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BOUNDED SYMMETRIC DOMAINS AND DERIVED GEOMETRIC STRUCTURES

Abstract. — Every homogeneous circular convex domain $D \subset \mathbb{C}^n$ (a bounded symmetric domain) gives rise to two interesting Lie groups: The group $G = \mathsf{Aut}(D)$ of all biholomorphic automorphisms of D and its isotropy subgroup $K \subset \mathsf{GL}(n, \mathbb{C})$ at the origin (a maximal compact subgroup of G). The group G acts in a natural way on the compact dual X of D (a certain compactification of $\mathbb{C}^n$ that generalizes the Riemann sphere in case D is the unit disk in $\mathbb{C}$). Various authors have studied the orbit structure of the G-space X, here we are interested in the *Cauchy-Riemann structure* of the G-orbits in X (which in general are only real-analytic submanifolds of X). Also, we discuss certain K-orbits in the Grassmannian of all linear subspaces of $\mathbb{C}^n$ that are closely related to the geometry of the bounded symmetric domain D.

Key words: Bounded symmetric domains; Lie groups; Jordan triple systems; CR-structures; orbit structures; Grassmannians.

1. Introduction

Let me start with a remark that seems totally unrelated to the title of this talk: For a given integer $n \geq 1$ let $E = \mathbb{C}^{n\times n}$ be the algebra of all complex $n \times n$-matrices and consider the subgroup

$$\mathit{\Gamma} := \{z \mapsto azb \ \text{ or } \ z \mapsto az'b \ : \ a, b \in \mathsf{GL}(n, \mathbb{C})\}$$

of the linear group $\mathsf{GL}(E)$, where z' denotes the transpose of z. Then there are known various conditions (mostly in the more general situation of an arbitrary field in place of $\mathbb{C}$) that force a linear operator L on E to be in $\mathit{\Gamma}$ – such as respecting the determinant function [6], invertibility [2], or the spectrum (see [1] and the references therein). Just for curiosity let us give a short proof involving holomorphy that also works for the more general space $E = \mathbb{C}^{n\times m}$ of rectangular matrices and $\mathit{\Gamma} \subset \mathsf{GL}(E)$ defined in the additional cases $n \neq m$ by

$$\mathit{\Gamma} := \{z \mapsto azb \ : \ a \in \mathsf{GL}(n, \mathbb{C}), b \in \mathsf{GL}(m, \mathbb{C})\} \ .$$

We claim that for all integers $1 \leq k \leq n \leq m$ and $E = \mathbb{C}^{n\times m}$ the group $\mathit{\Gamma}$ consists precisely of all linear operators L on E leaving the subset $E_k \subset E$ of all rank-k-matrices invariant. Then our argument is as follows (compare [12], every $z \in E$ will also be considered as a linear operator $\mathbb{C}^m \to \mathbb{C}^n$): First of all, it can be seen from $\overline{E}_k \backslash E_k = \overline{E}_{k-1}$ that our claim for $k = 1$ will also imply the case $k > 1$. Now, $z \mapsto \big(\mathrm{im}(z), \mathrm{im}(z')\big)$ gives a holomorphic $\mathbb{C}^*$-bundle $E_1 \to \mathbb{P}_{n-1} \times \mathbb{P}_{m-1}$, where for every integer q we denote by $\mathbb{P}_q$ the projective space of all complex lines in $\mathbb{C}^{q+1}$. The assumption on L implies that L is in $\mathsf{GL}(E)$ and hence induces a biholomorphic automorphism of $\mathbb{P}_{n-1} \times \mathbb{P}_{m-1}$. But these are all known to come from $\mathit{\Gamma}$.

The group $\varGamma$ is a reductive complex Lie group with maximal compact subgroup

$$K := \begin{cases} \{z \mapsto uzv \text{ or } z \mapsto uz'v : u, v \in \mathsf{U}(n)\} & n = m > 1 \\ \{z \mapsto uzv : u \in \mathsf{U}(n),\ v \in \mathsf{U}(m)\} & \text{otherwise} \end{cases}$$

where $\mathsf{U}(n) \subset \mathsf{GL}(n, \mathbb{C})$ is the unitary subgroup. Also, K is the fixed point set of the antiholomorphic group automorphism of $\varGamma$ defined by $[z \mapsto azb] \mapsto [z \mapsto (a^*)^{-1}z(b^*)^{-1}]$ (and in case $n = m > 1$ for the second connected component of $\varGamma$ in the same way).

The matrix domain

$$D := \{z \in \mathbb{C}^{n\times m} : (\mathbb{1} - zz^*) \text{ positive definite}\}$$

is an example of a bounded symmetric domain in $E = \mathbb{C}^{n\times m}$ and it is known that K coincides with the subgroup $\mathsf{GL}(D) \subset \mathsf{GL}(E)$ of all linear transformations mapping D onto itself. Every matrix $z \in E = \mathbb{C}^{n\times m}$ has as linear operator $\mathbb{C}^m \to \mathbb{C}^n$ an operator norm $\|z\|$, which is also the biggest singular value of z (i.e. the biggest eigenvalue of the hermitian matrix $\sqrt{zz^*}$). This makes E to a complex Banach space with open unit ball D, and K is also the group of all linear isometries of E. As a consequence we get for instance the following fact in linear algebra: *A linear operator on $\mathbb{C}^{n\times m}$ respects biggest singular values if and only if it belongs to the group K.*

The group $\varGamma$ can be embedded canonically in a complex Lie group Σ (see (1.1) below) associated with a certain compactification X of E. In case $n = 1$ (then $E = \mathbb{C}^m$, $K = \mathsf{U}(m)$ and $\varGamma = \mathsf{GL}(m, \mathbb{C})$) the space X is the projective space $\mathbb{P}_m$ containing $\mathbb{C}^m$ in the natural way. In case $n \geq 1$ arbitrary, X is the Grassmann manifold of all m-planes in $\mathbb{C}^{m+n}$ and E is embedded in X by identifying every $z \in E$ with its graph in $\mathbb{C}^m \times \mathbb{C}^n$. Then X is a compact connected complex manifold (the dual of D in the sense of hermitian symmetric spaces) and the boundary ∂E of the domain $E \subset X$ is a complex-analytic subset. The group $\Sigma := \mathsf{Aut}(X)$ of all biholomorphic automorphisms of X is a simple complex Lie group acting transitively, holomorphically and rationally on X. Its connected identity component is the group $\mathsf{PGL}(m + n, \mathbb{C})$ of all collineations, only in case $n = m > 1$ there is a second connected component (given by all correlations with respect to nondegenerate symmetric bilinear forms on $\mathbb{C}^{m+n}$).

