Classification of commutative algebras and tube realizations of hyperquadrics

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ABSTRACT: In this paper we classify up to affine equivalence all local tube realizations of real hyperquadrics in $\mathbb{C}^n$. We show that this problem can be reduced to the classification, up to isomorphism, of commutative nilpotent real and complex algebras. We also develop some structure theory for commutative nilpotent algebras over arbitrary fields of characteristic zero.

1. Introduction

It is a well-known fact that every real-analytic manifold M together with an involutive CR-structure (HM, J) admits at least locally a generic embedding into some $\mathbb{C}^n$, such that the CR-structure induced from the ambient space $\mathbb{C}^n$ coincides with the original one. A particularly important class of CR-submanifolds of $\mathbb{C}^n$ are the so-called CR-tubes, i.e., product manifolds $M = iF + \mathbb{R}^n \subset i\mathbb{R}^n \oplus \mathbb{R}^n = \mathbb{C}^n$ together with the inherited CR-structure, where $F \subset \mathbb{R}^n$ is a submanifold. One important point here is that the CR-structure of $iF + \mathbb{R}^n$ is closely related to real-geometric properties of the base F, which are often easier to deal with, see e.g. [9]. In general, a CR-manifold will not admit a local realization in $\mathbb{C}^n$ as a CR-tube. On the other hand, as shown by the example of the sphere $S = \{z \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 = 1\}$, it is not immediate, that S does admit several affinely inequivalent local tube realizations, see [5].

It is quite obvious that the existence of a CR-tube realization for a given CR-manifold (M, HM, J) is related to the presence of certain abelian subalgebras $\mathfrak{v}$ in $\mathfrak{hol}(M)$, the Lie algebra of infinitesimal CR-transformations, which are induced by all real translations $z \mapsto z + x$, $x \in \mathbb{R}^n$. It is perhaps a little bit more subtle to give sufficient and necessary conditions for abelian Lie subalgebras of $\mathfrak{hol}(M)$ to give local CR-tube realization of M. This has been worked out in [10]. For short, let us call every such abelian subalgebra a 'qualifying' subalgebra of $\mathfrak{hol}(M)$. Curiously, the notion of locally affine equivalence among various (germs of) tube realizations for a given M proved to be less appropriate for the study of CR-manifolds as it is too fine for many applications: Even a homogeneous CR-manifold may admit an affinely non-homogeneous tube realization, and in such a case the aforementioned equivalence relation will give rise to uncountable many equivalence classes of tube realizations. A coarser equivalence relation has been introduced in [10] which seems to be most natural in the context of CR-tubes. Moreover, it is quite surprising that under certain assumptions the pure geometric question of globally affine equivalence can be reduced to the purely algebraic problem of classifying conjugacy classes of certain maximal abelian subalgebras of $\mathfrak{hol}(M)$ with respect to a well-chosen group G.

The purpose of this paper is to give a full classification of all local CR-tube realizations of every hyperquadric

$$S_{p,q} = \big\{[z] \in \mathbb{P}(\mathbb{C}^m) : |z_1|^2 + \cdots + |z_p|^2 = |z_{p+1}|^2 + \cdots + |z_m|^2\big\}$$

in the complex projective space $\mathbb{P}(\mathbb{C}^m)$, where $m := p + q$ and $p, q \geq 1$, applying the general methods from [10]. The (compact) hyperquadric $S_{p,q}$ is the unique closed orbit of

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$\mathrm{SU}(p,q) \subset \mathrm{SL}(m,\mathbb{C})$ acting by biholomorphic transformations on $\mathbb{P}(\mathbb{C}^m)$. In this situation $\mathfrak{hol}(S_{p,q}) = \mathfrak{hol}(S_{p,q},a) \cong \mathfrak{su}(p,q)$ holds for every $a \in S_{p,q}$. It is well known and easy to see that $S_{p,q}$ is locally CR-equivalent to the affine real quadric in $\mathbb{C}^r$, $r := m-1$,

$$\Big\{z \in \mathbb{C}^r : \mathrm{Im}(z_r) = \sum_{1\le k<p} |z_k|^2 \;-\; \sum_{p\le k<r} |z_k|^2\Big\}$$

with non-degenerate Levi form of type $(p-1, q-1)$. Therefore the classification problem for local tube realizations for both classes is the same, compare also [5], [10], [11], [12], [13], [14], [20] for partial results in this context.

We have shown in [10] that every abelian subalgebra $\mathfrak{v} \subset \mathfrak{hol}(M)$, which yields a tube realization, determines an involution $\tau_{\mathfrak{v}} : \mathfrak{hol}(M) \to \mathfrak{hol}(M)$. For a given hyperquadric $S_{p,q}$ however, it turns out that all arising involutions $\tau_{\mathfrak{v}}$ are conjugate in $\mathfrak{g} := \mathfrak{hol}(S_{p,q}) \cong \mathfrak{su}(p,q)$. We therefore fix an involution $\tau : \mathfrak{g} \to \mathfrak{g}$ (with fixed point set $\mathfrak{g}^\tau \cong \mathfrak{so}(p,q)$) once and for all and reduce the classification of tube realizations to the algebraic classification of all maximal abelian subalgebras $\mathfrak{v}$ of $\mathfrak{g}$ contained in the (-1)-eigenspace of the non-riemannian symmetric pair $(\mathfrak{g},\tau)$ up to conjugation by $\mathrm{SU}(p,q)$ (in fact, up to conjugation by the normalizer G of $\mathrm{SU}(p,q)$ in $\mathrm{SL}(m,\mathbb{C})$, but these two groups differ only if $p=q$, and in this case the classification with respect to one group can easily be derived from the classification up to conjugation with respect to the other). In contrary to the special case of toral maximal subalgebras (i.e., Cartan subalgebras) $\mathfrak{t} \subset \mathfrak{g}$ only little is known about the general case of arbitrary abelian maximal subalgebras $\mathfrak{v} \subset \mathfrak{g}$. The key point here is that after some reduction procedures the conjugacy class of a maximal abelian subalgebra $\mathfrak{v} \subset \mathfrak{g}^{-\tau}$ is completely determined by its D-invariant (which is a first rough invariant of $\mathfrak{v}$, determined by its toral part, see (4.14) for more details) and a finite set of maximal abelian subalgebras $\mathfrak{n}_\jmath$, consisting of ad-nilpotent elements only in $\mathfrak{su}(p_\jmath,q_\jmath)$ and $\mathfrak{sl}(m_\jmath,\mathbb{C})$. Hence, the classification task reduces essentially to the classification of ad-nilpotent abelian subalgebras $\mathfrak{n}_\jmath$ up to conjugation in $\mathrm{SU}(p_\jmath,q_\jmath)$, resp. $\mathrm{SL}(m_\jmath,\mathbb{C})$. By our constructions, to every such $\mathfrak{n}_\jmath$ there is associated a finite-dimensional commutative associative nilpotent algebra $\mathcal{N}_\jmath$ over $\mathbb{F}=\mathbb{R}$, resp. $\mathbb{F}=\mathbb{C}$. Our main algebraic result is then the following

1.1 Theorem. Let $G = \mathrm{SU}(p,q)$ or $G = \mathrm{SL}(m,\mathbb{C})$ and $\tau : G \to G$ an involutive automorphism with $G^\tau \cong \mathrm{SO}(p,q)$, resp. $G^\tau \cong \mathrm{SO}(m,\mathbb{C})$. For any two maximal abelian ad-nilpotent subalgebras $\mathfrak{n}_1, \mathfrak{n}_2 \subset \mathfrak{g}$, contained in the (-1)-eigenspace of τ, the following conditions are equivalent:

(i) $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are conjugate by an element in G,
(ii) $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are conjugate by an element in G^τ,
(iii) the associated algebras $\mathcal{N}_1$ and $\mathcal{N}_2$ are isomorphic as abstract $\mathbb{F}$-algebras.

The commutative nilpotent algebras $\mathcal{N}$ occurring in the above theorem all have a 1-dimensional annihilator. On the other hand, given any nilpotent commutative $\mathbb{F}$-algebra $\mathcal{N}$ with 1-dimensional annihilator $\mathcal{A}$, we construct an invariant of $\mathcal{N}$ which is a certain non-degenerate symmetric 2-form $b_\pi : \mathcal{N}/\mathcal{A} \times \mathcal{N}/\mathcal{A} \to \mathcal{A}$. Depending on the type of b_π, the algebra $\mathcal{N}$ gives rise to a maximal abelian subalgebra in $\mathfrak{g}^{-\tau}$ ($\mathfrak{g}$ and τ as in the preceeding theorem) and in turn to a tube realization of a hyperquadric.

Summarizing, the classification of all local tube realizations of hyperquadrics is essentially equivalent to the classification of finite-dimensional nilpotent commutative $\mathbb{F}$-algebras up to isomorphism. In this paper we give an explicit classification for low values of p or q, i.e., we carry out all those cases (p,q) where there are only finitely many isomorphy classes. The classification in terms of explicit lists seems to be hopeless in the general case. For big values of p and q there are always uncountable many inequivalent nilpotent commutative $\mathbb{F}$-algebras.

As the algebraic results developed in this paper might be of broader interest, we collect in the Appendix all relevant results concerning the fine structure of nilpotent commutative algebras. These are formulated in a more general setup (e.g. over arbitrary fields of characteristic zero).

For certain applications one would like to have explicit defining equations for the various tube realizations, determined by qualifying subalgebras $\mathfrak{v} \subset \mathfrak{su}(p,q)$. One of our main geometric results is a procedure which produces for every qualifying $\mathfrak{v}$ an explicit defining equation which describes the tube realization $iF_{\mathfrak{v}} \oplus V_{\mathfrak{v}}$ of $S_{p,q}$. In that way we obtain quite transparent formulae, reflecting the algebraic structure of $\mathfrak{v}$.

The paper is organized as follows: In Section 2 we relate our results to existing results in the literature, in particular to those in [5], [11], [12]. In Section 3 we recall the necessary tools from [10] and give a short outline of the classification procedure. In particular we introduce certain abelian subalgebras $\mathfrak{v} \subset \mathfrak{su}(p,q)$ as qualifying MASAs – these are the algebraic objects to be classified. In Section 4 we split every MASA $\mathfrak{v}$ into its toral $\mathfrak{v}^{\mathrm{red}}$ and its nilpotent part $\mathfrak{v}^{\mathrm{nil}}$ and classify the centralizers of $\mathfrak{v}^{\mathrm{red}}$. Crucial for the classification is the decomposition given by Lemma 4.6 that leads to a combinatorial invariant $\boldsymbol{D}(\mathfrak{v})$ that we call the D-invariant of $\mathfrak{v}$. For fixed p,q the set $\mathcal{D}_{p,q}$ of all D-invariants of MASAs in $\mathfrak{su}(p,q)$ is finite, but still, in general there are infinitely many equivalence classes of MASAs in $\mathfrak{su}(p,q)$ with a fixed D-invariant. In Sections 5 and 6 we study MANSAs (maximal commutative associative nilpotent subalgebras) in $\mathfrak{su}(p_j,q_j)$ and $\mathfrak{sl}(\mathfrak{m}_j,\mathbb{C})$ as these are the building blocks for general MASAs in $\mathfrak{su}(p,q)$. In Section 7 we demonstrate briefly how for every MANSA $\mathfrak{v} \subset \mathfrak{su}(p,q)$ with corresponding tube realization $iF + \mathbb{R}^n \subset \mathbb{C}^n$ of $S_{p,q}$ the base $F \subset \mathbb{R}^n$ can be written in terms of a canonical equation. In section 9 we give two examples of MANSAs and in the Appendix 10 we collect several algebraic tools needed in the paper that might also be of independent interest.

2. Preliminaries

In the following we characterize algebraically the local tube realizations of the hyperquadric $S = S_{p,q} \subset \mathbb{P}_r := \mathbb{P}(\mathbb{C}^{r+1})$ with $p,q \geq 1$ and $r := p+q-1$, compare (3.3) in [10]. Since $S_{p,q}$ and $S_{q,p}$ only differ by a biholomorphic automorphism of the projective space $\mathbb{P}_r$ it would be enough to discuss the case $p \geq q$.

The local tube realizations of S up to affine equivalence in the cases $q = 1,2$ were obtained in the papers [5], [11] respectively by solving certain systems of partial differential equations coming from the Chern-Moser theory [3]. A classification of the case $q = 3$ has been announced in [12], proofs are intended to appear in the forthcoming book [14].

In this paper we give a classification for arbitrary p,q. It turns out that this, after several reducing steps, essentially boils down to the classification of abstract abelian nilpotent real and complex algebras $\mathcal{N}$ of dimension $r := p+q-1$ with 1-dimensional annihilator. For small values of q these can be determined explicitly while for large p,q this appears to be hopeless. On the other hand, we associate to every local tube realization of $S_{p,q}$ a combinatorial invariant $\boldsymbol{D}$ out of a finite set $\mathcal{D}_{p,q}$ in such a way that for any two local tube realizations the equations for the corresponding tube bases $F, \widetilde{F} \subset \mathbb{R}^r$ are essentially of the same type (up to some polynomial terms in the coordinates of $\mathbb{R}^r$ coming from the aforementioned different abelian nilpotent algebras arising naturally in this context).

To compare this with the known results in case $q \leq 3$ let us introduce the number $n := r-1 = p+q-2$, so that every local tube realization T of $S = S_{p,q}$ is a hypersurface in $\mathbb{C}^{n+1}$ with CR-dimension n, and r is the rank of the Lie algebra $\mathfrak{hol}(S)$. Let furthermore $c_{p,q}$ be the cardinality of all affine equivalence classes of closed tube submanifolds in $\mathbb{C}^r$ that are locally CR-equivalent to S.

In case $q=1$, that is the case of the standard sphere in $\mathbb{C}^p$, [5] implies $c_{p,1}=p+2=n+3$. In case $p\ge q=2$ we have $n=p$ and the explicit list of tube realizations in [11] implies the estimate $c_{p,2}\le p(p+9)/2$. Our considerations will give

$$c_{p,2}=5p+k(p-k)-\delta_{p,2}\ \text{ with }\ k:=\lceil p/2\rceil\,, \tag{2.1}$$

where for every $t\in\mathbb{R}$ the ceiling $\lceil t\rceil$ is the smallest integer $\ge t$ and δ is the Kronecker delta. Therefore, the list in the Theorem of [11] p. 442 must contain repetitions. Indeed, in type 7) for every s the parameters t and $\widetilde{t}:=n-2+s-t$ give affinely equivalent tube realizations. The same holds in case $p=2$ for $s=1$ in type 1) and $s=0$ in type 2).
In case $p\ge q=3$ it has been announced in [12] that $c_{p,3}$ is finite if and only if $p\le 5$.

In case $p,q\ge 4$ we show that $c_{p,q}$ always is infinite. Except for $c_{4,4,}$ this has already been announced in [12].

3. The algebraic setup

For fixed $p,q\ge 1$ with $m:=p+q\ge 3$ let $\mathbb{E}\cong\mathbb{C}^m$ be a complex vector space and $\mathsf{h}:\mathbb{E}\times\mathbb{E}\to\mathbb{C}$ a hermitian form of type (p,q) (p positive and q negative eigenvalues). Since any two hermitian forms of the same type on $\mathbb{E}$ are equivalent (up to a positive multiplicative constant) with respect to the group $L:=\mathrm{SL}(\mathbb{E})\cong\mathrm{SL}(m,\mathbb{C})$ it does not matter which h has been chosen above. More important for computational purposes is to choose a convenient vector basis of $\mathbb{E}$ in such a way that the corresponding matrix representation of h is optimally adapted.

