{p,q}$ is convex if and only if $M = H$, $M = -H$ or $M = iV$ (that is, if $\{p, q\} \subset \{0, r\}$).

6.9. Proposition. *In case M is not convex in E we have*

$$\text{Aut}(M) = \text{Aff}(M) = \begin{cases} \text{Aff}(H), & p \neq q, \\ \pm \text{Aff}(H), & p = q. \end{cases}$$

Proof. Case 1: M not open in E . Then $\text{hol}(M) = \mathfrak{h}$ and Lemma 6.8 imply $\text{aut}(M) = \mathfrak{a}$. As a consequence, every $g \in \text{Aut}(M) \subset \text{Aut}(\mathfrak{h})$ leaves \mathfrak{a} invariant, i.e. $g \in \pm \text{Aff}(H)$ by Proposition 6.3. In particular, $g \in \text{Aff}(M)$ and also $g \in \text{Aff}(H)$ if $M \neq -M$.

Case 2: M open in E . Then $pq \neq 0$ and it is easily seen that E is the convex hull of M . By [10, 2.5.10] every holomorphic function on M has a holomorphic extension to E , that is, $\text{Aut}(M) \subset \text{Aut}(E)$ by holomorphic extension. Without loss of generality we assume $p \leq q$ and fix $g \in \text{Aut}(M)$. Then either g maps the boundary part $M_{p-1,q}$ onto itself or maps $M_{p-1,q}$ to $M_{p,q-1}$. The latter case only happens if $p = q$ and then we replace g by $-g$ implying that g leaves $M_{p-1,q}$ invariant. By case 1 the restriction of g to $M_{p-1,q}$ extends to an affine transformation in $\text{Aff}(H)$, and the claim follows. \square

Finally, we come back to the tubes over future light cones: This corresponds to the rank-2 case $V = \mathcal{H}_2(\mathbb{K}_n)$ with $n \geq 1$. Put $m := n + 2$, identify the future cone in V with

$$\Omega = \left\{ x \in \mathbb{R}^m : x_1 > \sqrt{x_2^2 + \dots + x_m^2} \right\}$$

and let $e := (1, 0, \dots, 0) \in \Omega$ be fixed. Then $\text{GL}(\Omega)$ is the special Lorentz group $\text{O}(1, m-1)^+$, and the isotropy subgroup at e is the orthogonal group $\text{O}(m-1)$ acting in the canonical way on the orthogonal complement of e in \mathbb{R}^m . In particular, both groups have two connected components. As before let $H := \Omega \oplus i\mathbb{R}^m \subset \mathbb{C}^m$ be the corresponding right halfplane. It is known that the realization of H as a bounded symmetric domain in \mathbb{C}^m is the *Lie ball*

$$D = \left\{ z \in \mathbb{C}^m : (z|z) + \sqrt{(z|z)^2 - |\langle z, z \rangle|^2} < 1 \right\},$$

where $(z|w) = \sum z_k \bar{w}_k$ and $\langle z, w \rangle = \sum z_k w_k$ are the standard inner product and symmetric bilinear form on \mathbb{C}^m respectively. It is obvious that the orthogonal group $\text{O}(m)$ leaves D invariant and also that $\text{U}(1)$ acts on D by multiplication. Therefore, the direct product group $\text{U}(1) \times \text{O}(m)$ acts linearly on D , and it is known that actually $\text{GL}(D) = (\text{U}(1) \times \text{O}(m)) / \{\pm(1, \mathbb{1})\}$. In particular, the groups $\text{Aut}(D)$ and $\text{GL}(D)$ have two connected components if m is even and are connected otherwise. The compact dual Z of D is a complex quadric in the complex projective space \mathbb{P}_{m+1} .