Every $g \in \varGamma$ as well as every $g \in \mathsf{Aut}(D)$ extends to a biholomorphic automorphism of X. We may therefore identify $\mathsf{Aut}(D)$ with the subgroup $G := \{g \in \mathsf{Aut}(X) : g(D) = D\}$, and $\varGamma$ can be identified with the subgroup $\{g \in \mathsf{Aut}(X) : g|_E \in \mathsf{GL}(E)\}$. In this sense we get inclusions

$$\begin{array}{ccccc} G & \subset & \Sigma & & \\ \cup & & \cup & & \\ K & \subset & \varGamma & \subset & \mathsf{GL}(E) \end{array} \tag{1.1}$$

where $G = \mathsf{Aut}(D)$ is a real form of $\Sigma = \mathsf{Aut}(X)$ and all groups have the same number of connected components. Furthermore, K is the intersection and Σ is the group span of G and $\varGamma$. Every $g \in \mathsf{GL}(E)$ extends to a birational transformation of X, that is, we may consider $\mathsf{GL}(E)$ and Σ as subgroups of the birational transformation group of X. In this sense, $\varGamma = \Sigma \cap \mathsf{GL}(E)$ holds.

All orbits of K, Γ, G in X are connected locally-closed real submanifolds of X. The situation is similar for all other bounded symmetric domains (compare for instance [23], [21], [16], [22]).

In the following we want to give an introduction to the Jordan approach to bounded symmetric domains, even in infinite dimensions. Essentially no proofs are given, these may be found in the references or in forthcoming papers, e.g. [14]. New are the characterization of the Jordan functor in terms of a third partial derivative in section 2 and also the extended functional calculus in section 4.

2. The Jordan functor

Let E be a complex Banach space in the following. Then a bounded domain $D \subset E$ is called *symmetric* if for every $a \in D$ there is a holomorphic map $s : D \to D$ (called *symmetry* of D at a) with $s^2 = \mathsf{id}_D$ and a an isolated fixed point. The symmetry s is uniquely determined by a and will always be denoted by s_a. Let $G := \mathsf{Aut}(D)$ be the group of all biholomorphic automorphisms of D. Then G is a real (Banach) Lie group [24] acting transitively and analytically on D, and $a \mapsto s_a$ defines a real-analytic map $D \to G$. In particular, we may choose an arbitrary point $o \in D$ as base point in the following.

The bounded symmetric domains (with base points) form a category in a natural way: Define [17] on every such D a ‘multiplication’ $\mu : D \times D \to D$ by $(y, z) \mapsto y \cdot z := s_y(z)$. Then a holomorphic map $\varphi : D \to D'$ between bounded symmetric domains (with base point) is a *morphism* if it satisfies $\varphi(y \cdot z) = \varphi(y) \cdot \varphi(z)$ for all $y, z \in D$ (and maps base points into each other). Then every biholomorphic mapping $\varphi : D \to D'$ is an isomorphism in the category (since for every $a \in D$ the transformation $\varphi \circ s_a \circ \varphi^{-1}$ must be the symmetry at $\varphi(a)$).

In finite dimensions (i.e. $D \subset \mathbb{C}^n$ for some n) there exists a G-invariant hermitian metric on D, for instance the *Bergman metric*. This metric provides all the concepts of differential geometry like geodesics, curvature and so on. Some of these notions can also be introduced in the general situation in terms of symmetries: Call a subset $M \subset D$ *symmetric* (or *totally geodesic*) if $s_a(M) = M$ holds for all $a \in M$, where $s_a \in \mathsf{Aut}(D)$ is the symmetry of D at the point a. A *geodesic* then is a symmetric connected real submanifold of dimension 1 in D. It can be shown that any two points $a \neq b$ in D lie on a unique geodesic of D. Furthermore, $M \subset D$ is called *flat* if $\{s_a s_b : a, b \in M\}$ generates a commutative subgroup of $\mathsf{Aut}(D)$.

In infinite dimensions the Banach space E in general cannot be renormed to a Hilbert space, that is, in general there does not exist a hermitian metric on D at all. But instead of a Hilbert norm there exists a canonical G-invariant Banach norm on the tangent bundle TD, namely the Carathéodory norm defined in the following way: For every $a \in D$ and every tangent vector $v \in T_aD$ at a let $\|v\|$ be the supremum over all $|df_a(v)|$ where f runs over all holomorphic functions on D with $f(a) = 0$ and $|f| < 1$. All tangent spaces T_aD are homeomorphic to E, we may therefore assume without loss of generality in the following that $E = T_oD$ is the tangent space at the base point o and that its norm is the Carathéodory norm of D. Then a Riemann mapping type theorem [10] for bounded symmetric domains states the following:

There exists a biholomorphic mapping φ from D onto the open unit ball of the tangent space $E = T_oD$ with $\varphi(o) = 0$, and any two such mappings differ by a surjective linear isometry of E.

Because of this mapping theorem, unless otherwise stated, every bounded symmetric domain in the following will be realized as the open unit ball of a complex Banach space (uniquely determined by the domain up to linear isometry) and the base point will be the origin $0 \in D$. For symmetric unit balls $D \subset E$ and $D' \subset E'$ the following linearity properties can be shown:

(i) Every morphism $\varphi : D \to D'$ with $\varphi(0) = 0$ is the restriction to D of a linear operator $R : E \to E'$ with $\|R\| \le 1$ that induces a linear isometry $E/\ker(R) \to E'$.

(ii) Every morphism $\varphi : D \to D'$ can be extended to a holomorphic map $B \to E'$, where B in E is the open ball with radius $\|a\|^{-1}$ for $a := \varphi(0) \in D'$.

(iii) Every closed totally geodesic complex submanifold M of D with $0 \in M$ is of the form $M = D \cap F$ for a suitable closed linear subspace $F \subset E$ and hence is a bounded symmetric domain in F.