The complex Lie group L acts in a canonical way transitively on the complex projective space $Z:=\mathbb{P}(\mathbb{E})$ with finite kernel of ineffectivity (the center of L). The subgroup

$$G:=\{g\in L:\mathsf{h}(gz,gz)=\pm\mathsf{h}(z,z)\ \text{ for all }\ z\in\mathbb{E}\} \tag{3.1}$$

is a real Lie group with $(1+\delta_{p,q})$ connected components acting transitively on the hypersurface

$$S=S_{p,q}:=\{[z]\in\mathbb{P}(\mathbb{E}):\mathsf{h}(z,z)=0\}\,.$$

We write for the corresponding Lie algebras

$$\mathfrak{l}=\mathfrak{sl}(\mathbb{E})\quad\text{ and }\quad\mathfrak{g}:=\mathfrak{su}(\mathbb{E},\mathsf{h})=\{\xi\in\mathfrak{l}:\mathrm{Re}\,\mathsf{h}(\xi z,z)=0\ \text{ for all }\ z\in\mathbb{E}\}\,.$$

As a matter of fact, $\mathfrak{l}$ coincides with the complex Lie algebra $\mathfrak{hol}(Z)$ of holomorphic vector fields on Z. Further, for every $a\in S$ the canonical inclusions $\mathfrak{g}\hookrightarrow\mathfrak{hol}(S)\hookrightarrow\mathfrak{hol}(S,a)$ turn out to be isomorphisms, and therefore we identify $\mathfrak{hol}(S)$ with $\mathfrak{g}$. With $\sigma:\mathfrak{l}\to\mathfrak{l}$ we denote the antilinear involutive Lie automorphism with $\mathrm{Fix}(\sigma)=\mathfrak{g}$.

The hyperquadric $S=S_{p,q}$ satisfies the assumptions of Theorem 7.1 in [10], and the various tube realizations are, up to the global affine equivalence as defined in [10] Definition 6.1, in a 1-1-correspondence to $\mathrm{Glob}(S,a)$-conjugacy classes of certain abelian subalgebras in $\mathfrak{hol}(S,a)$ (see [10] for the definition of $\mathrm{Glob}(S,a)$ and its basic properties). In the case under consideration $\mathrm{Glob}(S,a)=\mathrm{Ad}(G)\subset\mathrm{Aut}(\mathfrak{hol}(S,a))$ for every $a\in S$; our task then will be to classify up to the action of $\mathrm{Ad}(G)=\mathrm{Ad}(N_L(\mathfrak{g}))$ on $\mathfrak{l}$ all σ-invariant abelian subalgebras $\mathfrak{e}\subset\mathfrak{l}$ which have an open orbit in Z. Every such $\mathfrak{e}$ automatically has complex dimension $r:=m-1$ and is maximal abelian in $\mathfrak{l}$ by Lemma 2.1 in [10].

Every involution $\tau_{\mathfrak{v}}:(S,a)\to(S,a)$ extends to a global involution $\widehat{\tau}_{\mathfrak{v}}:Z\to Z$. Moreover, any two such involutions are conjugate by an element of G (even of the connected identity component $\mathrm{SU}(\mathbb{E},\mathsf{h})$ of G), compare [10]. The search can therefore be restricted by fixing once and for all an involution τ of S whose fixed point set $S^\tau = \mathrm{Fix}(\tau)$ is not empty and has dimension $r-1$. Such a τ has a unique extension to an antiholomorphic involution of $\mathbb{P}(\mathbb{E})$ that comes from a conjugation $\mathbb{E} \to \mathbb{E}$, $z \mapsto \overline{z}$, that is, $\tau[z] = [\overline{z}]$ for all $[z] \in S$. By our results it is enough to classify up to conjugation by G all abelian Lie subalgebras $\mathfrak{v} \subset \mathfrak{g}^{-\tau}$ with $\varepsilon_a(\mathfrak{v}) = T_a^{-\tau}S$ for a given point $a \in S$. These $\mathfrak{v}$ are automatically maximal abelian in $\mathfrak{g} \cong \mathfrak{su}(p,q)$ and have dimension $r = m-1 = \mathrm{rank}(\mathfrak{g})$.

3.2 Setup For the rest of the paper we fix the following notation: For p,q and $m = p+q$ as above, $\mathbb{E}$ is a complex vector space of dimension m with (positive definite) inner product $(z|w)$ (complex linear in the first and antilinear in the second variable). Furthermore, $\tau : \mathbb{E} \to \mathbb{E}$, $z \mapsto \overline{z}$, is an (antilinear) conjugation on $\mathbb{E}$ with $(\overline{z}|\overline{w}) = (w|z)$ for all $z, w \in \mathbb{E}$. With the same symbol τ we also denote the induced antiholomorphic involution of $Z = \mathbb{P}(\mathbb{E})$ as well as of the complex Lie algebra $\mathfrak{l} := \mathfrak{sl}(\mathbb{E}) = \mathfrak{hol}(\mathbb{P}(\mathbb{E}))$. In addition, $(e_j)_{1\le j\le m}$ is an orthonormal basis of $\mathbb{E}$ with $e_j = \overline{e}_j$ for all j, and the hermitian form $h = h_{p,q}$ on $\mathbb{E}$ is given by

$$h(e_j, e_k) = \vartheta_{p,j}\delta_{j,k} \ \text{ with } \ \vartheta_{p,j} := \begin{cases} 1 & \text{if } p \ge j \\ -1 & \text{otherwise}. \end{cases} \tag{3.3}$$

The involution τ on Z leaves the hyperquadric $S = S_{p,q}$ invariant. Therefore also $\mathfrak{g} = \mathfrak{su}(p,q) = \mathfrak{hol}(S)$ is invariant under the involution τ of $\mathfrak{l}$. As before, σ is the involution of $\mathfrak{l}$ defining the real form $\mathfrak{g}$ of $\mathfrak{l}$. Clearly, the involutions σ, τ commute on $\mathfrak{l}$.

$\mathrm{SU}(\mathbb{E}, h) = \mathrm{SU}(p,q)$ is the connected identity component G^0 of the group G defined in (3.1). Only in case $p = q$ the group G is disconnected and then $\left(\begin{smallmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{smallmatrix}\right) \in \mathrm{SL}(2p, \mathbb{C})$ is contained in the second connected component of G.

With $\mathrm{End}(\mathbb{E})$ we denote the endomorphism algebra of $\mathbb{E}$, a unital complex associative algebra with involution $g \mapsto g^*$ (the adjoint with respect to the inner product). With respect to the Lie bracket $[f,g] = fg - gh$ it becomes a reductive complex Lie algebra that is denoted by $\mathfrak{gl}(\mathbb{E})$ and contains $\mathfrak{sl}(\mathbb{E})$ as semisimple part. For every $z, w \in \mathbb{E}$ we denote by $z \otimes w^* \in \mathfrak{gl}(\mathbb{E})$ the endomorphism $x \mapsto (x|w)z$. Then $(z \otimes w^*)^* = w \otimes z^*$ is obvious. We also consider adjoints with respect to h and write $g^\star$ for the endomorphism satisfying $h(gw, z) = h(w, g^\star z)$ for all $w, z \in \mathbb{E}$.

3.4 The task Let $G, \mathfrak{g}, \mathfrak{l} = \mathfrak{sl}(\mathbb{E}), \sigma$ be as before and let a compatible conjugation $\tau : \mathfrak{l} \to \mathfrak{l}$ be fixed once and for all, induced by $z \mapsto \overline{z}$ on $\mathbb{E}$. In order to classify all local tube realizations of $S = S_{p,q}$ up to globally affine equivalence (compare Section 6 in [10]) we have to classify all abelian subalgebras $\mathfrak{v} \subset \mathfrak{g} = \mathfrak{hol}(S_{p,q}) = \mathfrak{su}(p,q)$ up to conjugation with respect to $\mathrm{Ad}(G)$ which have the following property:

(A) The complexification $\mathfrak{v}^{\mathbb{C}}$ has an open orbit in $Z = \mathbb{P}(\mathbb{E})$, that is, $\varepsilon_a(\mathfrak{v}^{\mathbb{C}}) = T_a Z$ for some $a \in Z$ (and hence even for some $a \in S$).

This condition, justified by Proposition 4.2 in [10], is of geometric nature but implies the following purely algebraic properties:

(B) $\mathfrak{v}$ is maximal abelian in $\mathfrak{g}$ – we call every such subalgebra a MASA in $\mathfrak{g}$.
(C) $\dim \mathfrak{v} = \mathrm{rank}\, \mathfrak{g} = \dim Z \quad (= r := m-1)$.
(D) $\mathrm{Ad}(g)(\mathfrak{v}) \subset \mathfrak{g}^{-\tau}$ for some $g \in G$.

Instead of classifying all G-conjugation classes of $\mathfrak{v}$ with property (A) we classify more generally the classes of $\mathfrak{v}$ satisfying (B) and (D), let us call them qualifying MASAs in $\mathfrak{g}$ for the following. It will turn out a posteriori that these $\mathfrak{v}$ automatically satisfy (A) and hence also (C).

3.5 A short outline of the classification procedure We proceed by analyzing the algebraic structure of maximal abelian subalgebras $\mathfrak{v} \subset \mathfrak{g}$. We will need some well known facts from the structure theory of semisimple Lie algebras (we refer to [17] and [21] as general references). Write

$$N_{\mathfrak{a}}(\mathfrak{b}) := \{x \in \mathfrak{a} : [x, \mathfrak{b}] \subset \mathfrak{b}\} \text{ for the normalizer and}$$
$$C_{\mathfrak{a}}(\mathfrak{b}) := \{x \in \mathfrak{a} : [x, \mathfrak{b}] = 0\} \text{ for the centralizer}$$

of any subalgebra $\mathfrak{b}$ in a Lie algebra $\mathfrak{a}$. Also let $Z(\mathfrak{a}) := C_{\mathfrak{a}}(\mathfrak{a})$ be the center of $\mathfrak{a}$. The classification idea is based on the observation that each maximal abelian $\mathfrak{v} \subset \mathfrak{g} \subset \mathrm{End}(\mathbb{E})$ has a unique decomposition into toral and nilpotent part, i.e., $\mathfrak{v} = \mathfrak{v}^{\mathrm{red}} \oplus \mathfrak{v}^{\mathrm{nil}}$, where $\mathfrak{v}^{\mathrm{red}}$ consists of semisimple and $\mathfrak{v}^{\mathrm{nil}}$ of nilpotent elements in $\mathrm{End}(\mathbb{E})$. Each toral subalgebra, in particular $\mathfrak{v}^{\mathrm{red}}$ of a qualifying MASA $\mathfrak{v}$, gives rise to the real reductive subalgebra $C_{\mathfrak{g}}(\mathfrak{v}^{\mathrm{red}})$. On the other hand, the maximality of $\mathfrak{v}$ implies that $\mathfrak{v}^{\mathrm{red}} = Z(C_{\mathfrak{g}}(\mathfrak{v}^{\mathrm{red}}))$. Hence, there is a natural bijection between [the G-conjugacy classes of] toral parts $\mathfrak{v}^{\mathrm{red}}$ of qualifying MASAs $\mathfrak{v}$ and [the G-conjugacy classes of] certain reductive subalgebras $C_{\mathfrak{g}}(\mathfrak{v}^{\mathrm{red}})$. A particular class of qualifying MASAs is formed by those $\mathfrak{v}$-s for which $\mathfrak{v}^{\mathrm{nil}} = 0$, i.e., $\mathfrak{v}$ is a real Cartan subalgebra of $\mathfrak{g}$. It turns out that each of the $\min\{p, q\}+1$ conjugacy classes (with respect to G^0 - or equivalently to G) of real CSAs has qualifying representatives.

It is well-known that general centralizers of tori in complex semisimple Lie algebras can be characterized by subsets of simple roots. In our case however we have to classify real centralizers. An additional complication is that not all (conjugacy classes of) centralizers of tori are of the form $C_{\mathfrak{g}}(\mathfrak{v}^{\mathrm{red}})$ with a qualifying MASA $\mathfrak{v}$. In the first part of our classification we provide a combinatorial tool giving an explicit characterization of these conjugacy classes of centralizes which are related to qualifying MASAs.

The nilpotent part $\mathfrak{v}^{\mathrm{nil}}$ of a qualifying MASA is contained in the semisimple part of $C_{\mathfrak{g}}(\mathfrak{v}^{\mathrm{red}})$, more precisely, we have the following diagram:

$$(3.6)\qquad \begin{array}{ccccc} \mathfrak{v} & = & \mathfrak{v}^{\mathrm{red}} & \oplus & \mathfrak{v}^{\mathrm{nil}} \\ \cap & & \| & & \cap \\ C_{\mathfrak{g}}(\mathfrak{v}^{\mathrm{red}})^{-\tau} & = & Z(C_{\mathfrak{g}}(\mathfrak{v}^{\mathrm{red}})) & \oplus & C^{\mathrm{ss}}_{\mathfrak{g}}(\mathfrak{v}^{\mathrm{red}})^{-\tau} . \end{array}$$

Moreover, $\mathfrak{v}^{\mathrm{nil}}$ is a maximal abelian and nilpotent subalgebra of $C^{\mathrm{ss}}_{\mathfrak{g}}(\mathfrak{v}^{\mathrm{red}})$ – we call such subalgebras MANSAs. An important observation is that the simple factors occurring in $C^{\mathrm{ss}}_{\mathfrak{g}}(\mathfrak{v}^{\mathrm{red}})$ are not arbitrary real forms of $C_{\mathfrak{l}}(\mathfrak{v}^{\mathrm{red}})$: A simple factor in $C^{\mathrm{ss}}_{\mathfrak{g}}(\mathfrak{v}^{\mathrm{red}})$ is isomorphic either to $\mathfrak{su}(p', q')$ or $\mathfrak{sl}_m(\mathbb{C})$. Consequently, $\mathfrak{v}^{\mathrm{nil}}$ is a product of qualifying MANSAs in $\mathfrak{su}(p', q')$ and $\mathfrak{sl}_m(\mathbb{C})$. The classification of the last mentioned Lie subalgebras turns out to be equivalent to the classification of arbitrary real or complex associative commutative nilpotent algebras $\mathcal{N}$ with 1-dimensional annihilator.

We will analyze the consequences of condition (C) later on; one outcome is that $\dim \mathfrak{v}^{\mathrm{nil}} = \mathrm{rank}(C^{\mathrm{ss}}_{\mathfrak{g}}(\mathfrak{v}^{\mathrm{red}}))$, hence $\dim \mathfrak{v} = \mathrm{rank}\, \mathfrak{g}$ and each such $\mathfrak{v}^{\mathbb{C}}$ has an open orbit in Z. Summarizing, our task then is reduced to the solution of the following algebraic problems:

[R] Classify up to conjugation all reductive subalgebras $\mathfrak{r} \subset \mathfrak{g}$ which are centralizers of the reductive part of a qualifying MASA $\mathfrak{v} \subset \mathfrak{g}$ (compare 3.4).

[N] Given a τ-stable reductive subalgebra $\mathfrak{r} = Z(\mathfrak{r}) \oplus \mathfrak{r}^{\mathrm{ss}}$ of the above type, classify up to conjugation all maximal abelian nilpotent subalgebras $\mathfrak{n}$ of $\mathfrak{r}^{\mathrm{ss}}$ with $\mathfrak{n} \subset (\mathfrak{r}^{\mathrm{ss}})^{-\tau}$.

4. Classification of the centralizers $C_{\mathfrak{g}}(\mathfrak{v}^{\mathrm{red}})$

4.1 Qualifying Cartan subalgebras. Particular examples of MASAs $\mathfrak{v} \subset \mathfrak{g}$ are maximal toral subalgebras, i.e., Cartan subalgebras. This is precisely the case when $\mathfrak{v} = \mathfrak{v}^{\mathrm{red}}$ and $\mathfrak{v}^{\mathrm{nil}} = 0$. It is well-known that $\mathfrak{su}(p, q)$ (with $q \le p$) has $q + 1$ G^0–conjugacy classes of real Cartan subalgebras. We need to know that each such conjugacy class contains a qualifying MASA of $\mathfrak{g}$:

4.2 Lemma. (Maximal toral subalgebras) Let $\mathfrak{g}$, τ and $S = S_{p,q}$ with $p \geq q$ be as before. Then:

(i) Every complex Cartan subalgebra of $\mathfrak{sl}(\mathbb{E})$ has an open orbit in $\mathbb{P}(\mathbb{E})$.
(ii) Every G^0-conjugacy class of real Cartan subalgebras in $\mathfrak{g}$ has a representative contained in $\mathfrak{g}^{-\tau}$.
(iii) To every maximal abelian subalgebra $\mathfrak{v} \subset \mathfrak{g}^{-\tau}$ there exists a real Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a $g \in G$ with $\mathrm{Ad}(g)(\mathfrak{v}^{\mathrm{red}}) \subset \mathfrak{h} \subset \mathfrak{g}^{-\tau}$.