The boundary of D is the union $\partial D = S_0 \cup S_1$ of two $\text{Aut}(D)$ -orbits: The Shilov boundary

$$S_0 := \{ z \in \mathbb{C}^m : (z|z) = |\langle z, z \rangle| = 1 \}$$

is also a $\text{GL}(D)$ -orbit and coincides with the set of extreme points of the closed ball \bar{D} , while S_1 is the smooth boundary part of D . The CR-manifold S_1 is an $\text{Aut}(D)$ -equivariant disk bundle, where the fibers are the holomorphic arc components of S_1 in the sense of [30]. For instance, the analytic disk through $e_1 := (1/2, i/2, 0, \dots, 0) \in S_1$ is $\{e_1 + te_2 :$

$|t| < 1$ }, where $e_2 := (1/2, -i/2, 0, \dots, 0)$. The boundary of each such disk is contained in the orbit S_0 .

As before let $M := C \oplus i\mathbb{R}^n$ be the tube over the future light cone

$$C = \{x \in \mathbb{R}^m : x_1 > 0, x_1^2 = x_2^2 + \dots + x_m^2\}.$$

There is a transformation in $\text{Aut}(Z)$ (Cayley transformation) mapping H biholomorphically to D and mapping M to a dense open subset of S_1 . In particular, M and S_1 are locally equivalent as CR-manifolds.

In the following we specialize to $m = 3$. \mathbb{R}^3 and \mathbb{C}^3 are identified with the spaces V and E of symmetric matrices in $\mathbb{R}^{2 \times 2}$ and $\mathbb{C}^{2 \times 2}$ respectively. In particular, Ω is the cone of positive definite matrices in V and $e \in \Omega$ becomes the unit 2×2 -matrix. The group $\text{Aut}(H)$ is isomorphic to the real symplectic group

$$\text{Sp}(2, \mathbb{R}) := \{A \in \mathbb{R}^{4 \times 4} : A^t J A = J\},$$

where $J := \begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix}$. Then the action of $\text{Sp}(2, \mathbb{R})$ on H is more easily described if we replace the right halfplane H by Siegel's upper halfplane $iH = V \oplus i\Omega$: Write every $A \in \text{Sp}(2, \mathbb{R})$ in block form $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with 2×2 -blocks and put

$$A(z) := (az + b)(cz + d)^{-1} \quad \text{for all } z \in iH.$$

For every $s \in C$ the isotropy subgroup of $\text{Sp}(2, \mathbb{R})$ at the point $is \in iM$ is isomorphic to $\text{Aut}(M, s)$ and consists of the 5-dimensional group of all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}(2, \mathbb{R})$ satisfying the linear equations $as = sd$ and $b = -scs$ on $\mathbb{R}^{4 \times 4}$. Furthermore, there is a 1-parameter subgroup of $\text{Aut}(M, s)$ whose elements all have the same 1-jet at $s \in M$.

7. Appendix: Nondegeneracy conditions

In the following we recall the notion of *finite nondegeneracy* (see e.g. [3]) and give equivalent descriptions for a certain class that contains in particular all homogeneous CR-manifolds.

Let M be a smooth (abstract) CR manifold with tangent bundle TM and holomorphic subbundle $HM \subset TM$. The complex structure on every holomorphic tangent space $H_p M \subset T_p M$ will be denoted by J . Thus $J : HM \rightarrow HM$ is a smooth bundle transformation with $J^2 = -\text{id}$. Denote by $\mathbb{C}TM := \mathbb{C} \otimes TM$ the complexified tangent bundle of M that contains the complexification $\mathbb{C}HM := \mathbb{C} \otimes HM$ in a canonical way as a complex subbundle. Extend J to a complex linear bundle transformation of $\mathbb{C}HM$, which is then the direct sum of two complex subbundles $H^{1,0}M$ and $H^{0,1}M$, the eigenbundles of J corresponding to the eigenvalues i and $-i$.