In particular, (i) implies that the isotropy group

$$K := \{g \in G : g(0) = 0\} \quad = \quad \mathsf{GL}(D) \;\subset\; \mathsf{GL}(E) \tag{2.1}$$

at the origin consists of all surjective linear isometries of the Banach space E. Furthermore, if we denote by $\mathcal{D}$ the category of all bounded symmetric domains with base point and by $\mathcal{E}$ the category of all complex Banach spaces with linear contractions as morphisms we get a functor $\mathcal{F} : \mathcal{D} \to \mathcal{E}$ by $\mathcal{F}(D) := T_oD$ for every D in $\mathcal{D}$ with base point $o \in D$ and $\mathcal{F}(\varphi) := d\varphi_o$ for every morphism φ in $\mathcal{D}$. It is clear that the image of $\mathcal{F}$ consists precisely of those complex Banach spaces whose open unit balls are homogeneous under their biholomorphic automorphism groups (since every unit ball allows the symmetry $z \mapsto -z$ at the origin). The image $\mathcal{F}(\mathcal{D})$ can be characterized in Jordan algebraic terms in the following way: Let D be a bounded symmetric domain (realized as the open unit ball of a complex Banach space E according to our convention) and consider the multiplication map $\mu : D \times D \to D$ defined above. Then μ is real-analytic and the third partial derivative of μ at $(0,0) \in D \times D$

$$\Lambda := \frac{\partial^3 \mu(y,z)}{\partial z \partial y \partial z}\Big|_{(0,0)} \tag{2.2}$$

is an $\mathbb{R}$-trilinear mapping $E \times E \times E \to E$ which is symmetric complex bilinear in the outer variables since $\mu(y,z) = s_y(z)$ is holomorphic in z. For every $(a,b,c) \in E^3$ put

$$\{abc\} := -\frac{1}{4}\Lambda(a,b,c) \tag{2.3}$$

and call it the *Jordan triple product* of the vectors a, b, c (the negative sign is chosen in order to get positive spectrum in (iii) below). Furthermore, define the left multiplication operator $L(a,b)$ on E by $z \mapsto \{abz\}$. Then the following can be shown for all $a, b, x, y, z \in E$:

(i) $\{xyz\}$ is symmetric bilinear in (x, z) and conjugate linear in y,

(ii) $\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$,

(iii) $L(a, a)$ has spectrum ≥ 0 as linear operator on E,

(iv) $L(a, a)$ is hermitian and has norm $\|a\|^2$.

Notice that the first three properties (i) - (iii) are purely algebraic. Identity (ii) is called the *Jordan triple identity*. Condition (iv) involves the norm and in particular implies that the triple product is continuous. By definition, an operator $R \in \mathcal{L}(E)$ is called *hermitian* if $\exp(itR) \in \mathsf{GL}(E)$ is isometric for all $t \in \mathbb{R}$. It can be shown (not at all obvious, see [5]) that (i) - (iv) imply

(v) $\|\{xyz\}\| \leq \|x\|{\cdot}\|y\|{\cdot}\|z\|$ and $\|\{zzz\}\| = \|z\|^3$.

By definition, a complex Banach space E is called a *positive hermitian Jordan triple system*, or a *JB*-triple* for short, if there exists a triple product $\{\ \}$ on E satisfying conditions (i) - (iv). This is the case if and only if the open unit ball of E is symmetric and then the triple product on E is uniquely determined. The JB*-triples form a category – call a linear operator $R : E \to F$ between JB*-triples a morphism if it satisfies

$$R\{xyz\} = \{(Rx)(Ry)(Rz)\} \quad \text{for all} \quad x, y, z \in E \tag{2.4}$$

(it can be shown that every morphism is contractive and in particular continuous). The main result [10] on $\mathcal{F}$ (therefore also called the *Jordan functor*) now is:
The category $\mathcal{D}$ of all bounded symmetric domains with base point is equivalent via $\mathcal{F}$ to the category of all JB-triples.*

Usually, the functor $\mathcal{F}$ is obtained as follows. Let D be an arbitrarily given bounded symmetric domain with base point o and let $\mathfrak{g}$ be the Lie algebra of the Lie group $G = \mathsf{Aut}(D)$. Then the symmetry s_o acts by the adjoint representation on $\mathfrak{g}$ and the corresponding -1-eigenspace $\mathfrak{p} \subset \mathfrak{g}$ is a Lie triple system that, as a real vector space, can be identified with the complex Banach space $E := T_oD$. Then for all vectors $x, y, z \in E$ the Jordan triple product $\{xyz\}$ is the part of $\frac{1}{2}[[x, y], z]$ which is conjugate linear in y, more precisely, $4\{xyz\} = [[x, y], z] + i[[x, iy], z]$. It is obvious that in this approach the Jordan triple product depends on the complex structure of the tangent space T_oD. But the same is true for the approach using third partial derivatives – the dependence is hidden in the choice of the realization of D as open unit ball (third partial derivatives are not invariant against local changes of coordinates). Nevertheless, our approach also works without using the open ball realization (and also without using any Lie groups). The only fact we need is that the symmetry s_o at the base point has a square root j_o in the isotropy group K that gives the complex structure on the tangent space T_oD (i.e. its differential at o is the multiplication by $i \in \mathbb{C}$). Then, if we apply a local (holomorphic) change of coordinates around o in which j_o becomes linear (for instance $z \mapsto \sum_{k=1}^{4} i^{-k} j_o^k(z)$), then the third derivative (2.2) in these coordinates has an invariant meaning and provides by (2.3) the Jordan triple product on the Banach space T_oD.

On every JB*-triple E there exist various important operators. For every $a \in E$ define $Q(a) : E \to E$ by $z \mapsto \{aza\}$. Then $Q(a)$ is conjugate linear and depends quadratically on a.

For every $a, b \in E$ the (complex) linear operator

$$B(a,b) := \mathbb{1} - 2L(a,b) + Q(a)Q(b)$$

is called the *Bergman operator*. These operators satisfy (see [18])

$$Q\big(Q(a)z\big) = Q(a)Q(z)Q(a) \quad \text{and} \quad Q(B(a,b)z) = B(a,b)Q(z)B(b,a)$$

for all $a, b, z \in E$. The ball D just is the connected component of the open set $\{a \in E : B(a,a) \text{ invertible}\}$ that contains the origin. In finite dimensions [19], Bergman metric and kernel function of D are closely related to the Bergman operators.

Let us now consider two important examples. The first is the Banach space $E = \mathcal{L}(H)$ of all bounded linear operators on the complex Hilbert space H. Then E is a Banach algebra with involution $z \mapsto z^*$ (the adjoint of operators). The open unit ball $D \subset E$ is symmetric and the multiplication map μ on D is given by

$$s_y(z) = a \ - \ (\mathbb{1} - aa^*)^{1/2} \underbrace{(\mathbb{1} - za^*)^{-1} z}_{z + za^*z + za^*za^*z + \ldots} (\mathbb{1} - a^*a)^{1/2} \quad \text{for} \quad a := 2(\mathbb{1} + yy^*)^{-1} y \ .$$

In particular, $\{zaz\} = za^*z$ and $B(a,b)z = (\mathbb{1} - ab^*)z(\mathbb{1} - b^*a)$ for all $a, b, z \in E$. The JB*-subtriples of $E = \mathcal{L}(H)$ were introduced by Harris [7] under the name of *J*-algebras*. Examples are for instance all C*-algebras (i.e. closed *-invariant associative complex subalgebras of E) but also the space $F := \mathcal{L}(H_1, H_2)$ for arbitrary closed linear subspaces $H_1, H_2 \subset H$ (with $p : H \to H_1$ the canonical orthogonal projection and $j : H_2 \to H$ the canonical injection identify every $z \in F$ with the operator $jzp \in E$).