Proof. (i) is an easy consequence of the fact that there is only one conjugacy class of complex CSAs in $\mathfrak{sl}(\mathbb{E})$ and that the subspace of all diagonal matrices in $\mathfrak{sl}(\mathbb{E})$ is one of them. For the proof of (ii) fix an arbitrary real CSA $\mathfrak{h}$ of $\mathfrak{g}$. Then $\mathfrak{h}^{\mathbb{C}}$ is a complex CSA of $\mathfrak{sl}(E)$ and hence has an open orbit in $\mathbb{P}(\mathbb{E})$. Therefore, by Propositions 4.2 and 3.2 in [10], there is a point $a \in S$ and an involution θ of (S, a) with $\mathfrak{h} \subset \mathfrak{g}^{-\theta}$. Also, θ satisfies (3.1) in [10] and extends to an antiholomorphic involution of $\mathbb{P}(\mathbb{E})$. Therefore $\tau = g\theta g^{-1}$ for some $g \in G^0$, that is, $\mathrm{Ad}(g)(\mathfrak{h}) \subset \mathfrak{g}^{-\tau}$. Below, we also give an alternative, algebraic proof of (ii) without referring to results from [10]. Assertion (iii) follows from (ii) since there exists a CSA $\mathfrak{h}$ of $\mathfrak{g}$ with $\mathfrak{v}^{\mathrm{red}} \subset \mathfrak{h}$. □

Every toral subalgebra $\mathfrak{t} \subset \mathfrak{g} \subset \mathrm{End}(\mathbb{E})$ has a unique decomposition $\mathfrak{t} = \mathfrak{t}_+ \oplus \mathfrak{t}_-$ into its compact and its vector part, that is, all elements in $\mathfrak{t}_+$ ($\mathfrak{t}_-$ respectively) have imaginary (real respectively) spectrum as operators on $\mathbb{E}$. Clearly the dimensions of these parts are invariants of the G-conjugacy class of $\mathfrak{t}$ in $\mathfrak{g}$, and in the case of a semisimple Lie algebra $\mathfrak{g}$ of Hermitian type, (as for instance $\mathfrak{su}(p,q)$) $\dim \mathfrak{t}_+$ determines uniquely its conjugacy class. For later use we construct explicitly for every $\ell = 0, 1, \ldots, q$ a CSA ${}^\ell\mathfrak{h}$ of $\mathfrak{g}$ with ${}^\ell\mathfrak{h} \subset \mathfrak{g}^{-\tau}$ and $\ell = \dim({}^\ell\mathfrak{h}_-)$.

4.3 Diagonal bases. Consider on the integer interval $\{1, 2, \ldots, m\}$ the reflection defined by $j \mapsto j^\bullet := m+1-j$ and recall the choice of the orthonormal basis $(e_j)_{1\leq j\leq m}$ and of $\vartheta_{p,j}$ as in (3.3). Fix an integer ℓ with $0 \leq \ell \leq q$, a complex number ω with $2\omega^2 = i$ and define a new orthonormal basis $({}^\ell f_j)_{1\leq j\leq m}$ of $\mathbb{E}$ by

$$ {}^\ell f_j := \begin{cases} e_j & \text{if } \ell < j < \ell^\bullet \\ \omega e_j + \overline{\omega} e_{j^\bullet} & \text{otherwise} \end{cases}, \quad \text{for which } e_j = \begin{cases} {}^\ell f_j & \text{if } \ell < j < \ell^\bullet \\ \overline{\omega}\, {}^\ell f_j + \omega\, {}^\ell f_{j^\bullet} & \text{otherwise} \end{cases} $$

is easily verified. Then for all $1 \leq j \leq k \leq m$ we have

$$ h({}^\ell f_j, {}^\ell f_k) = \begin{cases} i\delta_{j,k^\bullet} & \text{if } j \leq \ell \\ \vartheta_{p,j}\delta_{j,k} & \text{if } \ell < j < \ell^\bullet \,. \end{cases} \tag{4.4} $$

Let ${}^\ell\mathfrak{h} \subset \mathfrak{g} = \mathfrak{su}(\mathbb{E}, h)$ be the abelian subalgebra of all endomorphisms that are diagonal with respect to $({}^\ell f_j)$. From

$$ \begin{aligned} {}^\ell f_j \otimes {}^\ell f_j^* + {}^\ell f_{j^\bullet} \otimes {}^\ell f_{j^\bullet}^* &= e_j \otimes e_j^* + e_{j^\bullet} \otimes e_{j^\bullet}^* \\ {}^\ell f_j \otimes {}^\ell f_j^* - {}^\ell f_{j^\bullet} \otimes {}^\ell f_{j^\bullet}^* &= ie_j \otimes e_{j^\bullet}^* - ie_{j^\bullet} \otimes e_j^* \,. \end{aligned} $$

for all $j \leq \ell$ we derive that the decomposition ${}^\ell\mathfrak{h} = {}^\ell\mathfrak{h}_+ \oplus {}^\ell\mathfrak{h}_-$ into compact and vector parts is given by

$$ \begin{aligned} {}^\ell\mathfrak{h}_- &= \bigoplus_{j\leq\ell} i\mathbb{R}\Big(e_j \otimes e_{j^\bullet}^* - e_{j^\bullet} \otimes e_j^*\Big) \\ {}^\ell\mathfrak{h}_+ &= \Big(\bigoplus_{j\leq\ell} i\mathbb{R}\big(e_j \otimes e_j^* + e_{j^\bullet} \otimes e_{j^\bullet}^*\big) \oplus \bigoplus_{\ell<j<\ell^\bullet} i\mathbb{R}\big(e_j \otimes e_j^*\big)\Big)_{\mathrm{tr}=0} \,. \end{aligned} \tag{4.5} $$

As a consequence, ${}^\ell\mathfrak{h} = C_{\mathfrak{g}}({}^\ell\mathfrak{h})$ is a CSA of $\mathfrak{g}$ with $\dim({}^\ell\mathfrak{h}_-) = \ell$ and ${}^\ell\mathfrak{h} \subset \mathfrak{g}^{-\tau}$. This gives a constructive proof for (ii) in Lemma 4.2. □

4.6 The general case $\mathfrak{v}^{\,\mathrm{red}} \subset \mathfrak{v}$. We proceed to the general case where $C_{\mathfrak{g}}(\mathfrak{v}^{\,\mathrm{red}})$ may contain $\mathfrak{v}^{\,\mathrm{red}}$ properly, that is $\mathfrak{v}^{\,\mathrm{nil}} \neq 0$. Given $\mathfrak{v}^{\,\mathrm{red}}$, or equivalently $C_{\mathfrak{g}}(\mathfrak{v}^{\,\mathrm{red}})$, after conjugating with an element of $\mathrm{SU}(\mathbb{E}, h)$ we may assume that $\mathfrak{v}^{\,\mathrm{red}} \subset {}^{\ell}\mathfrak{h} \subset \mathfrak{g}^{-\tau}$ for some $\ell \leq q$ as above. In the complex situation, (i.e., for the centralizer in $\mathfrak{g}^{\mathbb{C}}$, or equivalently in $\mathfrak{gl}(\mathbb{E})$) it is well known that there is a unique direct sum decomposition with summands $\mathbb{E}_{\jmath} \neq 0$

$$\mathbb{E} = \bigoplus_{\jmath \in \mathcal{J}} \mathbb{E}_{\jmath} \quad \text{such that} \quad C_{\mathfrak{gl}(\mathbb{E})}(\mathfrak{v}^{\,\mathrm{red}}) = \bigoplus_{\jmath \in \mathcal{J}} \mathfrak{gl}(\mathbb{E}_{\jmath}) \,. \tag{4.7}$$

The subspaces $\mathbb{E}_{\jmath}$ correspond to joint eigenspaces of the toral abelian subalgebra $\mathfrak{v}^{\,\mathrm{red}} \subset \mathfrak{sl}(\mathbb{E})$ with respect to certain functionals $\gamma \in (\mathfrak{v}^{\,\mathrm{red}})^{*}$. Since $\mathfrak{v}^{\,\mathrm{red}}$, and in turn $C_{\mathfrak{l}}(\mathfrak{v}^{\,\mathrm{red}})$ is invariant under the conjugation τ, we conclude that there is an involution $\jmath \mapsto \bar{\jmath}$ of the index set $\mathcal{J}$ with $\tau\mathbb{E}_{\jmath} = \mathbb{E}_{\bar{\jmath}}$ for all $\jmath \in \mathcal{J}$. One key point here is that for every $\jmath \in \mathcal{J}$ the restriction $h_{\jmath}$ of h to $\mathbb{E}_{\jmath} + \mathbb{E}_{\bar{\jmath}}$ is non-degenerate while in case $\bar{\jmath} \neq \jmath$ the spaces $\mathbb{E}_{\jmath}$ and $\mathbb{E}_{\bar{\jmath}}$ are totally h–isotropic and have zero intersection. (A priori, the non-degeneracy of the restrictions $h_{\jmath}$ does not follow from the mere τ–invariance of the decomposition $\mathbb{E} = \bigoplus \mathbb{E}_{\jmath}$. However, since $\mathfrak{v}^{\,\mathrm{red}} \subset {}^{\ell}\mathfrak{h}$ for some ℓ, one can show using root theory that every subspace $\mathbb{E}_{\jmath} \subset \mathbb{E}$ occurring in the above decomposition is invariant under every orthogonal projection ${}^{\ell}f_k \otimes {}^{\ell}f_k^{*}$, $k \in \mathcal{J}$. With 4.4 then the non-degeneracy of $h_{\jmath}$ follows).

4.8 Restrictions of σ and τ to the simple factors of the centralizer. Choose a subset $\mathcal{L} \subset \mathcal{J}$ such that $\mathcal{J} = \mathcal{K} \cup \mathcal{L} \cup \overline{\mathcal{L}}$ is a disjoint union for $\mathcal{K} := \{\jmath \in \mathcal{J} : \bar{\jmath} = \jmath\}$. For every $\jmath \in \mathcal{K}$ the subalgebra $\mathfrak{sl}(\mathbb{E}_{\jmath}) \subset \mathfrak{sl}(\mathbb{E})$ is invariant under τ as well as σ, that is

$$\mathfrak{sl}(\mathbb{E}_{\jmath})^{\sigma} = \mathfrak{su}(\mathbb{E}_{\jmath}, h_{\jmath}) \qquad \mathfrak{sl}(\mathbb{E}_{\jmath})^{\tau} = \mathfrak{sl}(\mathbb{E}_{\jmath}^{\tau})$$

(with $\mathfrak{su}(\mathbb{E}_{\jmath}, h_{\jmath}) = 0 = \mathfrak{sl}(\mathbb{E}_{\jmath})$ in case $\dim \mathbb{E}_{\jmath} = 1$). Also, for every $\jmath \in \mathcal{L}$ we have $\mathbb{E}_{\bar{\jmath}} = \tau(\mathbb{E}_{\jmath})$, $\sigma(\mathfrak{sl}(\mathbb{E}_{\jmath})) = \tau(\mathfrak{sl}(\mathbb{E}_{\jmath})) = \mathfrak{sl}(\mathbb{E}_{\bar{\jmath}})$ and $\sigma, \tau \in \mathrm{Aut}_{\mathbb{R}}(\mathfrak{sl}(\mathbb{E}_{\jmath}) \oplus \mathfrak{sl}(\mathbb{E}_{\bar{\jmath}}))$ are given by

$$\tau(x, y) = (\underline{\tau} y \underline{\tau}, \underline{\tau} x \underline{\tau}) \quad \text{and} \quad \sigma(x, y) = (-y^{\star}, -x^{\star}) \,,$$

where $x^{\star}, y^{\star}$ are the adjoints of x, y with respect to the hermitian form h. The symmetric complex bilinear form $\beta : \mathbb{E}_{\jmath} \times \mathbb{E}_{\jmath} \to \mathbb{C}$ defined by $\beta(x, y) = h(x, \tau y)$ is non-degenerate and

$$\big(\mathfrak{sl}(\mathbb{E}_{\jmath}) \oplus \mathfrak{sl}(\mathbb{E}_{\bar{\jmath}})\big)^{\sigma} \cong \mathcal{R}^{\mathbb{C}}_{\mathbb{R}}\big(\mathfrak{sl}(\mathbb{E}_{\jmath})\big) \quad \big(\mathfrak{sl}(\mathbb{E}_{\jmath}) \oplus \mathfrak{sl}(\mathbb{E}_{\bar{\jmath}})\big)^{\sigma\tau} \cong \mathcal{R}^{\mathbb{C}}_{\mathbb{R}}\big(\mathfrak{so}(\mathbb{E}_{m_{\jmath}}, \mathbb{C})\big) \,, \tag{4.9}$$

where $\mathcal{R}^{\mathbb{C}}_{\mathbb{R}}$ is the forgetful functor restricting scalars from $\mathbb{C}$ to $\mathbb{R}$. With these ingredients we can state:

4.10 Lemma. For the decomposition (4.7) we have

$$\begin{aligned}
\mathfrak{v}^{\,\mathrm{red}} = Z(C_{\mathfrak{g}}(\mathfrak{v}^{\,\mathrm{red}})) &= \Big(\bigoplus_{\mathcal{K} \cup \mathcal{L}} i\mathbb{R}\, \mathrm{id}_{\mathbb{E}_{\jmath} + \mathbb{E}_{\bar{\jmath}}}\Big)_{\mathrm{tr}=0} \oplus \bigoplus_{\mathcal{L}} \mathbb{R}(\mathrm{id}_{\mathbb{E}_{\jmath}} - \mathrm{id}_{\mathbb{E}_{\bar{\jmath}}}) \\
C^{\mathrm{ss}}_{\mathfrak{g}}(\mathfrak{v}^{\,\mathrm{red}}) &= \bigoplus_{\mathcal{K}} \mathfrak{su}(\mathbb{E}_{\jmath}, h_{\jmath}) \oplus \bigoplus_{\mathcal{L}} \big(\mathfrak{sl}(\mathbb{E}_{\jmath}) \oplus \mathfrak{sl}(\mathbb{E}_{\bar{\jmath}})\big)^{\sigma} \\
&\cong \bigoplus_{\mathcal{K}} \mathfrak{su}(p_{\jmath}, q_{\jmath}) \oplus \bigoplus_{\mathcal{L}} \mathfrak{sl}(m_{\jmath}, \mathbb{C}) \,,
\end{aligned}$$

where $m_{\jmath} = \dim(\mathbb{E}_{\jmath})$ and $(p_{\jmath}, q_{\jmath})$ for $\jmath \in \mathcal{K}$ is the type of the restriction $h_{\jmath}$ on $\mathbb{E}_{\jmath}$. If $\mathfrak{v}^{\,\mathrm{red}} \subset {}^{\ell}\mathfrak{h}$ for a Cartan subalgebra as in (4.5) then each $\mathbb{E}_{\jmath}$ is spanned by some of the vectors in the basis $({}^{\ell}f_j)$. For each fixed $\mathfrak{g} \cong \mathfrak{su}(p, q)$ there are only finitely many G^0-conjugacy classes of centralizers $C_{\mathfrak{g}}(\mathfrak{v}^{\,\mathrm{red}})$ as $\mathfrak{v} \subset \mathfrak{g}$ varies through all qualifying MASAs in $\mathfrak{g}$.