Consider the subbundles $H^{0,1}M \subset \mathbb{C}HM$ of $\mathbb{C}TM$ and denote by $A^{1,0}M \supset A^0M$ the corresponding annihilator subbundles in the complexified cotangent bundle $\mathbb{C} \otimes T^*M$. For every $p \in M$, $A_p^{1,0}M$ consists of all linear forms on $\mathbb{C}T_p M$ that are J -linear on $H_p M$. As shorthand let us also write $\mathcal{A}^0 := \Gamma(M, A^0M)$, $\mathcal{A}^{1,0} := \Gamma(M, A^{1,0}M)$, $\mathcal{H}^{1,0} := \Gamma(M, H^{1,0}M)$ and $\mathcal{H}^{0,1} := \Gamma(M, H^{0,1}M)$ for the corresponding spaces of

smooth sections over M . Clearly, all these are in a natural way modules over the ring $\mathcal{F} := C^\infty(M, \mathbb{C})$ of smooth complex-valued functions on M .

For every vector field $X \in \Gamma(M, \mathbb{C}TM)$ and every complex k -form ω on M the contraction $\iota_X \omega$ is the $(k-1)$ -form defined by

$$(\iota_X \omega)(Y_2, \dots, Y_k) = \omega(X, Y_2, \dots, Y_k) \quad \text{if } k > 0, \quad \iota_X \omega = 0 \quad \text{if } k = 0.$$

Also, the Lie derivative with respect to X on the space of all complex exterior differential forms is defined by

$$L_X := d \circ \iota_X + \iota_X \circ d.$$

For all $X \in \mathcal{H}^{0,1}$ and $\omega \in \mathcal{A}^{1,0}$ we have $\iota_X \omega = \omega(X) = 0$ and hence

$$(L_X \omega)(Y) = d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]). \tag{7.1}$$

The integrability condition $[\mathcal{H}^{0,1}, \mathcal{H}^{0,1}] \subset \mathcal{H}^{0,1}$ therefore implies $(L_X \omega)(Y) = 0$ for all $Y \in \mathcal{H}^{0,1}$ and X, ω as above, that is, the linear subspace $\mathcal{A}^{1,0} \subset \Gamma(M, \mathbb{C} \otimes T^*M)$ is L_X -invariant for every $X \in \mathcal{H}^{0,1}$. As a consequence, we can define \mathcal{A}^{k+1} , $k \geq 0$, inductively to be the smallest linear subspace of $\mathcal{A}^{1,0}$ that contains \mathcal{A}^k and $L_X(\mathcal{A}^k)$ for every $X \in \mathcal{H}^{0,1}$. Now M is called *finitely nondegenerate* at $p \in M$ if

$$\mathcal{A}_p^k := \{Y_p : Y \in \mathcal{A}^k\} = \mathcal{A}_p^{1,0}$$

for some k , and is called *k-nondegenerate* at p if k is minimal with this property. Furthermore we say that M has *constant degeneracy* if $\dim \mathcal{A}_p^k$ does not depend on $p \in M$ for every k . This property is for instance satisfied if M is *locally homogeneous*, that is, for every $x, y \in M$ there are open neighbourhoods U of x and V of y together with a CR-diffeomorphism $\varphi : U \rightarrow V$ satisfying $\varphi(x) = y$.

For the rest of the section we assume that M has constant degeneracy. For manifolds of this type we give an equivalent approach to finite nondegeneracy using Lie brackets of vector fields rather than Lie derivatives (cf. also [8]).