The other example is associated with the two famous exceptional bounded symmetric domains of dimensions 16 and 27 (for further details compare [19]). Consider the real Cayley division algebra $\mathbb{O}$, which has dimension 8 over the reals and contains $\mathbb{R}$ as a subalgebra in such a way that $1 \in \mathbb{R}$ is also a multiplicative identity of $\mathbb{O}$. Furthermore, there is an algebra involution $x \mapsto \overline{x}$ with $\{x \in \mathbb{O} : \overline{x} = x\} = \mathbb{R}$ and $x\overline{x} > 0$ for all $x \neq 0$ from $\mathbb{O}$. Now consider in $\mathbb{O}^{3\times 3}$ the real linear subspace $V := \mathcal{H}_3(\mathbb{O})$ of all hermitian matrices $a = (a_{jk})$, that is $a_{kj} = \overline{a_{jk}}$ for all $1 \leq j, k \leq 3$. Obviously, V has dimension 27 and it is well known that it becomes a formally real Jordan algebra with respect to the symmetrized matrix product $x \circ y := (xy + yx)/2$. The product extends to a complex Jordan product on the complexification $U := V^{\mathbb{C}} = V \oplus iV$ and $T(\Omega) := \{x + iy \in V \oplus iV : y \in \Omega\}$ is the (unbounded) tube domain realization of the exceptional domain of dimension 27 where $\Omega = \exp(V)$ is an open convex cone in V that also coincides with the interior of $\{x^2 : x \in V\}$. On the other hand, the convex hull $\mathsf{ch}(\exp(iV))$ of the 'generalized unit circle' $\exp(iV) \subset U$ is the closed unit ball with respect to a norm that makes U a JB*-triple. The open unit ball D of U is biholomorphically equivalent to $T(\Omega)$, the corresponding triple product on $U = \mathcal{H}_3(\mathbb{O})^{\mathbb{C}}$ now is given in the standard way as

$$\{xyw\} = (x \circ y^*) \circ w + (w \circ y^*) \circ x - (x \circ w) \circ y^*$$

where $(u + iv)^* = u - iv$ for all $u, v \in V$. The linear subspace $W \subset V$ of all matrices a with $a_{11} = a_{22} = a_{33} = a_{23} = 0$ has as complexification in U a subtriple of complex dimension 16

that also cannot be realized as a subtriple of $\mathcal{L}(H)$ for H any complex Hilbert space. But the following Gelfand-Naimark theorem for JB*-triples is known [5]:
Every JB-triple can be realized as closed JB*-subtriple of $\mathcal{L}(H) \oplus \mathcal{C}\big(S, \mathcal{H}_3(\mathbb{O})^{\mathbb{C}}\big)$, where H is a suitable complex Hilbert space, S is a suitable compact topological space and the second summand is the Banach space of all continuous $\mathcal{H}_3(\mathbb{O})^{\mathbb{C}}$-valued functions on S with sup-norm and pointwise defined triple product.*

3. Associated Grassmannians

In the following we fix an arbitrary bounded symmetric domain D, realized as the open unit ball of the associated JB*-triple E. As before, we denote by $G := \mathsf{Aut}(D)$ the group of all biholomorphic automorphisms of D and by $K := \{g \in G : g(0) = 0\} = \mathsf{GL}(D)$ the isotropy subgroup at the origin. The group K can also be given by (real-analytic) cubic equations in $\mathsf{GL}(E)$ since it is also the automorphism group of the JB*-triple E, that is,

$$K = \big\{g \in \mathsf{GL}(E) : g\{xyz\} = \{(gx)(gy)(gz)\} \;\; \forall x, y, z \in E\big\}$$

is a (real) linear algebraic group of degree ≤ 3 in the sense of [8] and hence a real (Banach) Lie group in the topology induced from $\mathsf{GL}(E)$. The Lie algebra $\mathfrak{k} \subset \mathcal{L}(E)$ of K is the space of all triple derivations of E. The sum $\mathfrak{k} \oplus i\mathfrak{k}$ is direct in $\mathcal{L}(E)$ and is a closed complex Lie subalgebra. Now consider the so called *structure group* of the Jordan triple system

$$\varGamma = \big\{g \in \mathsf{GL}(E) : \exists\, \widetilde{g} \in \mathsf{GL}(E) \text{ with } g\{xyz\} = \{(gx)(\widetilde{g}y)(gz)\} \;\; \forall x, y, z \in E\big\}$$

(which is essentially the automorphism group of the Jordan pair [18] associated with the Jordan triple system E). For every $g \in \varGamma$ the transformation $\widetilde{g}$ is uniquely determined and again is an element of Γ. The group $\varGamma$ is a complex Lie group with Lie algebra $\mathfrak{k} \oplus i\mathfrak{k}$ and $g \mapsto \widetilde{g}$ defines an antiholomorphic group automorphism of $\varGamma$ with fixed point set K. Therefore, $\varGamma \subset \mathsf{GL}(E)$ is a complexification of the real Lie group K. For instance, every $g = \exp(L(a, b)) \in \mathsf{GL}(E)$ with $a, b \in E$ is contained in $\varGamma$ and $\widetilde{g} = \exp(-L(b, a))$. Furthermore, for all $a, b \in D$ the Bergman operator $B(a, b)$ is in $\varGamma$, and also for every $g \in \mathsf{Aut}(D)$ the derivative dg_a at $a \in D$ belongs to the structure group $\varGamma$.

So far we have also for arbitrary JB*-triples all ingredients of the diagram (1.1) except Σ. Let us therefore briefly sketch how E can be extended to a complex manifold by adding points at infinity in a way that reflects the triple structure on E [10]: There exists a complex (Banach) manifold X containing E as an open subset together with a complex (Banach) Lie group Σ acting holomorphically and transitively on X such that

(i) For every $a \in E$ there exists a transformation $t_a \in \Sigma$ with $t_a(z) = z + a \in E$ for all $z \in E$. The translation group $T := \{t_a : a \in E\}$ is a closed complex Lie subgroup of Σ isomorphic to the complex vector group E via $a \leftrightarrow t_a$.

(ii) $\{g \in \Sigma : g|_E \in \mathsf{GL}(E)\}$ is a closed complex Lie subgroup of Σ isomorphic to $\varGamma$ via $g \mapsto g|_E$.

(iii) $\{g \in \Sigma : g(D) = D\}$ is a closed real Lie subgroup isomorphic to $G = \mathsf{Aut}(D)$ via $g \mapsto g|_D$.