For the sake of clarity let us mention that in general there are infinitely many conjugacy classes of qualifying MASAs $\mathfrak{v}$ while the above Lemma asserts that there are only finitely many conjugacy classes of the corresponding toral parts $\mathfrak{v}^{\,\mathrm{red}}$. The point here is that for a fixed $\mathfrak{v}^{\,\mathrm{red}}$ there may be infinitely many non-conjugate qualifying MANSAs in $C_{\mathfrak{g}}(\mathfrak{v}^{\,\mathrm{red}})^{\mathrm{ss}}$.

The above lemma describes the structure of $\mathfrak{v}^{\,\mathrm{red}}$ and its centralizer $C_{\mathfrak{g}}(\mathfrak{v}^{\,\mathrm{red}})$ in an elementary-geometric way and shows that both determine each other uniquely. For the description of $\mathfrak{v} = \mathfrak{v}^{\,\mathrm{red}} \oplus \mathfrak{v}^{\,\mathrm{nil}}$ it is therefore enough to determine all possible $\mathfrak{v}^{\,\mathrm{nil}}$. These split into a direct sum

$$(4.11)\qquad \begin{aligned} &\mathfrak{v}^{\,\mathrm{nil}} = \bigoplus_{\mathcal{K}\cup\mathcal{L}} \mathfrak{n}_{\jmath} \quad \text{with} \\ &\mathfrak{n}_{\jmath} := \mathfrak{v}^{\,\mathrm{nil}} \cap \mathfrak{s}_{\jmath} \quad \text{for} \quad \mathfrak{s}_{\jmath} := \begin{cases} \mathfrak{su}(\mathbb{E}_{\jmath}, h_{\jmath}) & \jmath \in \mathcal{K} \\ \big(\mathfrak{sl}(\mathbb{E}_{\jmath}) \oplus \mathfrak{sl}(\mathbb{E}_{\bar\jmath})\big)^{\sigma} & \jmath \in \mathcal{L}\,. \end{cases} \end{aligned}$$

Every $\mathfrak{n}_{\jmath}$ is an abelian ad-nilpotent subalgebra of $\mathfrak{s}_{\jmath}$. In case $\jmath \in \mathcal{K}$ the algebra $\mathfrak{n}_{\jmath}$ has dimension $p_{\jmath} + q_{\jmath} - 1$. As a consequence, $p_{\jmath} = 0$ is possible only if $q_{\jmath} = 1$ (since in case $q_{\jmath} > 1$ the form $h_{\jmath}$ is definite and every ad-nilpotent element in $\mathfrak{s}_{\jmath}$ is zero). In the same way $q_{\jmath} = 0$ implies $p_{\jmath} = 1$. In case $\jmath \in \mathcal{L}$ the algebra $\mathfrak{n}_{\jmath}$ has dimension $2m_{\jmath} - 2$.

Before we turn to the corresponding classification result we need to extract some invariants from the equations in 4.10. For every set A denote by $\mathcal{F}(A)$ the free commutative monoid over A. We write the elements of $\mathcal{F}(A)$ in the form $\sum_{\alpha\in A} n_{\alpha}{\cdot}\alpha$ with $n_{\alpha} \in \mathbb{N}$ and $\sum_{A} n_{\alpha} < \infty$. Here we use the free monoids over the following sets, where $\mathbb{N} = \{0, 1, 2, \ldots\}$:

$$(4.12)\qquad \begin{aligned} &\boldsymbol{K} := \{(s,t) \in \mathbb{N}^2 : (st = 0) \Rightarrow (s + t = 1)\}\,, \\ &\boldsymbol{L} := \mathbb{N}\backslash\{0\} \quad \text{and} \quad \boldsymbol{J} := \boldsymbol{K} \cup \boldsymbol{L}\,. \end{aligned}$$

Then $$\mathcal{D} := \mathcal{F}(\boldsymbol{J}) = \mathcal{F}(\boldsymbol{K}) + \mathcal{F}(\boldsymbol{L})\,,$$
and the permutation of $\boldsymbol{J}$ defined by $(s,t) \mapsto (t,s)$ on $\boldsymbol{K}$ and the identity on $\boldsymbol{L}$ induces an involution $\boldsymbol{D} \mapsto \boldsymbol{D}^{\mathrm{opp}}$ of $\mathcal{D}$. As an example, the opposite of $\boldsymbol{D} = 4{\cdot}(3,5) + 2{\cdot}\boldsymbol{7}$ is $\boldsymbol{D}^{\mathrm{opp}} = 4{\cdot}(5,3) + 2{\cdot}\boldsymbol{7}$. Notice that $2{\cdot}\boldsymbol{7}$ and $7{\cdot}\boldsymbol{2}$ are different elements in $\mathcal{F}(\boldsymbol{L}) \subset \mathcal{D}$. The set $\boldsymbol{J}$ can be considered in a canonical way as subset of $\mathcal{F}(\boldsymbol{J})$ by identifying $\boldsymbol{j} \in \boldsymbol{J}$ with $1{\cdot}\boldsymbol{j} \in \mathcal{D}$, but for better distinction we write $\boldsymbol{j}$ instead of $1{\cdot}\boldsymbol{j}$ only if no confusion is likely. Also, for better distinction we write the natural numbers in $\mathbb{N}\backslash\{0\}$ in boldface if we consider them as element of $\boldsymbol{L}$.

4.13 Definition. For every $p, q \in \mathbb{N}$ we denote by $\mathcal{D}_{p,q} \subset \mathcal{D}$ the subset of all

$$\boldsymbol{D} = \sum_{\boldsymbol{j}\in\boldsymbol{J}} n_{\boldsymbol{j}}{\cdot}\boldsymbol{j} \in \mathcal{F}(\boldsymbol{J}) \qquad \text{satisfying}$$

$$p = \sum_{\boldsymbol{j}=(s,t)\in\boldsymbol{K}} n_{\boldsymbol{j}} s + \sum_{\boldsymbol{j}\in\boldsymbol{L}} n_{\boldsymbol{j}} j \quad \text{and} \quad q = \sum_{\boldsymbol{j}=(s,t)\in\boldsymbol{K}} n_{\boldsymbol{j}} t + \sum_{\boldsymbol{j}\in\boldsymbol{L}} n_{\boldsymbol{j}} j\,,$$

where j is the natural number underlying $\boldsymbol{j}$. Then $\mathcal{D}_{p,q}^{\mathrm{opp}} = \mathcal{D}_{q,p}$ and $\mathcal{D}_{p,q} + \mathcal{D}_{p',q'} \subset \mathcal{D}_{p+p',q+q'}$ are obvious.

To every qualifying MASA $\mathfrak{v} \subset \mathfrak{su}(p,q)$ we associate an element $\boldsymbol{D}(\mathfrak{v})$ of $\mathcal{D}$ that only depends on the $\mathrm{SU}(p,q)$-conjugation class of $\mathfrak{v}$ and is called the D-invariant of $\mathfrak{v}$:

Suppose that $\mathfrak{v}$ (after a suitable conjugation) gives rise to the equations (4.7) and (4.10). Then just put

$$\boldsymbol{D}(\mathfrak{v}) := \sum_{\jmath\in\mathcal{K}} 1{\cdot}(p_\jmath, q_\jmath) + \sum_{\jmath\in\mathcal{L}} 1{\cdot}\boldsymbol{m}_\jmath\,. \tag{4.14}$$

For instance, the CSAs in $\mathfrak{g} = \mathfrak{su}(p,q)$ are precisely the qualifying MASAs $\mathfrak{v} \subset \mathfrak{g}$ with $\boldsymbol{D}(\mathfrak{v}) \in \mathcal{F}(A)$ with $A := \{(1,0),(0,1),\mathbf{1}\}$. Indeed, in the notation of (4.5) we have $\boldsymbol{D}({}^{\ell}\mathfrak{h}) = (p-\ell){\cdot}(1,0) + (q-\ell){\cdot}(0,1) + \ell{\cdot}\mathbf{1}$.

The relevance of the D-invariants for our classification problem is demonstrated by the following two results. Recall that $G = N_{\mathrm{SL}(p+q,\mathbb{C})}(\mathfrak{su}(\mathbb{E}, h_{p,q}))$ and $G^0 = \mathrm{SU}(\mathbb{E}, h_{p,q})$ with $G \neq G^0$ only if $p = q$.

4.15 Proposition. $\boldsymbol{D} \in \mathcal{D}$ is the D-invariant of a qualifying MASA $\mathfrak{v}$ in $\mathfrak{su}(p,q)$ if and only if $\boldsymbol{D} \in \mathcal{D}_{p,q}$.

4.16 Proposition. Let $\mathfrak{v}_1, \mathfrak{v}_2$ be two qualifying MASAs in $\mathfrak{g} = \mathfrak{su}(p,q)$. Then the toral parts $\mathfrak{v}_1^{\mathrm{red}}, \mathfrak{v}_2^{\mathrm{red}}$ (and hence also the corresponding centralizers) are G^0-conjugate in $\mathfrak{g}$ if and only if $D(\mathfrak{v}_1) = D(\mathfrak{v}_2)$. In case $p = q$ the toral parts $\mathfrak{v}_1^{\mathrm{red}}, \mathfrak{v}_2^{\mathrm{red}}$ are G-conjugate if and only if $D(\mathfrak{v}_1) = D(\mathfrak{v}_2)$ or $D(\mathfrak{v}_1) = D(\mathfrak{v}_2)^{\mathrm{opp}}$ (in this case $\mathrm{SU}(\mathbb{E}, h_{p,q})$ has index two in G).

Our classification problem now reduces to the following task: For every $p, q \geq 1$ with $p+q \geq 3$ and every $\boldsymbol{D}$ in the finite set $\mathcal{D}_{p,q}$ determine all G-conjugacy classes of qualifying MASAs $\mathfrak{v} \subset \mathfrak{su}(p,q)$ with $\boldsymbol{D}(\mathfrak{v}) = \boldsymbol{D}$.

4.17 Explicit classification for small values of q: For $\mathcal{D}_{p,q}$ in the cases $p \geq q = 1,2$ we have the following explicit lists (without repetitions).
$\mathcal{D}_{p,1}$ consists of all invariants
(i) $(p-s){\cdot}(1,0) + (s,1)$ for $s = 1, 2, \dots, p$.
(ii) $(p-1){\cdot}(1,0) + \mathbf{1}$ and $p{\cdot}(1,0) + (0,1)$.

$\mathcal{D}_{p,2}$ consists of all invariants
(iii) $\mathbf{1} + \mathcal{D}_{p-1,1}$,
(iv) $(p-s-t){\cdot}(1,0) + (s,1) + (t,1)$ for all $1 \leq s \leq t$ with $s+t \leq p$,
(v) $(p-s){\cdot}(1,0) + (s,2)$, $(p-s){\cdot}(1,0) + (0,1) + (s,1)$ for $1 \leq s \leq p$,
(vi) $(p-2){\cdot}(1,0) + \mathbf{2}$, $p{\cdot}(1,0) + 2{\cdot}(0,1)$.
Notice that in $\mathcal{D}_{2,2}$ there exists an invariant that is not self-opposite, e.g. $(1,0) + (1,2)$. As a consequence, in case $\mathfrak{g} \cong \mathfrak{su}(2,2)$ there are eleven $\mathrm{SU}(2,2)$-conjugation classes of centralizers in contrast to the only ten G-conjugation classes in this case.

4.18 Nilpotent parts of qualifying MASAs. So far we have given a description of conjugacy classes of the toral parts $\mathfrak{v}^{\mathrm{red}}$ of qualifying MASAs in $\mathfrak{g}$. For the description of $\mathfrak{v} = \mathfrak{v}^{\mathrm{red}} \oplus \mathfrak{v}^{\mathrm{nil}}$ it is therefore sufficient to determine all possible nilpotent parts $\mathfrak{v}^{\mathrm{nil}}$. According to 3.5, each such $\mathfrak{v}^{\mathrm{nil}}$ is a maximal abelian nilpotent subalgebra (MANSA) of $C^{\mathrm{ss}}_{\mathfrak{g}}(\mathfrak{v}^{\mathrm{red}})$, compare (3.6). On the other hand, given any qualifying MASA $\mathfrak{v} \subset \mathfrak{g}$, each MANSA $\mathfrak{n}$ of $C^{\mathrm{ss}}_{\mathfrak{g}}(\mathfrak{v}^{\mathrm{red}})$ in the -1-eigenspace of τ gives rise to a qualifying MASA $\mathfrak{v}_{\mathfrak{n}} := \mathfrak{v}^{\mathrm{red}} \oplus \mathfrak{n}$ in $\mathfrak{g}$. Given one of the finitely may conjugacy classes of centralizers $C_{\boldsymbol{D}}$ with $\boldsymbol{D} \in \mathcal{D}_{p,q}$, our task is therefore to classify the MANSAs in the semisimple part $C^{\mathrm{ss}}_{\boldsymbol{D}}$ of $C_{\boldsymbol{D}}$. As already explained, $C^{\mathrm{ss}}_{\boldsymbol{D}}$ decomposes uniquely into simple ideals $\mathfrak{g}_\jmath$, each of them being isomorphic either to $\mathfrak{su}(p_\jmath, q_\jmath)$ or to $\mathfrak{sl}_{m_\jmath}(\mathbb{C})$. Consequently each qualifying MANSA $\mathfrak{n} \subset C^{\mathrm{ss}}_{\boldsymbol{D}}$ has the unique decomposition $\mathfrak{n} = \bigoplus_{\mathcal{J}_1} \mathfrak{n}_\jmath$ with $\mathcal{J}_1 := \{\jmath \in \mathcal{K} \cup \mathcal{L} : \dim \mathbb{E}_\jmath > 1\} \subset \mathcal{J}$, compare

4.10. Each of the factors $\mathfrak{n}_\jmath$ is a qualifying MANSA in $\mathfrak{g}_\jmath$, more precisely:

$$
\begin{array}{lcccl}
C_{\mathfrak{g}}(\mathfrak{v}^{\text{red}}) = C_{\boldsymbol{D}} = & \mathfrak{v}^{\text{red}} & \oplus & \bigoplus_{\mathcal{J}_1} \mathfrak{g}_\jmath & \qquad \mathfrak{g}_\jmath \cong \mathfrak{su}(p_\jmath, q_\jmath) \\
\cup \qquad \| & & & \cup & \qquad \text{or} \\
\mathfrak{v}_{\mathfrak{n}} = & \mathfrak{v}^{\text{red}}_{\mathfrak{n}} & \oplus & \bigoplus_{\mathcal{J}_1} \mathfrak{n}_\jmath & \qquad \mathfrak{g}_\jmath \cong \mathcal{R}^{\mathbb{C}}_{\mathbb{R}}(\mathfrak{sl}(m_\jmath, \mathbb{C}))\ .
\end{array}
\tag{4.19}
$$

Summarizing the results of the present subsection, our next task is to determine maximal abelian ad-nilpotent subalgebras $\mathfrak{n}_\jmath \subset \mathfrak{g}_\jmath$ with either $\mathfrak{g}_\jmath = \mathfrak{su}(\mathbb{E}_\jmath, h_\jmath) \cong \mathfrak{su}(p_\jmath, q_\jmath)$ or $\mathfrak{g}_\jmath = (\mathfrak{sl}(\mathbb{E}_\jmath) \oplus \mathfrak{sl}(\mathbb{E}_{\bar\jmath}))^\sigma \cong \mathcal{R}^{\mathbb{C}}_{\mathbb{R}}(\mathfrak{sl}_{m_\jmath}(\mathbb{C}))$. Here we can restrict to subalgebras that are contained in the (-1)-eigenspace of an involution τ coming from a conjugation on the vector spaces $\mathbb{E}_\jmath$ and $\mathbb{E}_\jmath \oplus \mathbb{E}_{\bar\jmath}$ respectively. In the following we discuss the cases $\mathfrak{sl}(\mathbb{E})^\sigma = \mathfrak{su}(\mathbb{E}, h) \cong \mathfrak{su}(p, q)$ and $\big(\mathfrak{sl}(\mathbb{E}) \oplus \mathfrak{sl}(\overline{\mathbb{E}})\big)^\sigma \cong \mathfrak{sl}(m, \mathbb{C})$ separately.