For the ascending chain $(\mathcal{A}^k)_{k \geq 0}$ we have the descending dual chain of kernels

$$\mathcal{H}^k := \{Y \in \mathcal{H}^{1,0} : \omega(Y) = 0 \text{ for all } \omega \in \mathcal{A}^k\}$$

with $\mathcal{H}^0 = \mathcal{H}^{1,0}$. It is clear that M is finitely nondegenerate at $p \in M$ if and only if $\mathcal{H}^k = 0$ for some k . The \mathcal{F} -modules \mathcal{H}^k can also be characterized in a direct way. For this put $\mathcal{H}^{-1} := \Gamma(M, \mathbb{C}TM)$ and define for every $k \geq 0$ the \mathcal{F} -bilinear map

$$\beta^k : \mathcal{H}^{0,1} \times \mathcal{H}^k \longrightarrow \mathcal{H}^{-1}/(\mathcal{H}^{0,1} \oplus \mathcal{H}^k) \tag{7.2}$$

by $\beta^k(X, Y) = \pi^k([X, Y])$, where $\pi^k : \mathcal{H}^{-1} \rightarrow \mathcal{H}^{-1}/(\mathcal{H}^{0,1} + \mathcal{H}^k)$ is the canonical projection.

7.3. Lemma. *For every $k \geq 0$,*

$$\mathcal{H}^{k+1} = \{Y \in \mathcal{H}^k : \beta^k(\mathcal{H}^{0,1}, Y) = 0\}$$

is the right β^k -kernel. In particular, \mathcal{H}_p^1 is the Levi kernel at $p \in M$. Furthermore, in case $k \geq 1$ the map β^k takes values in the linear subspace

$$(\mathcal{H}^{0,1} \oplus \mathcal{H}^{k-1})/(\mathcal{H}^{0,1} \oplus \mathcal{H}^k) \cong \mathcal{H}^{k-1}/\mathcal{H}^k.$$

Proof. Fix $k \geq 0$ and assume $\mathcal{A}^k(\mathcal{H}^k) = 0$ and $[\mathcal{H}^{0,1}, \mathcal{H}^k] \subset \mathcal{H}^{0,1} + \mathcal{H}^{k-1}$ as induction hypothesis. Notice that these assumptions are automatically satisfied in case $k = 0$. For every $X \in \mathcal{H}^{0,1}$, $Y \in \mathcal{H}^k$ and $\omega \in \mathcal{A}^k$ the equality (7.1) and $\omega(X) = \omega(Y) = 0$ imply

$$(L_X\omega)(Y) = -\omega([X, Y]).$$

From the induction hypothesis we therefore get, for every $Y \in \mathcal{H}^k$,

$$\begin{aligned} Y \in \mathcal{H}^{k+1} &\Leftrightarrow \omega([X, Y]) = 0 \text{ for all } X \in \mathcal{H}^{0,1}, \omega \in \mathcal{A}^k \\ &\Leftrightarrow [X, Y] \in (\mathcal{H}^{0,1} + \mathcal{H}^k) \text{ for all } X \in \mathcal{H}^{0,1} \\ &\Leftrightarrow \beta^k(X, Y) = 0 \text{ for all } X \in \mathcal{H}^{0,1}. \end{aligned}$$

Thus \mathcal{H}^{k+1} is the right β^k -kernel and also $[\mathcal{H}^{0,1}, \mathcal{H}^{k+1}] \subset \mathcal{H}^{0,1} + \mathcal{H}^k$. Finally, the mapping

$$H_p^{1,0}M \times H_p^{1,0}M \rightarrow \mathbb{C}T_pM, \quad (X_p, Y_p) \mapsto (\beta^0(\bar{X}, Y))_p,$$

is a multiple of the Levi form at $p \in M$, that is, \mathcal{H}_p^1 is the Levi kernel at p . \square

Finally, we mention that using the natural isomorphisms between HM , $H^{1,0}M$ and $H^{0,1}M$, we can also regard $H_p^kM := \mathcal{H}_p^k$ as a complex (that is, J -invariant) subspace of H_pM and β_p^k as a map $H_pM \times H_pM \rightarrow H_p^{k-1}M/H_p^kM$ between real tangent spaces, given by the part of the Lie bracket which is J -antilinear in the first and J -linear in the second argument. We used this interpretation in Section 5 as a criterion for 2-nondegeneracy.

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