(iv) There is an antiholomorphic group automorphism $g \mapsto \widetilde{g}$ of period 2 on Σ that extends the one on Γ defined above. $\widetilde{t}_a(z) = (\mathbb{1} + L(z,a))^{-1}z \in E$ holds for all $a, z \in E$ with $\|a\| \cdot \|z\| < 1$.

(v) $\{g \in \Sigma : g(E) = E\} = \Gamma T = T\Gamma$ and $\{g \in \Sigma : g(0) = 0\} = \Gamma\widetilde{T} = \widetilde{T}\Gamma$.

(vi) Σ is generated by the subgroups T, Γ, $\widetilde{T}$.

We call X the *compact type extension* of the JB*-triple E. In general, X and with it the complex Lie group Σ is uniquely determined by E only up to coverings. Uniqueness is forced, for instance, by requiring that X is simply connected, or to the other extreme, that Σ has trivial center. In finite dimensions, X is a simply connected, compact projective algebraic manifold (called the dual of the bounded symmetric domain D) and $\partial E = X \backslash E$ is a complex-analytic subset of X.

Because of (ii) the group Σ acts effectively on X, we therefore consider it as a subgroup of $\mathsf{Aut}(X)$. Also, both groups in (ii) as well as in (iii) will be identified henceforth.

As with every Banach space [3], there is associated the Grassmann manifold $\mathbb{G} = \mathbb{G}(E)$ with the JB*-triple E. This is a complex (Banach) manifold on which the complex Lie group $\mathsf{GL}(E)$ acts holomorphically. Furthermore, there is a canonical metric on $\mathbb{G}$ invariant under the group of all linear isometries of E, that is under the group K. As set $\mathbb{G}$ consists of all closed linear subspaces $U \subset E$ that have a complement (that is a closed linear subspace $V \subset E$ with $U \cap V = 0$ and $U + V = E$). As an example, if m and n are arbitrary cardinal numbers and if H is the Hilbert space of dimension $n + m$, then the subset $\mathbb{G}_{m,n} \subset \mathbb{G}(H)$ of all closed linear subspaces of dimension m and codimension n in H is a connected component of $\mathbb{G}(H)$, on which the unitary group of H acts transitively by isometries.

On $\mathbb{G} = \mathbb{G}(E)$ the groups K and Γ as well as their connected identity components K^0 and Γ^0 act and one may ask: *What are the corresponding orbits for a point $F \in \mathbb{G}$ that, considered as linear subspace $F \subset E$, is related to the structure of the bounded symmetric domain.* Possible properties of $F \subset E$ (in decreasing generality) could be for instance: (1) F is a subtriple, i.e $\{FFF\} \subset F$, (2) F is an inner ideal, i.e. $\{FEF\} \subset F$, (3) F is a principal inner ideal, i.e. $F = \{aEa\}$ for some $a \in E$, (4) $F = E_1(e)$ is the Peirce-1-space with respect to a tripotent $e \in E$, (5) F is a minimal inner ideal (in the space of all non-zero inner ideals partially ordered by inclusion). Let us here concentrate on the cases (4) and (5) (for more information also on the other cases compare [12] and also [4]):
By definition, $e \in E$ is a *tripotent* if $\{eee\} = e$ holds, and two tripotents e, c are called *orthogonal* if $L(e,c) = 0$ (or equivalently $L(c,e) = 0$) holds. Every tripotent $e \in E$ gives a decomposition (called *Peirce decomposition*)

$$E = E_1(e) \oplus E_{1/2}(e) \oplus E_0(e)$$

where $E_k(e)$ for every k is the k-eigenspace of the operator $L(e,e)$ in E. Every Peirce space $E_k(e)$ is a subtriple and represents a point in $\mathbb{G}$, since there is a complement due to the Peirce decomposition. In addition, $E_1(e) = \{eEe\}$ is a principal inner ideal, and e is called a *minimal tripotent* if $E_1(e)$ is of dimension 1. Inner ideals in Jordan triple systems are the analogues of one-sided ideals in associative algebras (outer ideals could be defined but seem to be of no relevance). In the JB*-triple E inner ideals can be characterized for instance geometrically in the following way: The closed linear subspace $F \subset E$ is an inner ideal if and only if for every $g \in \mathsf{Aut}(D)$ the image $g(F \cap D)$ is of the form $A \cap D$ for some closed affine subspace $A \subset E$. It is known that for every linear subspace $F \subset E$ the following conditions are equivalent: (i) F is a minimal inner ideal, (ii) $F = E_1(e) = \mathbb{C}e$ for some tripotent $e \neq 0$ and (iii) F is a subtriple of dimension 1. This implies for instance that the set M of all minimal inner ideals in E can be realized in the projective space $\mathbb{P}(E)$ by the homogeneous (real-analytic) equation $z \wedge \{zzz\} = 0$, more precisely, by the family of all (real-analytic) homogeneous scalar equations $\lambda(z) \cdot \nu(\{zzz\}) = \lambda(\{zzz\}) \cdot \nu(z)$, where λ, ν run over all bounded linear forms on E.

It can be shown more generally [12] that the subset $P \subset \mathbb{G}$ of all Peirce-1-spaces with respect to tripotents is a Γ-invariant closed complex submanifold of $\mathbb{G}$. Every connected component N of P is a K^0-orbit and for every $F = E_1(e) \in N$ the tangent space $T_F N$ is the Peirce-1/2-space $E_{1/2}(e)$. Furthermore, N is a compact type extension of the JB*-triple $E_{1/2}(e)$.

Let us consider as an example the exceptional JB*-triple $E = \mathcal{H}_3(\mathbb{O})^{\mathbb{C}}$ of dimension 27 and let $e \in E$ be the matrix with $e_{11} = 1$ and $e_{jk} = 0$ for all other j, k. Then $E_1(e) = \mathbb{C}e$ and $E_{1/2}(e)$ is the exceptional JB*-triple of dimension 16 that occurred at the end of section 2. The compact complex manifold $M \subset \mathbb{P}_{26}$ of all minimal inner ideals in E is connected, contains the point $\mathbb{C}e$ and therefore is the exceptional compact hermitian symmetric space of dimension 16 (the projective plane over the complexified octonions $\mathbb{O}^{\mathbb{C}}$, compare also [20]).

4. K-orbits and functional calculus

In this section we specialize to a case that is close to the finite dimensional situation. As before fix a bounded symmetric domain D realized as the open unit ball of the corresponding JB*-triple E. Let X, G, K, Γ have the same meaning as before.