5. MANSAs in $\mathfrak{su}(p, q)^{-\tau}$

For notational simplicity let us drop the subscript $\jmath$ for the rest of this section and write $\mathbb{E} = \mathbb{E}_\jmath$ as well as $(p, q) = (p_\jmath, q_\jmath)$ for the type of the restriction of h to $\mathbb{E}_\jmath$ and also $\mathfrak{g} = \mathfrak{su}(\mathbb{E}, h)$. As before we denote by σ the conjugation on the complexification $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(\mathbb{E})$ with $\mathrm{Fix}(\sigma) = \mathfrak{g}$. For every $z \in \mathrm{End}(\mathbb{E})$ we denote by $z^\star \in \mathrm{End}(\mathbb{E})$ the adjoint with respect to h.

Without loss of generality we assume $pq \neq 0$, since otherwise $p + q = 1$ and thus $\mathfrak{su}(E, h) = 0$, see (4.12). As before, $m = p + q$ and $r = m - 1$.

In order to classify the maximal nilpotent Lie subalgebras $\mathfrak{n} = \mathfrak{v}^{\text{nil}} \subset \mathfrak{su}(E, h)$ we relate them to nilpotent commutative and associative $\mathbb{R}$-algebras, see the Appendix for the terminology.

5.1 Proposition. Let $\mathfrak{n} \subset \mathfrak{g} = \mathfrak{su}(\mathbb{E}, h)$ be a Lie subalgebra. Then the following conditions are equivalent:

(i) $\mathfrak{n}$ is maximal among all abelian Lie subalgebras $\mathfrak{a}$ of $\mathfrak{g}$ such that every element of $\mathfrak{a}$ is a nilpotent endomorphism of $\mathbb{E}$.

(ii) $\mathfrak{n}$ is maximal among all abelian subalgebras of $\mathfrak{g}$ that are ad-nilpotent in $\mathfrak{g}$.

(iii) The complexification $\mathfrak{n}^{\mathbb{C}}$ is σ-stable and is maximal among all abelian and nilpotent associative subalgebras of $\mathrm{End}(\mathbb{E})$.

If these conditions are satisfied then $\mathfrak{n}^{\mathbb{C}} \oplus\ \mathbb{C}\,\mathrm{id}_{\mathbb{E}}$ is a maximal abelian subalgebra of the complex associative algebra $\mathrm{End}(\mathbb{E})$. Further, in the notation of 9.3 ff. for every $\mathfrak{n} \subset \mathfrak{g}$ satisfying one (and hence all) of the conditions (i) - (iii) the following holds:

(iv) The subspaces $\mathbb{K}_{\mathfrak{n}^{\mathbb{C}}}$ and $\mathbb{B}_{\mathfrak{n}^{\mathbb{C}}}$ are h-orthogonal in $\mathbb{E}$, implying $d_1 = d_3$.

(v) There exists an $\mathfrak{n}^{\mathbb{C}}$-adapted decomposition $\mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3$ of $\mathbb{E}$ such that $\mathbb{E}_1$ and $\mathbb{E}_3 = \mathbb{K}$ are h-isotropic and $\mathbb{E}_2 \perp_h (\mathbb{E}_1 \oplus \mathbb{E}_3)$. If, in addition, $\mathfrak{n}$ is τ-stable then the adapted decomposition can be chosen to be τ-stable, too. The restriction of h to $\mathbb{E}_2$ is of type $(p - d_1, q - d_1)$.

Now the conjugation τ of $\mathbb{E}$ comes into play. For $\mathbb{V} := \mathbb{E}^\tau \cong \mathbb{R}^m$ we have $\mathbb{E} = \mathbb{V} \oplus i\mathbb{V}$, and we identify $\mathrm{End}(\mathbb{V})$ in the obvious way with the real subalgebra $\mathrm{End}(\mathbb{E})^\tau$ of $\mathrm{End}(\mathbb{E})$. A crucial observation is the following refinement of Proposition 9.7.

5.2 Proposition. Assume that $\mathfrak{n} \subset \mathfrak{su}(\mathbb{E}, h)^{-\tau}$ is maximal among abelian and ad-nilpotent subalgebras of $\mathfrak{su}(E, h)$. As before, let $\mathcal{N} := i\,\mathfrak{n} \subset \mathrm{End}(\mathbb{E})$. Then, with the notation of Proposition 9.7, the following holds:

(i) $\dim \operatorname{Ann}(\mathcal{N}) = 1$. In particular, $\dim \mathbb{V}_1 = \dim \mathbb{V}_3 = 1$ for any $\mathcal{N}$-adapted decomposition of $\mathbb{V}$.

(ii) Fix generators $v_1 \in \mathbb{V}_1$ and $v_3 \in \mathbb{V}_3$ with $h(v_1, v_3) = 1$. This yields canonical identifications $\mathbb{V}_1 = \mathbb{R} = \mathbb{V}_3$, $\operatorname{Ann}(\mathcal{N}) = \mathbb{R}$, $\mathcal{N}_{21} = \operatorname{Hom}(\mathbb{R}, \mathbb{V}_2) = \mathbb{V}_2$ and $\mathcal{N}_{32} = \operatorname{Hom}(\mathbb{V}_2, \mathbb{R}) = \mathbb{V}_2^*$ (the dual of $\mathbb{V}_2$). The map $J : \mathbb{V}_2 \to \mathbb{V}_2^*$ is given by $J(y)(x) = h(x, y)$ for all $x, y \in \mathbb{V}_2$. With all these identifications the matrix presentation in Proposition 9.7 reads

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ y & N(y) & 0 \\ t & J(y) & 0 \end{pmatrix} : y \in \mathbb{V}_2,\, t \in \mathbb{R} \right\} \subset \mathrm{S}(\mathbb{V}, h) \subset \operatorname{End}(\mathbb{E}) ,$$

where $\mathrm{S}(\mathbb{V}, h) \subset \operatorname{End}(\mathbb{V})$ is the linear subspace of all h-selfadjoint operators on $\mathbb{V}$.

(iii) The restriction of h to $\mathbb{V}_2$ has type $(p-1, q-1)$ and $N(y) \in \mathrm{S}(\mathbb{V}_2, h)$ for every $y \in \mathbb{V}_2$.

Proof. (ii): From 5.1 and 9.7.(ii) follows that for a maximal abelian and ad-nilpotent subalgebra $\mathfrak{n} \subset \mathfrak{su}(\mathbb{E}, h)$ we have $\operatorname{Ann}(i\,\mathfrak{n}) = \{x \in \operatorname{Ann}(\mathfrak{n}^{\mathbb{C}}) = \operatorname{Hom}(\mathbb{E}_1, \mathbb{E}_3) : x = x^\star\}$. Since at the same time $\mathfrak{n}$ is contained in the (-1)-eigenspace of τ the first part of the lemma together with 9.7.(ii) imply $\operatorname{Ann}(i\,\mathfrak{n}) = \operatorname{Hom}(\mathbb{V}_1, \mathbb{V}_3) = \operatorname{Hom}(\mathbb{E}_1, \mathbb{E}_3)^\tau$. This is only possible if $\dim \mathbb{E}_1 = \dim \mathbb{E}_3 = 1$. □

The next proposition shows that the classification of maximal nilpotent subalgebras $\mathfrak{n} \subset \mathfrak{su}(\mathbb{E}, h)$, contained also in $\mathfrak{sl}(\mathbb{E})^{-\tau}$, reduces to the classification of abstract associative nilpotent subalgebras (of $\operatorname{End}(\mathbb{E})$) with 1-dimensional annihilator. Crucial for the following theorem is the construction of a non-degenerate 2-form $b = b_\pi$ depending on a suitable projection π, see 9.12. Keeping also in mind Proposition 9.10 and Lemma 9.14 we have:

5.3 Theorem. Let $\mathcal{N}$ be an arbitrary commutative associative and nilpotent $\mathbb{R}$-algebra with $\dim \operatorname{Ann}(\mathcal{N}) = 1$ and let $\mathbb{V} := \mathcal{N}^0$ be its unital extension. Fix an identification $\operatorname{Ann}(\mathcal{N}) = \mathbb{R}$ and a projection π on $\mathbb{V}$ with range $\operatorname{Ann}(\mathcal{N}) = \mathbb{R}$ satisfying $\pi(\mathbb{1}) = 0$. Then for the left regular representation L of $\mathbb{V} = \mathcal{N}^0$ and the symmetric real 2-form $b : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ we have:

(i) $L(\mathcal{N})$ is a maximal nilpotent and abelian subalgebra of $\operatorname{End}(\mathbb{V})$ contained in $\mathrm{S}(\mathbb{V}, b)$.

(ii) Let $\mathbb{E} := \mathbb{V} \oplus i\mathbb{V}$ be the complexification of $\mathbb{V}$ and τ the conjugation on $\mathbb{E}$ with $\mathbb{E}^\tau = \mathbb{V}$. Furthermore, denote the unique hermitian extension of b to $\mathbb{E} \times \mathbb{E}$ by the same symbol b. Then $\mathfrak{n} := iL(\mathcal{N})$ is a subset of $\mathfrak{su}(\mathbb{E}, b)^{-\tau}$ and is a maximal abelian and ad-nilpotent Lie subalgebra of $\mathfrak{su}(\mathbb{E}, b)$. Finally, $\exp \mathfrak{n}^{\mathbb{C}} \subset \mathrm{SL}(\mathbb{E})$ has an open orbit in $\mathbb{P}(\mathbb{E})$.

(iii) Every maximal abelian and ad-nilpotent subalgebra of $\mathfrak{su}(\mathbb{E}, h)$ which is also contained in $\mathfrak{sl}(\mathbb{E})^{-\tau}$ for τ as in 3.2 is equivalent to some $iL(\mathcal{N})$ as above.

Suppose that for $k = 1, 2$ there are given two abelian nilpotent associative $\mathbb{R}$-algebras $\mathcal{N}_k$ each with 1-dimensional annihilator $\mathcal{A}_k$ and assume that the corresponding 2-forms b_k on the corresponding unital extensions have types (p_k, q_k) with respect to the linear isomorphisms $\lambda_k : \mathcal{A}_k \cong \mathbb{R}$. Then for $\mathcal{I} := \{(x, y) \in \mathcal{A}_1 \oplus \mathcal{A}_2 : \lambda_1(x) = \lambda_2(y)\}$ the quotient algebra $\mathcal{N} := (\mathcal{N}_1 \oplus \mathcal{N}_2)/\mathcal{I}$ is an abelian nilpotent associative algebra with 1-dimensional annihilator $\mathcal{A} := (\mathcal{A}_1 \oplus \mathcal{A}_2)/\mathcal{I}$, and the 2-form b on $\mathcal{N}$ induced by $b_1 \times b_2$ on $\mathcal{N}_1 \times \mathcal{N}_2$ has type $(p_1 + p_2 - 1, q_1 + q_2 - 1)$.

5.4 MANSAs in $\mathfrak{su}(p, q)$ for low values of q. By our above considerations, to determine all MANSAs $\mathfrak{n} \subset \mathfrak{su}(p, q)$ up to $\mathrm{SU}(p, q)$-conjugacy it is equivalent to determine up to isomorphism all real nilpotent abelian associative algebras $\mathcal{N}$ with annihilator $\mathcal{A}$ such that for some linear isomorphism $\mathcal{A} \cong \mathbb{R}$ the form b has type (p,q) on $\mathcal{N}^0$. For low values of q this can be done:

q = 1: There is precisely 1 equivalence class of MANSAs in $\mathfrak{su}(p,1)^{-\tau}$ for every $p \geq 1$. Indeed, for every $\mathcal{N}$ with annihilator $\mathcal{A} \cong \mathbb{R}$ the factor algebra $\mathcal{N}/\mathcal{A}$ must be a zero product algebra.

q = 2: There are precisely $\min(p,3)$ equivalence classes of MANSAs in $\mathfrak{su}(p,2)^{-\tau}$ for every $p \geq 1$. Representing algebras $\mathcal{N}$ are obtained as follows: For every $n \geq 1$ with $n \leq \min(p,3)$ let $\mathcal{N}_1$ be the cyclic abelian algebra of dimension n, compare Example 9.40, and identify $t \in \mathbb{R}$ with $t\xi^n$ in the annihilator $\mathcal{A}_1$ of $\mathcal{N}_1$. Then the corresponding form b_1 has type $(1,1)$, $(2,1)$, $(2,2)$ for $n = 1,2,3$ respectively. Next choose an abelian nilpotent algebra $\mathcal{N}_2$ with 1-dimensional annihilator $\mathcal{A}_2$ such that the construction $\mathcal{N} := \mathcal{N}_1 \oplus \mathcal{N}_2/\mathcal{I}$ as above leads to a nilpotent algebra with 1-dimensional annihilator $\mathcal{A}$ such that the corresponding 2-form b on $\mathcal{N}^0$ has type $(p,2)$. This is always possible since $\mathcal{N}_2/\mathcal{A}_2$ must be a zero product algebra.

6. MANSAs in $\mathfrak{sl}(m,\mathbb{C})$

In this section we deal with the simple factors $\mathfrak{g}_\jmath \cong \mathfrak{sl}(m_\jmath,\mathbb{C})$ in (4.10). We retain our convention from the last section and drop the index $\jmath$ from our notation, that is, we consider abelian nilpotent subalgebras of $\mathrm{End}(\mathbb{E})$ that are contained in $\mathfrak{g} = \mathfrak{sl}(m,\mathbb{C})$, where the latter space is considered as a real Lie algebra. Recall that in this case the restriction of the involution τ to $\mathfrak{g}$ is given by the map $x \mapsto -x'$, where x' is the adjoint with respect to the complex bilinear non-degenerate symmetric 2-form β given by $\beta(v,w) := h(v,\tau w)$ for all $v,w \in \mathbb{E}$, compare (4.9). Note that the complexification $\mathfrak{g}^{\mathbb{C}}$ is isomorphic to the product $\mathfrak{sl}(m,\mathbb{C}) \times \mathfrak{sl}(m,\mathbb{C})$.

6.1 Proposition. Let $\mathfrak{g} = \mathcal{R}^{\mathbb{C}}_{\mathbb{R}}(\mathfrak{sl}(\mathbb{E}))$ and let $\mathfrak{n} \subset \mathfrak{g}$ be a Lie subalgebra. Then the following conditions (i) – (iii) are equivalent:

(i) $\mathfrak{n}$ is maximal among ad-nilpotent and abelian $\mathbb{R}$-subalgebras of $\mathfrak{sl}(\mathbb{E})$.

(ii) $\mathfrak{n}$ is maximal among all abelian $\mathbb{C}$-subalgebras of $\mathfrak{sl}(\mathbb{E})$ that are ad-nilpotent in $\mathfrak{sl}(\mathbb{E})$.

(iii) $\mathfrak{n}$ is maximal among abelian and nilpotent subalgebras of the associative complex algebra $\mathrm{End}(\mathbb{E})$.

If these conditions are satisfied, then $\mathfrak{n}^{\mathbb{C}} \oplus \mathbb{C}\cdot \mathrm{id}_{\mathbb{E}}$ is a maximal abelian subalgebra of the complex associative algebra $\mathrm{End}(\mathbb{E})$. Furthermore we have:

(iv) Let $\mathfrak{n}$ be maximal among ad-nilpotent and abelian $\mathbb{C}$-subalgebras of $\mathfrak{sl}(\mathbb{E})$ contained in $\mathfrak{sl}(\mathbb{E})^{-\tau}$. Then $\dim \mathrm{Ann}(\mathfrak{n}) = 1$, i.e., $1 = \dim \mathbb{E}_1 = \dim \mathbb{E}_3$ for any $\mathfrak{n}$-adapted decomposition of $\mathbb{E}$ (dimensions over $\mathbb{C}$).

(v) Identifying $\mathbb{C} = \mathbb{E}_1 = \mathbb{E}_3 = \mathrm{Ann}(\mathfrak{n})$, the matrix presentation in Proposition 9.7 reads

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ y & N(y) & 0 \\ t & J(y) & 0 \end{pmatrix} : y \in \mathbb{E}_2,\, t \in \mathbb{C} \right\} \subset \mathrm{S}(\mathbb{E},\beta) \subset \mathrm{End}(\mathbb{E}) ,$$

where $\mathrm{S}(\mathbb{E},\beta) \subset \mathrm{End}(\mathbb{E})$ is the linear subspace of all β-selfadjoint operators on $\mathbb{E}$.

Proof. (iv): Proposition 9.7 implies that $\mathrm{Ann}(\mathfrak{n}) = \mathrm{Hom}(\mathbb{E}_1,\mathbb{E}_3)$ for a given $\mathfrak{n}$-adapted decomposition of $\mathbb{E}$. On the other hand, $\mathfrak{n}$ is also contained in $\mathfrak{sl}(\mathbb{E})^{-\tau}$, i.e., $x = x'$ for all $x \in \mathfrak{sl}(\mathbb{E})$. Hence, we can choose an $\mathfrak{n}$-adapted decomposition $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3$ such that $\mathbb{E}_1$ and $\mathbb{E}_3$ are β-isotropic and $(\mathbb{E}_1 \oplus \mathbb{E}_3)$, $\mathbb{E}_2$ are β-orthogonal. In the matrix presentation of $\mathfrak{n}$ as in Proposition 9.7 the two above conditions imply $\mathrm{Ann}(\mathfrak{n}) = \mathrm{Hom}(\mathbb{E}_1,\mathbb{E}_3) = \{x \in \mathrm{Hom}(\mathbb{E}_1,\mathbb{E}_3) : x = x'\}$. This is only possible if $\dim E_1 = \dim \mathbb{E}_3 = 1 = \dim \mathrm{Ann}(\mathfrak{n})$. □

Similar to Theorem 5.3 we have

6.2 Theorem. Let $\mathcal{N}$ be an arbitrary associative $\mathbb{C}$-algebra which is commutative nilpotent and has annihilator $\mathrm{Ann}(\mathcal{N})$ of dimension 1. Let furthermore π be an arbitrary (complex linear) projection on its unital extension $\mathbb{E} := \mathcal{N}^0$ with range $\mathrm{Ann}(\mathcal{N})$ and $\pi(\mathbb{1}) = 0$, and fix an identification $\mathrm{Ann}(\mathcal{N}) = \mathbb{C}$. Then $L(\mathcal{N})$ is maximal in the class of all nilpotent and abelian subalgebras $A \subset \mathrm{End}(\mathbb{E})$ which are contained in $\mathrm{S}(\mathbb{E}, \mathsf{b})_{\mathrm{tr}=0} = \mathfrak{sl}(\mathbb{E})^{-\tau}$. Here $\mathsf{b} = \mathsf{b}_\pi$ is as in 9.12, x' is the adjoint with respect to the complex bilinear 2-form b and $\tau : \mathrm{End}(\mathbb{E}) \to \mathrm{End}(\mathbb{E})$ is given by $x \mapsto -x'$. On the other hand, every maximal abelian and ad-nilpotent subalgebra of $\mathfrak{sl}(\mathbb{E})$ which is also contained in $\mathfrak{sl}(\mathbb{E})^{-\tau}$ for τ as in 4.9 is equivalent to some $L(\mathcal{N})$ as above.

Note that the complex nilpotent group $N^{\mathbb{C}}$, corresponding to $\mathfrak{n}^{\mathbb{C}} \cong \mathfrak{n} \times \mathfrak{n} \subset \mathfrak{sl}(\mathbb{E}) \times \mathfrak{sl}(\mathbb{E}) \cong (\mathfrak{sl}(\mathbb{E}))^{\mathbb{C}}$ does not have an open orbit in $\mathbb{P}(\mathbb{E}\oplus\mathbb{E})$, but the subgroup corresponding to the following subalgebra does

$$(\mathfrak{n} \times \mathfrak{n}) \oplus \mathbb{C}(\mathrm{id}, -\mathrm{id}) = (\mathfrak{v}^{\mathrm{nil}})^{\mathbb{C}} \oplus (\mathfrak{v}^{\mathrm{red}})^{\mathbb{C}} \subset C_{\mathfrak{sl}(\mathbb{E}\oplus\mathbb{E})}(\mathfrak{v}^{\mathrm{red}}) \subset \mathfrak{sl}(\mathbb{E} \oplus \mathbb{E}) .$$

MANSAs in $\mathfrak{sl}(\mathfrak{m}, \mathbf{C})$ for low values of m. There exist exactly 1,1,1,2,3 equivalence classes of qualifying MANSAs in $\mathfrak{sl}(\mathfrak{m}, \mathbb{C})$ for $m = 1, 2, 3, 4, 5$ (see also the more detailed description at the end of the following Section 7). With our construction of nilpotent algebras out of cubic forms c in case $\mathbb{F} = \mathbb{C}$ (compare Proposition 9.36) it follows that there are infinitely many equivalence classes of qualifying MANSAs in $\mathfrak{sl}(\mathfrak{m}, \mathbb{C})$ for every $m \geq 8$.

7. Normal forms for equations

Every local tube realization $T_F = V + iF \subset E := V \oplus iV$ of $S_{p,q}$ is characterized by a qualifying MASA $\mathfrak{v} \subset \mathfrak{su}(p, q)$. In addition, the base F of the tube can always be chosen to be a closed (real-analytic) hypersurface in the real vector space V, see [10]. In the following we want to find canonical real-analytic real valued functions ψ on V with $d\psi \neq 0$ everywhere such that $F = \{x \in V : \psi(x) = \psi(0)\}^0$, where the upper index ^{0} means to take the connected component containing the origin. For this we consider the D-invariant $\boldsymbol{D}(\mathfrak{v})$ of $\mathfrak{v} = \mathfrak{v}^{\mathrm{red}} \oplus \mathfrak{v}^{\mathrm{nil}}$, see (4.14), and start with the special case that $\boldsymbol{D}(\mathfrak{v}) \in \boldsymbol{J} = \boldsymbol{K} \cup \boldsymbol{L}$. The general case with $\boldsymbol{D}(\mathfrak{v}) \in \mathcal{F}(\boldsymbol{J})$ arbitrary then is obtained by putting these special equations together.

1. Case $\boldsymbol{D}(\mathfrak{v}) \in \boldsymbol{K}$: Let $\boldsymbol{j} := (p, q)$ and $n := p + q - 1$. Furthermore let $\widehat{V}_{\boldsymbol{j}} := \mathbb{R}^{p+q}$ with coordinates $(x_0, x_1, \ldots, x_n)$ and define the linear form $\lambda_{\boldsymbol{j}}$ on $\widehat{V}_{\boldsymbol{j}}$ by $\lambda_{\boldsymbol{j}}(x) = (p + q)x_0$. Also, identify $V_{\boldsymbol{j}} := \mathbb{R}^n$ with coordinates $(x_1, \ldots, x_n)$ in the obvious way with the hyperplane $\{x \in \widehat{V}_{\boldsymbol{j}} : \lambda_{\boldsymbol{j}}(x) = 0\}$. We define $\Psi_{\boldsymbol{j}}$ as the set of all real-analytic functions $\psi(x) = e^{x_0} f(x_1, \ldots, x_n)$ on $\widehat{V}_{\boldsymbol{j}}$ where f is an extended real nil-polynomial on $V_{\boldsymbol{j}}$ and the second derivative of ψ at the origin of $\widehat{V}_{\boldsymbol{j}}$ has type (p, q), compare the Appendix for the notion of a nil-polynomial.

It is clear that for every real $t > 0$ and every $g \in \mathrm{GL}(V_{\boldsymbol{j}})$ with $e^{x_0} f(x)$ also the function $t\, e^{x_0} f(g(x))$ is contained in $\Psi_{\boldsymbol{j}}$. Furthermore, $\Psi_{\boldsymbol{j}^{\mathrm{opp}}} = -\Psi_{\boldsymbol{j}}$ is evident for the opposite $\boldsymbol{j}^{\mathrm{opp}} = (q, p)$. In particular, $\Psi_{(1,0)} = \{te^{x_0} : t > 0\}$, $\Psi_{(1,1)} = \{te^{x_0} x_1 : t \neq 0\}$ and $\Psi_{(2,1)}$ is the orbit of $\psi = e^{x_0}(x_2 + x_1^2)$ under the group $\mathbb{R}^* \times \mathrm{GL}(2, \mathbb{R})$.

2. Case $\boldsymbol{D}(\mathfrak{v}) \in \boldsymbol{L}$: Let $\boldsymbol{j} := \boldsymbol{m}$ for some integer $m \geq 1$ and put $n := m - 1$. Consider $\widehat{V}_{\boldsymbol{m}} := \mathbb{C}^m$ with complex coordinates $(z_0, z_1, \ldots, z_n)$ as real vector space and define the linear form $\lambda_{\boldsymbol{m}}$ on $\widehat{V}_{\boldsymbol{m}}$ by $\lambda_{\boldsymbol{m}}(z) := m(z_0 + \overline{z}_0)$. Furthermore, consider $W_{\boldsymbol{m}} := \mathbb{C}^n$ with coordinates $(z_1, \ldots, z_n)$ in the obvious way as linear subspace of $\widehat{V}_{\boldsymbol{m}}$. We define $\Psi_{\boldsymbol{m}}$

as the set of all real-analytic functions $\psi(z) = \mathrm{Re}\big(e^{z_0} f(z_1,\dots,z_n)\big)$ on $\widehat{V}_{\boldsymbol{j}}$ where f is an extended complex nil-polynomial on $W_{\boldsymbol{m}}$. Then the second derivative of every $\psi \in \Psi_{\boldsymbol{m}}$ at the origin has type (m,m).

The group $\mathbb{C}^* \times \mathrm{GL}(W_{\boldsymbol{m}})$ acts in a canonical way on the extended complex nil-polynomials on $W_{\boldsymbol{m}}$ and thus also on $\Psi_{\boldsymbol{m}}$. In particular, $\Psi_{\boldsymbol{m}}$ is the orbit of the functions $\mathrm{Re}(e^{z_0})$, $\mathrm{Re}(e^{z_0} z_1)$ and $\mathrm{Re}(e^{z_0}(z_2+z_1^2))$ for $m = 1,2,3$ respectively.

3. Case $\boldsymbol{D}(\mathfrak{v})$ arbitrary: Then $\boldsymbol{D} := \boldsymbol{D}(\mathfrak{v}) \in \mathcal{D}_{p,q}$ for integers $p,q \ge 1$ with $p+q \ge 3$, and there exists a unique sum representation $\boldsymbol{D} = \sum_{\alpha\in A} \boldsymbol{j}_\alpha$ with $(\boldsymbol{j}_\alpha)_{\alpha\in A}$ a finite family in $\boldsymbol{J} = \boldsymbol{K} \cup \boldsymbol{L}$. Put $\widehat{V}_{\boldsymbol{D}} := \bigoplus_{\alpha\in A} \widehat{V}_{\boldsymbol{j}_\alpha}$ and define the linear form $\lambda_{\boldsymbol{D}}$ on $\widehat{V}_{\boldsymbol{D}}$ by $(x_{\boldsymbol{j}_\alpha}) \mapsto \sum_\alpha \lambda_{\boldsymbol{j}_\alpha}(x_{\boldsymbol{j}_\alpha})$. Furthermore, let $\Psi_{\boldsymbol{D}}$ be the space of all functions

$$\psi:\ (x_{\boldsymbol{j}_\alpha})_\alpha \longmapsto \sum_\alpha \psi_\alpha(x_{\boldsymbol{j}_\alpha})$$

on $\widehat{V}_{\boldsymbol{D}}$, where $(\psi_\alpha)_{\alpha\in A}$ is an arbitrary family of functions $\psi_\alpha \in \Psi_{\boldsymbol{j}_\alpha}$. The second derivative of every $\psi \in \Psi_{\boldsymbol{D}}$ at the origin then has type (p,q).

The relevance of the vector spaces $\widehat{V}_{\boldsymbol{D}}$ with linear form $\lambda_{\boldsymbol{D}}$ and function space $\Psi_{\boldsymbol{D}}$ is the following: Consider in $\widehat{V}_{\boldsymbol{D}}$ the hyperplane $V := \{x \in \widehat{V}_{\boldsymbol{D}} : \lambda_{\boldsymbol{D}}(x) = 0\}$. Then for every $\psi \in \Psi_{\boldsymbol{D}}$ the analytic hypersurface

$$F := \big\{x \in V : \psi(x) = \psi(0)\big\}^0 \tag{7.1}$$

is the base of a local tube realization of $S_{p,q}$ with D-invariant $\boldsymbol{D}$, and every local tube realization of $S_{p,q}$ with D-invariant $\boldsymbol{D}$ occurs this way up to affine equivalence. Indeed, every local tube realization of $S_{p,q}$ is associated with a qualifying MASA $\mathfrak{v} \subset \mathfrak{su}(p,q)$. In particular, for $\mathbb{E} := \mathbb{C}^{p+q}$ the complexification $\mathfrak{v}^{\mathbb{C}} := \mathfrak{v} \oplus i\mathfrak{v} \subset \mathfrak{sl}(\mathbb{E})$ has an open orbit O in the projective space $\mathbb{P}(\mathbb{E})$ that is the image of the locally biholomorphic map $\varphi : \mathfrak{v}^{\mathbb{C}} \to \mathbb{P}(\mathbb{E}),\ \xi \mapsto \exp(\xi)a$, where a is a suitable point in $\mathbb{P}(\mathbb{E})$. Then every connected component M of $\varphi^{-1}(S_{p,q})$ is a closed tube submanifold of $\mathfrak{v}^{\mathbb{C}}$, lets take the one that contains the origin. Then the base $F := M \cap i\,\mathfrak{v}$ of the tube manifold M has (7.1) as defining equation if we put $V := i\,\mathfrak{v} \subset \mathfrak{sl}(\mathbb{E})$ and $\psi := \varphi_{|V}$. Now consider the extended space $\widehat{V} := \mathbb{R}{\cdot}\,\mathrm{id} \oplus i\mathfrak{v} \subset \mathfrak{gl}(\mathbb{E})$ and let tr be the trace functional on $\widehat{V}$. Also extend ψ to $\widehat{V}$ by $t{\cdot}\,\mathrm{id} \oplus x \mapsto e^t\psi(x)$ and denote the extension by the same symbol ψ.
Now the D-invariant $\boldsymbol{D}(\mathfrak{v})$ and the corresponding decomposition of $\mathfrak{v}^{\mathrm{red}}$ in Lemma 4.10 comes into play. We can identify $\widehat{V}$ with $\widehat{V}_{\boldsymbol{D}}$ and tr with $\lambda_{\boldsymbol{D}}$ in a canonical way. Since ψ is defined in terms of $\exp$ it is compatible with the decomposition in Lemma 4.10 and we only have to discuss defining equations for the case that there is only one summand in (4.7), that is, that $\boldsymbol{D}(\mathfrak{v}) \in \boldsymbol{J}$:

1. Case $\boldsymbol{D}(\mathfrak{v}) \in \boldsymbol{K}$: Let $\boldsymbol{D}(\mathfrak{v}) := (p,q)$. In case $pq = 0$ we have $V = 0$ and we have up to a positive factor $\psi(x) = (p-q)e^x$ on $\widehat{V} = \mathbb{R}$. We therefore assume $p,q \ge 1$ in the following. But then $\mathcal{N} := V = i\,\mathfrak{v}$ is an associative commutative nilpotent real subalgebra of $\mathrm{End}(\mathbb{E})$ with 1-dimensional annihilator $\mathcal{A}$, compare the Appendix. Choose a pointing ω on $\mathcal{N}$ such that the associated symmetric 2-form $h(x,y) = \omega(xy)$ has type (p,q) on $\mathcal{N}^0$. Then as base for a tube realization associated with $\mathfrak{v}$ we can take

$$F = \{x \in \mathcal{N} : f(x) = 0\} \quad \text{with} \quad f(x) := h(\exp x/2, \exp x/2) = \omega(\exp x)\,.$$

But f is an extended real nil-polynomial on $\mathcal{N}$ and F is a smooth algebraic hypersurface.