For E (and equivalently for D) one has the notion of rank $r \in \mathbb{N} \cup \infty$. This can be defined in various (equivalent) ways: It is for instance the maximal dimension of flat real smooth submanifolds $M \subset D$. Algebraically, it is characterized as the maximal length of a proper chain $I_1 \subset I_2 \subset \ldots$ of non-zero principal inner ideals in E. The rank is finite if and only if E as a Banach space is reflexive and then it can be read off the closed unit ball $\overline{D}$ in the following way: Define inductively M_k for every $k \geq 0$ to be the set of all smooth points in $\overline{D} \backslash (M_0 \cup M_1 \ldots \cup M_{k-1})$. Then every M_k is a (locally closed) G-invariant real submanifold of E and the rank r is the maximal k with $M_k \neq \emptyset$. For instance, $r = 1$ holds if and only if E is a complex Hilbert space of positive dimension.

Now suppose for the rest of the section that D has finite rank r. Then X is simply connected, E is dense in X, and $\Sigma = \mathsf{Aut}(X)$ as well as $G = \{g \in \Sigma : \widetilde{g} = g\}$ hold. Assume furthermore throughout that D is irreducible (i.e. not a direct product of two bounded symmetric domains of lower dimensions – this assumption is not really restrictive, the general case for finite rank is only a trivial extension).

Every $a \in E$ has a representation

$$a = \lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_r e_r \,, \qquad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0 \,, \tag{4.1}$$

with $(e_1, \ldots, e_r)$ an r-tuple of pairwise orthogonal minimal tripotents (called a *frame* in E). The coefficient tuple $\sigma(a) := (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathbb{R}^r$ is uniquely determined by a, the λ_k-s are called the *singular values* of a. Also, for every $t > 0$ the tripotent $\sum_{\lambda_k = t} e_k$ is uniquely determined by a. The maximal $k \geq 0$ with $\lambda_k \neq 0$ is called the *rank* of a. This is also the rank of the principal inner ideal $\{aEa\}$ generated by a. For every p with $1 \leq p \leq \infty$ and every integer $k = 1, 2, \ldots, r$ put

$$\|a\|_p := \|(\lambda_1, \lambda_2, \ldots, \lambda_r)\|_p \quad \text{and} \quad \|\!|a|\!\|_k := \lambda_1 + \lambda_2 + \ldots + \lambda_k \,.$$

Every $\|\;\|_p$ and every $\|\!|\;|\!\|_k$ is an equivalent K-invariant norm on E with $\|\;\|_\infty = \|\;\| = \|\!|\;|\!\|_1$. Furthermore $\|\;\|_2$ is a Hilbert norm. The set $\{z \in \overline{D} : \|z\|_p = r^{1/p}\}$ does not depend on $p < \infty$, coincides with $\{z \in \overline{D} : \|\!|z|\!\|_r = r\}$ and is the Shilov boundary of D. The geometric relevance of the norms $\|\!|\;|\!\|_k$ becomes clear for instance from the fact that $z \in E$ is in the convex hull of the orbit $K(a)$ if and only if $\|\!|z|\!\|_k \leq \|\!|a|\!\|_k$ holds for all $k \leq r$.

Two points $a, b \in E$ are in the same K-orbit if and only if $\sigma(a) = \sigma(b)$, that is, if and only if a, b have the same singular values (with multiplicities counted). More generally, we want to characterize the K, G, Γ-orbits in the bigger space X also by certain invariants. By the irreducibility assumption all these orbits are connected. In the simplest possible case we have $D = \Delta$ the open unit disk in the line $E = \mathbb{C}$ and $X = \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ the Riemann sphere([1]). For the r-th power we have $D = \Delta^r$ the polydisk, $\mathbb{C}^r$ with triple product $\{xyz\} = (x_1\overline{y}_1 z_1, x_2\overline{y}_2 z_2, \ldots, x_r\overline{y}_r z_r)$ the polyline and $\overline{\mathbb{C}}^r$ the polysphere, all of dimension r.

Now return to the general case and choose a frame $(e_1, e_2, \ldots, e_r)$ in E. Then

$$z \;\mapsto\; z_1 e_1 + z_2 e_2 + \ldots + z_r e_r$$

defines a triple morphism $\mathbb{C}^r \hookrightarrow E$ that extends to an injective $\mathsf{Aut}(\overline{\mathbb{C}}^r)$-equivariant embedding $\varphi : \overline{\mathbb{C}}^r \hookrightarrow X$. Consider in $\overline{\mathbb{C}}^r$ the compact subset

$$[0, \infty]^r_+ := \{\lambda \in [0, \infty]^r : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r\} \,.$$

Then to every $a \in X$ there is a unique $\lambda \in [0, \infty]^r_+$ with $\varphi(\lambda) \in K(a)$. We put $\sigma(a) := \lambda$ and call the coordinates of $\sigma(a)$ the *singular values* of a (which may include ∞). In this way we get

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([1]) Here and in the following the bar over a set always means the closure with respect to the extended space X. The bar over single elements means conjugation.

a K-invariant map $\sigma : X \to [0,\infty]^r_+$ that establishes a homeomorphism from the orbit space X/K onto $[0,\infty]^r_+$ and is called the *singular value map*.

Let $\overline{\mathbb{R}} := \mathbb{R} \cup \infty$ be the closure of $\mathbb{R}$ in $\overline{\mathbb{C}}$ and call with $-\infty := \infty$ a function $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ *odd* if it satisfies $f(-t) = -f(t)$ for all $t \in \overline{\mathbb{R}}$ and $f(0) = 0$. Every odd f induces a K-equivariant mapping $\boldsymbol{f} : X \to X$ in the following way: Write every $a \in X$ in the form (4.1) and put

$$\boldsymbol{f}(a) := f(\lambda_1)e_1 + f(\lambda_2)e_2 + \ldots + f(\lambda_r)e_r\,,$$

which is to be understood in terms of the embedding $\varphi : \overline{\mathbb{C}}^r \hookrightarrow X$ above and does not depend on the choice of the frame $(e_1, \ldots, e_r)$. Then it is clear that for every further odd function g on $\overline{\mathbb{R}}$ and $h := f \circ g$ we have $\boldsymbol{h} = \boldsymbol{f} \circ \boldsymbol{g}$ for the induced mappings on X (that will always be denoted by the corresponding boldface letters). We call $f \mapsto \boldsymbol{f}$ the *odd functional calculus* on X. The mapping $\boldsymbol{f}$ is of class $\mathcal{C}^k$ (or real-analytic) on E if the function f has the same property on the manifold $\overline{\mathbb{R}}$. As an example, the function $f(t) = -t$ induces on X the symmetry $s_0 \in K \subset \mathsf{Aut}(X)$ of D at the origin. Another example is the function $f(t) = 1/t$ for $t \neq 0$ and $f(0) = 0$. In the special case $E = \mathbb{C}^{n\times m}$ (compare section 1), for every matrix $a \in E$ the transposed conjugate matrix $\boldsymbol{f}(a)^* \in \mathbb{C}^{m\times n}$ is the Penrose pseudo-inverse of a.

An important example comes from the cube function $c(t) = t^3$ on $\overline{\mathbb{R}}$ with ∞ the value at infinity. The induced mapping $\boldsymbol{c}$ is a real-analytic homeomorphism of X and is called the *cube mapping on* X. For every $z \in E$ obviously $\boldsymbol{c}(z) = \{zzz\}$ holds. The pair $(X, \boldsymbol{c})$ should be considered as a dynamical system. For instance, $0 \in X$ is an attractor and D is its basin of attraction. The fixed point set $\mathsf{Fix}(\boldsymbol{c})$ (generalized tripotents in a way) in X is a real-analytic submanifold and consists precisely of all $a \in X$ with $\sigma(a) \in \{\infty, 1, 0\}^r$. In particular, $\mathsf{Fix}(\boldsymbol{c})$ is the disjoint union of $\binom{r+2}{2}$ K-orbits, $r+1$ of which are in E. These can be indexed as $S_0, S_1, \ldots, S_r$ where $S_k \subset E$ is the space of all tripotents of rank k.

For every $n \in \mathbb{Z}$ we have the powers $\boldsymbol{c}^n : X \to X$ and also pointwise limits $\boldsymbol{c}^{+\infty} := \lim_{n\to+\infty} \boldsymbol{c}^n$ and $\boldsymbol{c}^{-\infty} := \lim_{n\to-\infty} \boldsymbol{c}^n$, which are induced by the odd functions satisfying for $t \in [0, \infty]$

$$c^{+\infty}(t) = \begin{cases} 0 & t < 1 \\ 1 & t = 1 \\ \infty & t > 1 \end{cases} \quad \text{and} \quad c^{-\infty}(t) = \begin{cases} 0 & t = 0 \\ 1 & 0 < t < \infty \\ \infty & t = \infty \end{cases} \quad .$$

Both, forward limit $\boldsymbol{c}^{+\infty}$ as well as backward limit $\boldsymbol{c}^{-\infty}$, map X onto the fixed point set $\mathsf{Fix}(\boldsymbol{c}) \subset X$. For $\sigma^+ := \sigma \circ \boldsymbol{c}^{+\infty}$ and $\sigma^- := \sigma \circ \boldsymbol{c}^{-\infty}$ any two points $a, b \in X$ are in the same G-orbit (respectively Γ-orbit) if and only if $\sigma^+(a) = \sigma^+(b)$ (respectively $\sigma^-(a) = \sigma^-(b)$). In particular, there are again precisely $\binom{r+2}{2}$ G-orbits and also $\binom{r+2}{2}$ Γ-orbits in X, from which in both cases there are $r+1$ contained in E. In case of Γ these are the spaces

$$E_k := \Gamma(S_k)$$

of all rank-k-elements in E, and in case of G these are the spaces

$$M_k := G(S_k) \subset \overline{D}$$

for $0 \le k \le r$ that already occurred in a geometric way at the beginning of this section. Notice that $\boldsymbol{c}^{+\infty} : M_k \to S_k$ and $\boldsymbol{c}^{-\infty} : E_k \to S_k$ are K-equivariant real-analytic fibrations.

5. CR-structure of orbits

We are interested in the holomorphic structure of K- and G-orbits in the complex manifold X (which in general are not complex submanifolds) and introduce some notation. Let $S \subset X$ be an arbitrary subset and let N be a simply connected Riemann surface. Then a subset $C \subset S$ is called a *holomorphic N-component of S* if

(i) Every holomorphic map $\varphi : N \to X$ with $\varphi(N) \subset S$ satisfies $\varphi(N) \subset C$ or $\varphi(N) \cap C = \emptyset$.

(ii) C is minimal with respect to (i).

It is clear that every point of S lies in a unique holomorphic N-component of S. In case $N = \Delta \subset \mathbb{C}$ is the open unit disk, holomorphic Δ-components are the same as the *holomorphic arc components* in the sense of [23]. The other two possibilities for N are $\mathbb{C}$ and the Riemann sphere $\overline{\mathbb{C}} = \mathbb{P}_1$. Now suppose that $S \subset X$ is a smooth (locally closed) real submanifold and that Y is an arbitrary complex (Banach) manifold. Then a smooth mapping $f : S \to Y$ is called *Cauchy-Riemann* (CR for short) if for every $a \in S$ and $b := f(a)$ the restriction of the differential $df_a : T_aS \to T_bY$ to the maximal complex subspace $T_aS \cap iT_aS \subset T_aX$ is complex linear. Then it is obvious that every $\mathbb{C}$-valued CR-function on S is constant on every holomorphic $\overline{\mathbb{C}}$-component of S.

Throughout this section let D be an irreducible bounded symmetric domain of finite rank r realized as open unit ball in the JB*-triple E. Also X, K, G, Γ as well as the singular value map $\sigma : X \to [0,\infty]^r_+$ and the cube mapping $\boldsymbol{c} : X \to X$ have the same meaning as in the last section. Then X is a holomorphic $\overline{\mathbb{C}}$-component of itself and E is dense in X.

A special role is played by the domains D of tube type. These can be characterized in terms of the triple product in the following way: There is a tripotent $e \in E$ with $\{eex\} = x$ for all $x \in E$ (examples are $\mathbb{C}^{r\times r}$ or $\mathcal{H}_3(\mathbb{O})^{\mathbb{C}}$ with e the unit matrix). Then E becomes a complex Jordan algebra with unit e with respect to the Jordan product $z \circ w := \{zew\}$. It turns out that E_r coincides with the subset of all invertible elements in the Jordan algebra sense and that the inversion $z \mapsto z^{-1}$ on E_r extends to an automorphism $j \in \mathsf{Aut}(X)$ of period 2. For every $a \in X$ with $\sigma(a) = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ the identity $\sigma(j(a)) = (\lambda_r^{-1}, \ldots, \lambda_2^{-1}, \lambda_1^{-1})$ holds. As a consequence, j permutes all K, G, Γ-orbits in X. In particular, $j(D) \subset X$ is another open G-orbit biholomorphically equivalent to a bounded domain. Furthermore, the involution $z \mapsto \{eze\}$ on E extends to an antiholomorphic automorphism $z \mapsto z^*$ of period 2 on X. The composition of both, i.e. $z \mapsto j(z)^* = j(z^*)$, has precisely the Shilov boundary S_r of D as fixed point set (so it is the reflection at this generalized unit circle). Among all bounded symmetric domains those of tube type are characterized by the property that the symmetry $s_0 \in K$ has a second isolated fixed point in X besides $0 \in D$.