2. Case $\boldsymbol{D}(\mathfrak{v}) \in \boldsymbol{L}$: Let $\boldsymbol{D}(\mathfrak{v}) := \boldsymbol{m}$ and put $n := m-1$. Then $\mathfrak{v} = \mathbb{R}\cdot \mathrm{id} + \mathcal{N} \subset \mathfrak{gl}(m,\mathbb{C}) \subset \mathfrak{u}(m,m)$, where $\mathcal{N} \subset \mathfrak{sl}(m,\mathbb{C})$ is a complex MANSA and at the same time a commutative associative nilpotent complex subalgebra of $\mathrm{End}(\mathbb{C}^m)$. In case $m=1$ we have $\mathcal{N} = 0$ and $\psi(z) = \mathrm{Re}(e^z)$ on $\widehat{V} = \mathbb{C}$. Let us therefore assume $m > 1$ in the following. Then $\mathcal{N}$ has annihilator $\mathcal{A}$ of complex dimension 1. Let ω be a pointing on $\mathcal{N}$. Then the real symmetric 2-form $h(z,w) = \mathrm{Re}\,\omega(zw)$ has type (m,m) on the complex unital extension $\mathcal{N}^0 = \mathbb{C}\cdot \mathrm{id} + \mathcal{N} \subset \mathrm{End}(\mathbb{C}^m)$. We have to consider the complexification $\mathfrak{v}^{\mathbb{C}} = \mathfrak{v} \oplus i\mathfrak{v}$ and to restrict the exponential mapping to $i\mathfrak{v}$. For this, we may identify the $\mathbb{R}$-linear space $V := i\mathfrak{v}$ with $i\mathbb{R}\cdot \mathrm{id} + \mathcal{N} \subset \mathcal{N}^0$ and get as base for a tube realization associated to $\mathfrak{v}$ the hypersurface

$$F = \{w \in i\mathbb{R}\cdot \mathrm{id} + \mathcal{N} : h(\exp w/2, \exp w/2) = 0\}\,.$$

Writing $w = is\cdot \mathrm{id} + z$ with $s \in \mathbb{R}$, $z \in \mathcal{N}$ we have

$$h(\exp w/2, \exp w/2) = \mathrm{Re}(e^{is} f(z)) \quad \text{for} \quad f(z) := \omega(\exp z)\,,$$

that is, f is an extended complex nil-polynomial on $\mathcal{N} \cong \mathbb{C}^n$, and F is affinely equivalent to the non-algebraic hypersurface

$$\{(s,z) \in \mathbb{R} \oplus \mathbb{C}^n : \mathrm{Re}\big(e^{is} f(z_1, \ldots, z_n)\big) = 0\}^0\,. \qquad \square$$

7.2 Local tube realizations corresponding to Cartan subalgebras of $\mathfrak{su}(p,q)$: By the above we know $\Psi_{\boldsymbol{j}}$ for all $\boldsymbol{j} \in \{(1,0),(0,1),\mathbf{1}\}$ and thus we can explicitly write down the normal form equations of every CSA in $\mathfrak{su}(p,q)$: For fixed $p,q \geq 1$ and every $\ell \geq 0$ with $\ell \leq \min(p,q)$ consider the CSA ${}^{\ell}\mathfrak{h}$ as defined in Section 4. Then we have

$$\boldsymbol{D} := \boldsymbol{D}({}^{\ell}\mathfrak{h}) = (p-\ell)\cdot(1,0) + (q-\ell)\cdot(0,1) + \ell\cdot\mathbf{1}.$$

With $d := p+q-2\ell$ then

$$V_{\boldsymbol{D}} = \Big\{(z,t) \in \mathbb{C}^{\ell} \oplus \mathbb{R}^d : \sum_{k=1}^{\ell}(z_k + \overline{z}_k) + \sum_{k=1}^{d} t_k = 0\Big\}$$

and as tube base we can take a connected component of the set of all $(z,t) \in V_{\boldsymbol{D}}$ satisfying

$$\sum_{k=1}^{\ell} \mathrm{Re}(e^{z_k}) \;+\; \sum_{k=1}^{p-\ell} e^{t_k} \;=\; \sum_{k=1}^{q-\ell} e^{t_{p-\ell+k}}\,. \tag{7.3}$$

7.4 Comparing with the equations of Isaev-Mishchenko:

It is easy to write down explicitly $\Psi_{\boldsymbol{j}}$ for all $\boldsymbol{j} = (p,q)$ with $q \leq 2$, getting back the classifications in [5] and [11]. In the following we compare the equations obtained in [11] with ours, where $\boldsymbol{D}$ is the corresponding D-invariant and $n = p$:

types 1), 4), 5): $\boldsymbol{D} = s\cdot(1,0) + (n-s,2)$. The corresponding MANSA in $\mathfrak{su}(n-s,2)$ has nil-index 2,3,4 respectively.

type 2): $\boldsymbol{D} = s\cdot(1,0) + (0,1) + (n-s,1)$.

type 3): $\boldsymbol{D} = s\cdot(1,0) + (n-s-1,1) + \mathbf{1}$,

type 6): $\boldsymbol{D} = (n-2)\cdot(1,0) + \mathbf{2}$.

type 7): $\boldsymbol{D} = s\cdot(1,0) + (t,1) + (n-s-t,1)$.

types 8), 9), 10): These types correspond to the three Cartan subalgebras of $\mathfrak{su}(n,2)$ and are affinely equivalent to the equations (7.3) for $\ell = 0,2,1$ respectively.

8. Some Examples

We will give applications of Proposition 9.36 in the real as well as in the complex case. We start with the real version.

8.1 Examples obtained from real cubic forms Let W be a real vector space of dimension $2n$ and q a quadratic form of type (n,n) on W. Then there exists a decomposition $W = W' \oplus W''$ into totally isotropic linear subspaces. Let furthermore c be a cubic form on W' and define the function f on $V := W \oplus \mathrm{I\!R}$ by

$$(8.2) \qquad f(x,y,t) := t + \mathsf{q}(x+y) + \mathsf{c}(x) \quad \text{for all} \quad t \in \mathrm{I\!R}, x \in W', y \in W''\,.$$

Then f is an extended nil-polynomial on V and the hypersurface $F := \{v \in V : f(v) = 0\}$ in V is the base of a local tube realization for $S_{p,p}$ with $p = n+1$. On the other hand, F is affinely homogeneous and also the complement $V \backslash F$ is affinely homogeneous, compare the end of the Appendix. The complement $V \backslash F$ decomposes into two affinely homogeneous domains $D^{\pm}$, the tube domains over these domains are complex affinely homogeneous domains in $V^{\mathbb{C}} = V \oplus iV$. With (9.44) we see that there exists a real $\binom{n}{3}$-parameter family of cubic forms on W' leading to pairwise affinely inequivalent examples (notice that property $(*)$ of Proposition 9.43 is satisfied for all $\alpha(t_j)$ in (9.44) near c_0). In particular this shows that there are infinitely many affinely non-equivalent local tube realizations for $S_{4,4}$.

8.3 Examples obtained from complex cubic forms We start with a more general situation: Suppose that V is a complex vector space and $f : V \to \mathbb{C}$ is a holomorphic submersion with $f(0) = 0$. Then

$$F := \{z \in V : f(z) = 0\}^0$$

is a complex hypersurface in V, the complement $D := V \backslash F$ is a domain in V and

$$(8.4) \qquad F := \big\{(t,z) \in \mathrm{I\!R} \oplus V : \mathrm{Re}\big(e^{it} f(z)\big) = 0\big\}^0$$

is a real hypersurface in $\mathrm{I\!R} \oplus V$. With respect to the canonical projection $\mathrm{pr} : F \to V$, $(t,z) \mapsto z$, the surface F is a covering over the domain D and a trivial real line bundle over H. In fact, $\widetilde{F} := \mathrm{pr}^{-1}(F)$ is a connected real hypersurface in F, while the open subset $\widetilde{D} := \mathrm{pr}^{-1}(D)$ in F in general is not connected.

Now assume that $V = W \oplus \mathbb{C}$ and that f is an extended complex nil-polynomial of degree ≤ 3 on V as considered in Proposition 9.36. As a consequence of Proposition 9.22 the group $\mathsf{A} := \{g \in \mathrm{Aff}(V) : f \circ g = f\}$ acts transitively on every level set $f^{-1}(c)$ in V. Also, for every $s \in \mathbb{C}$ the linear transformation $\theta_s := e^s \,\mathrm{id}_{W'} \oplus e^{2s} \,\mathrm{id}_{W''} \oplus e^{3s} \,\mathrm{id}_{\mathbb{C}}$ satisfies $f \circ \theta_s = e^{3s} f$, see also (9.41). The group $\mathbb{C} \times \mathsf{A}$ acts by the affine transformations

$$(t,z) \mapsto \big(t - 3\,\mathrm{Im}(s)\,,\, \theta_s\, g(z)\big), \qquad s \in \mathbb{C}, g \in \mathsf{A},$$

on F and has precisely three orbits there – the closed orbit $\widetilde{F}$ and the two connected components $\widetilde{D}^{\pm}$ of the domain $\widetilde{D}$. Also, the translation $(t,z) \mapsto (t+\pi, z)$ interchanges these two domains $\widetilde{D}^{+}$, $\widetilde{D}^{-}$ in F.

Putting things together we got the following: Let W be a complex vector space of dimension $2n$ and q a non-degenerate quadratic form on W. Then there exists a decomposition $W = W' \oplus W''$ into totally isotropic linear subspaces. Let furthermore c be a cubic form on W' and define the function f on $V := W \oplus \mathbb{C}$ by (8.2) with $\mathrm{I\!R}$ replaced by $\mathbb{C}$. Then f is an extended complex nil-polynomial and F as defined in (8.4) is the base of a local tube realization $M \subset U^{\mathbb{C}} = U \oplus iU$ for $S_{p,p}$, where $U := \mathrm{I\!R} \oplus V$ and $p = 2(n+1)$. Furthermore, the real hypersurface M in $U^{\mathbb{C}}$ contains an affinely homogeneous domain. For every $n \geq 3$ we get a complex $\binom{n}{3}$-parameter family of pairwise affinely inequivalent examples.

9. Appendix – Nilpotent commutative algebras

In this Appendix, all occurring algebras are either associative or Lie. Throughout, $\mathbb{F}$ is an arbitrary base field of characteristic zero. For every associative algebra A, every $x \in A$ and every integer $k \geq 1$ we put

$$(9.1) \qquad x^{(k)} := \frac{1}{k!}x^k \quad \text{and} \quad x^{(0)} := \mathbb{1} \text{ if } A \text{ has a unit } \mathbb{1}\,.$$

We collect several purely algebraic statements that are used in the paper and might be of independent interest. Some of them are probably known to the experts. Since we could not find a reference in the literature we state it here. Recall e.g. our convention that $\mathrm{End}(\mathbb{E})$ is the associative endomorphism algebra while $\mathfrak{gl}(\mathbb{E})$ is the same space, but endowed with the corresponding Lie product. We start with a standard definition.

9.2 Definition. Let $\mathcal{N}$ be an commutative associative algebra over $\mathbb{F}$ and define the ideals $\mathcal{N}^k \subset \mathcal{N}$ inductively by $\mathcal{N}^1 = \mathcal{N}$ and $\mathcal{N}^{k+1} = \langle \mathcal{N}\mathcal{N}^k\rangle$. Then $\mathcal{N}$ is called nilpotent if $\mathcal{N}^{k+1} = 0$ for some $k \geq 0$, and the minimal k with this property is called the nil-index of $\mathcal{N}$. Furthermore, $\mathcal{A} := \mathrm{Ann}(\mathcal{N}) := \{x \in \mathcal{N} : x\mathcal{N} = 0\}$ is called the annihilator of $\mathcal{N}$.

The general embedded case

In the following, let $\mathbb{E}$ be a vector space of finite dimension $m \geq 2$ over $\mathbb{F}$. For every subalgebra $\mathcal{N} \subset \mathrm{End}(\mathbb{E})$ define the following characteristic subspaces of $\mathbb{E}$:

$$(9.3) \qquad \mathbb{B} := \mathbb{B}_{\mathcal{N}} := \langle \mathcal{N}(v) : v \in \mathbb{E}\rangle \quad \text{and} \quad \mathbb{K} := \mathbb{K}_{\mathcal{N}} := \{v \in \mathbb{E} : \mathcal{N}(v) = 0\}\,.$$

9.4 Proposition. Suppose that $\mathcal{N}$ is maximal among all commutative and nilpotent subalgebras of $\mathrm{End}(\mathbb{E})$. Then

(i) $0 \neq \mathbb{K}_{\mathcal{N}} \subset \mathbb{B}_{\mathcal{N}} \neq \mathbb{E}$. Also, $\mathbb{K}_{\mathcal{N}} = \mathbb{B}_{\mathcal{N}}$ holds if and only if $\mathcal{A} = \mathrm{Ann}(\mathcal{N})$ has nil-index 1.

(ii) $\mathcal{N} \oplus \mathbb{F}\cdot \mathrm{id}$ is maximal among all commutative subalgebras of $\mathrm{End}(\mathbb{E})$.

(iii) $\mathcal{N}$ is irreducible on $\mathbb{E}$, i.e., for every $\mathcal{N}$-invariant decomposition $\mathbb{E} = \mathbb{E}' \oplus \mathbb{E}''$ either $\mathbb{E}' = 0$ or $\mathbb{E}'' = 0$.

9.5 $\mathcal{N}$-adapted decompositions and matrix presentations. For every $\mathcal{N}$ satisfying the assumptions in Proposition 9.4 we select subspaces $\mathbb{E}_1, \mathbb{E}_2$ of $\mathbb{E}$ such that $\mathbb{E}_1 \oplus \mathbb{B}_{\mathcal{N}} = \mathbb{E}$ and $\mathbb{E}_2 \oplus \mathbb{K}_{\mathcal{N}} = \mathbb{B}_{\mathcal{N}}$. Then, for $\mathbb{E}_3 := \mathbb{K}_{\mathcal{N}}$, we have the decomposition

$$(9.6) \qquad \mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3 \quad \text{with} \quad d_j := \dim \mathbb{E}_j \ \text{ for } \ j = 1, 2, 3\,,$$

that we also call an $\mathcal{N}$-adapted decomposition. Every $x \in \mathrm{End}(\mathbb{E})$ can be written as 3×3-matrix (x_{jk}) with $x_{jk} \in \mathrm{Hom}(\mathbb{E}_k, \mathbb{E}_j)$. With $\pi_{jk} : \mathrm{End}(\mathbb{E}) \to \mathrm{Hom}(\mathbb{E}_k, \mathbb{E}_j)$ we denote the projection $x \mapsto x_{jk}$.

We call two subalgebras $\mathcal{N} \subset \mathrm{End}(\mathbb{E})$ and $\mathcal{N}' \subset \mathrm{End}(\mathbb{E}')$ conjugate if there exists an invertible $\Psi \in \mathrm{Hom}(\mathbb{E}, \mathbb{E}')$ such that $\mathcal{N}' = \Psi \circ \mathcal{N} \circ \Psi^{-1}$. One of our goals is to decide under which conditions two isomorphic subalgebras (isomorphic as abstract $\mathbb{F}$-algebras) are already conjugate. In general, there exist isomorphic subalgebras which are not conjugate.