We know already that the groups Γ and G have the same finite number of orbits in X. Actually, there is a certain duality between G-orbits and Γ-orbits in X: Every G-orbit is of the form $G(a)$ with $a \in \mathsf{Fix}(\boldsymbol{c})$ – then $G(a) \mapsto \Gamma(a)$ induces an (everywhere discontinuous) bijection from X/G onto X/Γ that maps open to closed and closed to open subsets. In particular, G has $r+1$ open and 1 closed orbit (namely the Shilov boundary of D, which is contained in the closure of every other G-orbit) whereas Γ has $r+1$ closed and 1 open orbit (namely E_r, which contains every other Γ-orbit in its closure).

Next consider the K-orbits in X that are complex submanifolds. It is easily seen that these are precisely those orbits that are Γ-orbits at the same time. There exist precisely $r+1$ of them, say $X_0, X_1, \ldots, X_r$, where X_k is the set of all $a \in X$ having only $\{0, \infty\}$ as singular values and 0 with multiplicity k. The X_k-s are also the connected components of the fixed point set of the symmetry $s_0 \in K$ in X. Clearly, $X_0 = \{0\}$ and $X_k \subset \partial E = X \backslash E$ for $k > 0$. Consider the odd function π on $\overline{\mathbb{R}}$ satisfying $\pi(t) = \infty$ for all $t \neq 0$. Then $\boldsymbol{\pi}$ is holomorphic and Γ-equivariant on every Γ-orbit, and in particular,

$$\boldsymbol{\pi} : E_k \to X_k \quad \text{and} \quad \boldsymbol{\pi} : S_k \to X_k \tag{5.1}$$

are CR-mappings for every k. Geometrically, the spaces X_k have the following meaning: Two points in E_k have the same $\boldsymbol{\pi}$-image if and only if they generate the same principal inner ideal in E. Therefore, X_k can be identified with the complex manifold of all principal inner ideals of rank k in E. This identification has also the following more geometric interpretation. Let $\widetilde{E} \subset E$ be a principal inner ideal of rank k and let $\widetilde{X}$ be the corresponding compact type extension of $\widetilde{E}$. Then $\widetilde{E}$ is also the Peirce-1-space in E of a tripotent $e \in \widetilde{E}$ and hence is of tube type. In particular, the symmetry $\widetilde{z} \mapsto -\widetilde{z}$ on $\widetilde{E}$ has a unique isolated fixed point $\widetilde{a} \in \partial\widetilde{E} \subset \widetilde{X}$. On the other hand, the inclusion $\widetilde{E} \hookrightarrow E$ extends to a holomorphic inclusion $\widetilde{X} \hookrightarrow X$ and $\widetilde{a}$ may be considered as a point of X. Furthermore, the fiber $\boldsymbol{\pi}^{-1}(\widetilde{a})$ in E is the domain $\widetilde{E}_k = \widetilde{E} \cap E_k$ of all invertible elements of the unital Jordan algebra $\widetilde{E}$. Also, the fiber of the bundle $S_k \to X_k$ is the Shilov boundary of the bounded symmetric domain $\widetilde{D} = D \cap \widetilde{E}$ in $\widetilde{E}$. In this sense, $S_k \to X_k$ generalizes the Hopf fibration of the unit sphere in $\mathbb{C}^n$.

As a complex manifold, X_k is the compact type extension of the Peirce space $E_{1/2}(e)$, where $e \in E_k$ is an arbitrary tripotent (compare also section 4). In particular, every X_k is a holomorphic $\overline{\mathbb{C}}$-component. As an example, consider $E = \mathbb{C}^{n\times m}$. Then $r = \min(n, m)$ and for every $k \leq r$ we have $X_k = \mathbb{G}_{k,n-k} \times \mathbb{G}_{k,m-k}$ as direct product of Grassmannians. The bundle map $\boldsymbol{\pi}$ in (5.1) then is given by $z \mapsto \big(\mathrm{im}(z), \mathrm{im}(z')\big)$ (compare also section 1).

Let us now consider the analogue for the group G in place of Γ. Let M be an arbitrary G-orbit and put $S := M \cap \mathsf{Fix}(\boldsymbol{c})$. Fix $a \in S$ and denote by k the multiplicity of the singular value 1 for a. Then M is open in X if and only if $k = 0$, or equivalently, if $S = K(a)$ is a complex submanifold of X. In case S is a complex manifold of positive dimension, for every $b \in M$

there is a transformation $g \in G$ with $a, b \in g(S)$ which implies that M is a holomorphic $\overline{\mathbb{C}}$-component and hence has only constant holomorphic functions. Now consider the odd function τ on $\overline{\mathbb{R}}$ defined by $\tau(t) = t$ if $|t| = 1$ and $\tau(t) = 0$ otherwise. The corresponding G-equivariant fibration

$$\boldsymbol{\tau} : M \to S_k \tag{5.2}$$

is CR if and only if $k = 0$, i.e. if S_k consists of a single point. For every $e \in S_k$ the fiber $\boldsymbol{\tau}^{-1}(e) \subset M$ is a complex submanifold of X having the following description: Let $\widetilde{E} := E_1(e) \subset E$ be the Peirce-1-space of the tripotent e and denote by $\widetilde{X} \subset X$ the corresponding compact type extension of $\widetilde{E}$. Then for $\widetilde{D} := \widetilde{E} \cap D$ and $\widetilde{G} := \mathsf{Aut}(\widetilde{D}) \subset \mathsf{Aut}(\widetilde{X})$ the fiber $\boldsymbol{\tau}^{-1}(e) = M \cap \widetilde{X}$ is an open $\widetilde{G}$-orbit in $\widetilde{X}$ and also a holomorphic arc component of M. Therefore, (5.2) is the fibration of M by its arc components. This can be extended to the closure of M: $\boldsymbol{\tau}\big(\overline{M}\big) = S_k \cup S_{k+1} \cup \cdots \cup S_r$ holds and the fibers of the map $\boldsymbol{\tau} : \overline{M} \to X$ are also the holomorphic arc components of $\overline{M}$.

In the special situation $M = M_k$ there are many CR-functions on M (since $M_k \subset \partial D$) and every holomorphic arc component of M_k is a bounded symmetric domain of possibly lower dimension. For M_k the convex hull coincides with the polynomial convex hull and is given by

$$\mathsf{ch}(M_k) \;=\; M_0 \cup M_1 \cup \ldots \cup M_k \;=\; \mathsf{pch}(M_k)$$

which contains $D = M_0$. In [14] it will be shown that every smooth CR-function on M_k has a continuous extension to $\mathsf{ch}(M_k)$, which is CR on every M_j with $j \le k$. More generally, in [14] the CR-structure of arbitrary K-orbits $S := K(a)$, $a \in E$, will be studied. In particular, convex hulls, polynomial convex hulls and holomorphic envelopes of S will be determined.

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