It is obvious that a nilpotent subalgebra $\mathcal{N} \subset \mathrm{End}(\mathbb{E})$ contains only nilpotent endomorphisms. By a theorem of Engel the converse is also true: A subalgebra $\mathcal{N} \subset \mathrm{End}(\mathbb{E})$ consisting of nilpotent endomorphisms only is nilpotent and there is a full flag

$$F_1 \subset F_2 \subset \cdots \subset F_m = \mathbb{E}, \qquad \dim F_k = k,$$

which is stable under $\mathcal{N}$, i.e., with respect to a suitable basis of $\mathbb{E}$ the algebra $\mathcal{N}$ consists of strictly lower-triangular matrices in $\mathbb{F}^{m\times m}$. With this notation we can state

9.7 Proposition. Suppose that $\mathcal{N} \subset \mathrm{End}(\mathbb{E})$ satisfies the assumptions of 9.4 and has nil-index ν. For a fixed $\mathcal{N}$-adapted decomposition $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3$ and all $1 \leq j, k \leq 3$ put $\mathcal{N}_{jk} := \pi_{jk}(\mathcal{N})$. Then:

(i) There exists a linear bijection $J : \mathcal{N}_{21} \to \mathcal{N}_{32}$ and a linear map $N : \mathcal{N}_{21} \to \mathcal{N}_{22}$ such that

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ y & N(y) & 0 \\ t & J(y) & 0 \end{pmatrix} : y \in \mathcal{N}_{21},\, t \in \mathrm{Hom}(\mathbb{E}_1, \mathbb{E}_3) \right\} .$$

(ii) $\mathcal{A} := \{x \in \mathcal{N} : x_{21} = 0\} \cong \mathrm{Hom}(\mathbb{E}_1, \mathbb{E}_3)$ is the annihilator of $\mathcal{N}$.

(iii) $\mathcal{N}_{21} \times \mathcal{N}_{21} \to \mathcal{N}_{31},\ (x, y) \mapsto J(x) \circ y$, is a non-degenerate symmetric 2-form (in fact, is equivalent to the restriction of the form b defined in (9.12) after the obvious identifications).

(iv) $\mathcal{N}_{22}$ is a nilpotent commutative subalgebra of $\mathrm{End}(\mathbb{E}_2)$ with nil-index $\leq \max(\nu - 2, 1)$.

(v) $\mathbb{E}_2 = \langle \mathcal{N}_{23}(\mathbb{E}_1) \rangle$ and $\{z \in \mathbb{E}_2 : y(z) = 0 \text{ for all } y \in \mathcal{N}_{32}\} = 0$.

(vi) $d_1 d_3 + \lceil d_2/\mu \rceil \leq \dim \mathcal{N} \leq [m^2/4]$ for $m := d_1 + d_2 + d_3 = \dim \mathbb{E}$ and $\mu := \min(d_1, d_3)$. In particular, if $d_1 = 1$ then $\dim \mathcal{N} = d_2 + d_3 = m - 1$.

9.8 Remark. The linear maps N and J in (i) above satisfy for all $x, y \in \mathbb{E}$ the relations:

(a) $N(x)^k = 0$ for some integer k,

(b) $N(x)y = N(y)x, \quad J(x)y = J(y)x$ and $J(x)N(y) = J(y)N(x)$,

(c) $N(N(x)y) = N(x)N(y)$.

On the other hand, let a vector space $\mathbb{W}$ over $\mathbb{F}$ be given. Every pair $N : \mathbb{W} \to \mathrm{End}(\mathbb{W})$, $J : \mathbb{W} \overset{\approx}{\to} \mathbb{W}^*$ of linear maps satisfying (a) - (c) gives rise by (i) above to a commutative maximal nilpotent subalgebra $\mathcal{N} \subset \mathrm{End}(\mathbb{E})$ with $\mathbb{E} = \mathbb{F} \oplus \mathbb{W} \oplus \mathbb{F}$, i.e., $\mathbb{E}_1 = \mathbb{F} = \mathbb{E}_3$, $\mathbb{E}_2 = \mathbb{W}$ and $\mathcal{N}$ has 1-dimensional annihilator. In particular, $N \equiv 0$ and J given by any symmetric and non-degenerate scalar product on $\mathbb{W}$ trivially satisfy (a) - (c) and define a maximal nilpotent subalgebra $\mathcal{N}$ with $\dim \mathcal{N} = \dim \mathbb{E} - 1$ and nil-index 2. In this case $\mathcal{N}/\mathcal{A}$ is the zero product algebra.

9.9 Remark. The upper bound in inequality (vi) is sharp as the nilpotent subalgebra

$$\{x \in \mathrm{End}(\mathbb{E}) : x_{jk} = 0 \text{ if } (j, k) \neq (3, 1)\}$$

for $d_2 = d_3 = \lceil m/2 \rceil$ shows. It is much harder to find better lower bounds for $\dim \mathcal{N}$, not to speak of sharp ones. For infinitely many values of m (starting with $m = 14$) there exist maximal commutative and nilpotent subalgebras of $\mathrm{End}(\mathbb{F}^m)$ with $\dim \mathcal{N} < m - 1$, see [18].

Abstract commutative nilpotent algebras

The left-regular representation of a nilpotent (associative) algebra is not faithful, contrary to the case of any unital algebra. For every nilpotent algebra $\mathcal{M}$ denote by $\mathcal{M}^0 := \mathbb{F}\cdot\mathbb{1} \oplus \mathcal{M}$ its unital extension. Such extensions of nilpotent algebras are precisely those $\mathbb{F}$-algebras, which contain a maximal ideal of codimension 1 consisting of nilpotent elements only. Denote by $L : \mathcal{M}^0 \to \mathrm{End}(\mathcal{M}^0)$ the corresponding left regular representation. It is obvious that the algebras $\mathcal{M}$ and $L(\mathcal{M}) \subset \mathrm{End}(\mathcal{M}^0)$ are isomorphic via L.

9.10 Proposition. Let $\mathcal{M}$ be a commutative nilpotent $\mathbb{F}$-algebra of finite dimension. Then

(i) $L(\mathcal{M})$ is maximal among all commutative nilpotent subalgebras of $\mathrm{End}(\mathcal{M}^0)$ and consists entirely of nilpotent endomorphisms. Furthermore, $\mathbb{K}_{L(\mathcal{M})} = \mathrm{Ann}(\mathcal{M})$ and $\mathbb{B}_{L(\mathcal{M})} = \mathcal{M}$. The image of the unital extension, $L(\mathcal{M}^0)$ is maximal among all commutative subalgebras of $\mathrm{End}(\mathcal{M}^0)$.

(ii) A maximal commutative nilpotent subalgebra $\mathcal{N} \subset \mathrm{End}(\mathbb{E})$ is conjugate to the image $L(\mathcal{M})$ of some commutative nilpotent algebra $\mathcal{M}$ as above if and only if $\mathrm{codim}_{\mathbb{E}} \mathbb{B}_{\mathcal{N}} = 1$, see (9.3) for the notation.

(iii) Let $\mathcal{N} \subset \mathrm{End}(\mathbb{E})$ and $\mathcal{M} \subset \mathrm{End}(\mathbb{F})$ be two maximal commutative nilpotent subalgebras. If $\mathrm{codim}_{\mathbb{E}} \mathbb{B}_{\mathcal{N}} = \mathrm{codim}_{\mathbb{F}} \mathbb{B}_{\mathcal{M}} = 1$ then $\mathcal{M}$ and $\mathcal{N}$ are conjugate (in the above defined sense) if and only if $\mathcal{M}$ and $\mathcal{N}$ are isomorphic as abstract algebras. There exist non-conjugate subalgebras $\mathcal{N}, \mathcal{M} \subset \mathrm{End}(\mathbb{E})$ with $\mathrm{codim}_{\mathbb{E}} \mathbb{B}_{\mathcal{N}} > 1$, which are isomorphic as abstract algebras.

9.11 Associated 2-forms. Let $\mathcal{N}$ be an commutative and nilpotent $\mathbb{F}$-algebra and $\mathcal{A} := \mathrm{Ann}(\mathcal{N})$ its annihilator. On the unital extension $\mathcal{N}^0 = \mathbb{F}{\cdot}\mathbb{1} \oplus \mathcal{N}$ fix a projection $\pi = \pi^2 \in \mathrm{End}(\mathcal{N}^0)$ with range $\pi(\mathcal{N}) = \mathcal{A}$ and $\pi(\mathbb{1}) = 0$. Then

$$b_\pi : \mathcal{N}^0 \times \mathcal{N}^0 \to \mathcal{A}, \qquad (x, y) \longmapsto \pi(xy) \tag{9.12}$$

defines an $\mathcal{A}$-valued symmetric 2-form. Clearly, the restriction of b_π to $\mathcal{N}$ factorizes through $\mathcal{N}/\mathcal{A} \times \mathcal{N}/\mathcal{A}$ and we write also b_π for the corresponding $\mathcal{A}$-valued 2-form on $\mathcal{N}/\mathcal{A}$. Furthermore, π determines the decomposition

$$\begin{aligned} &\mathcal{N}^0 = \mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \mathcal{N}_3 \quad \text{with} \\ &\mathcal{N}_1 = \mathbb{F}{\cdot}\mathbb{1}, \qquad \mathcal{N}_2 = \mathcal{N} \cap \ker \pi \cong \mathcal{N}/\mathcal{A} \quad \text{and} \quad \mathcal{N}_3 = \mathcal{A} = \mathrm{Ann}(\mathcal{N})\ . \end{aligned} \tag{9.13}$$

The canonical isomorphism $\cong$ in (9.13) makes $\mathcal{N}_2$ to an algebra that we denote by $\mathcal{N}_2^\pi$. In terms of π and the algebra structure on $\mathcal{N}$ the product on $\mathcal{N}_2^\pi$ is given by $(x, y) \mapsto (\mathrm{id} - \pi)(xy)$ for $x, y \in \mathcal{N}_2^\pi$.

9.14 Lemma. Let $\mathcal{N} \neq 0$ be a commutative nilpotent associative $\mathbb{F}$-algebra with annihilator $\mathcal{A} := \mathrm{Ann}(\mathcal{N})$ and let $L : \mathcal{N}^0 \hookrightarrow \mathrm{End}(\mathcal{N}^0)$ be the left regular representation of the unital extension $\mathcal{N}^0$. Fix a projection π on $\mathcal{N}^0$ with range $\mathcal{A}$ as above and let b_π be the associated 2-form (9.12). Then

(i) The decomposition (9.13) is an $L(\mathcal{N})$-adapted decomposition of $\mathcal{N}^0$.

(ii) b_π is nondegenerate.

(iii) b_π is associative, or equivalently, every $L(y) \in \mathrm{End}(\mathcal{N}^0)$ is b_π-selfadjoint.

(iv) The subspaces $\mathcal{N}_1$ and $\mathcal{N}_3$ are b_π-isotropic while the subspaces $(\mathcal{N}_1 \oplus \mathcal{N}_3)$ and $\mathcal{N}_2$ are b_π-orthogonal to each other. Consequently, the restriction of b_π to the algebra $\mathcal{N}_2^\pi \cong \mathcal{N}/\mathcal{A}$ is a non-degenerate associative $\mathcal{A}$-valued 2-form.

(v) In the particular case $\mathbb{F} = \mathbb{R}$ and $\dim \mathcal{A} = 1$ the following holds after fixing a linear isomorphism $\psi : \mathcal{A} \to \mathbb{R}$: The type (p, q) of the real 2-form $\psi \circ b_\pi$ on $\mathcal{N}^0$ does not depend on the choice of the projection π.

Commutative nilpotent algebras consisting of selfadjoint endomorphisms

As indicated in the main part of this paper the classification of the various local tube realizations of $S_{p,q}$ is equivalent to the classification of maximal abelian subalgebras $\mathfrak{v}$ in $\mathfrak{su}(p, q) = \mathfrak{su}(\mathbb{E}, h)$, contained in the (-1)-eigenspace of the involution τ of the Lie algebra $\mathfrak{su}(p, q)$, up to conjugation by elements from $G := N_{\mathrm{SL}(\mathbb{E})}(\mathfrak{su}(p, q))$, compare 3.4.

Every such abelian subalgebra $\mathfrak{v}$ admits a unique decomposition into its ad-reductive and its ad-nilpotent part, $\mathfrak{v} = \mathfrak{v}^{\mathrm{red}} \oplus \mathfrak{v}^{\mathrm{nil}}$. Also, we have the further decomposition $\mathfrak{v}^{\mathrm{nil}} = \bigoplus_j \mathfrak{n}_j$, where the building blocks $\mathfrak{n}_j$ are various ad-nilpotent abelian subalgebras $\mathfrak{n}_j$ maximal in $\mathfrak{su}(p_j, q_j)$ or $\mathfrak{sl}(m_j, \mathbb{C})$. In turn, such Lie algebras (also contained in $\mathrm{Fix}(-\tau)$) are in a 1-1-correspondence to associative commutative and nilpotent subalgebras of $\mathrm{End}(\mathbb{V}_j)$, that consist of selfadjoint endomorphisms with respect to a symmetric non-degenerate 2-form h on $\mathbb{V}$ over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. This is one motivation to investigate a symmetric version of maximal commutative and nilpotent subalgebras in $\mathrm{End}(\mathbb{E})$.

In this subsection let $\mathbb{E}$ be an arbitrary vector space of finite dimension over $\mathbb{F}$ and $h : \mathbb{E}\times\mathbb{E} \to \mathbb{F}$ a symmetric non-degenerate 2-form. With $\mathrm{S}(\mathbb{E}, h) \subset \mathrm{End}(\mathbb{E})$ we denote the linear subspace of all operators a that are selfadjoint with respect to h (that is, $h(ax, y) = h(x, ay)$ for all $x, y \in \mathbb{E}$). Given $\mathbb{V} \subset \mathbb{E}$ we write $\mathbb{V}^{\perp}$ for the orthogonal complement with respect to h. Note that in general $\mathbb{V} \cap \mathbb{V}^{\perp} \neq 0$. For (maximal) nilpotent commutative subalgebras in $\mathrm{End}(\mathbb{E})$, contained in $\mathrm{S}(\mathbb{E}, h)$, there are $\mathcal{N}$-adapted decompositions which are in addition related to the 2-form h:

9.15 Lemma. Let $h : \mathbb{E}\times\mathbb{E} \to \mathbb{F}$ be a non-degenerate symmetric 2-form and $\mathcal{N} \subset \mathrm{End}(\mathbb{E})$ a maximal nilpotent commutative subalgebra, contained in $\mathrm{S}(\mathbb{E}, h)$. Let $\mathbb{K}_{\mathcal{N}}$, $\mathbb{B}_{\mathcal{N}}$ be the characteristic subspaces as defined in 9.3. Then:

(i) $\mathbb{K}_{\mathcal{N}} = \mathbb{B}_{\mathcal{N}}^{\perp}$.

(ii) $\dim \mathrm{Ann}(\mathcal{N}) = 1$

(iii) There exists an $\mathcal{N}$-adapted decomposition $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3$ such that

- (a) $\dim \mathbb{E}_1 = \dim \mathbb{E}_3 = 1$.
- (b) $h\big|_{\mathbb{E}_3} = 0$, $h\big|_{\mathbb{E}_1} = 0$, and the pairing $h : \mathbb{E}_1 \times \mathbb{E}_3 \to \mathbb{F}$ as well as the restriction $h\big|_{\mathbb{E}_2}$ are non-degenerate.
- (c) $\mathbb{E}_2 = (\mathbb{E}_1 \oplus \mathbb{E}_3)^{\perp}$.

9.16 Definition. Let $\mathcal{N} \subset \mathrm{End}(\mathbb{E})$ be as in the previous Lemma. We call every decomposition $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3$ which satisfies the condition (ii) in Lemma 9.15 an $(\mathcal{N}, h)$-*adapted decomposition of* $\mathbb{E}$.

In the following let $\mathcal{N} \subset \mathrm{S}(\mathbb{E}, h)$ be a maximal nilpotent commutative subalgebra of $\mathrm{End}(\mathbb{E})$ and $\mathcal{A} := \mathrm{Ann}(\mathcal{N})$ its annihilator (which is 1-dimensional according to the above lemma). Next, we relate the 2-form $b_{\pi} : \mathcal{N}^0 \times \mathcal{N}^0 \to \mathcal{A} := \mathrm{Ann}(\mathcal{N})$ to the symmetric 2-form $h : \mathbb{E} \times \mathbb{E} \to \mathbb{F}$. Recall that the choice of the projection $\pi : \mathcal{N} \to \mathcal{A}$ is equivalent to the choice of a linear subspace $\mathcal{N}_2 \subset \mathcal{N}$ with $\mathcal{N} = \mathcal{N}_2 \oplus \mathcal{A}$. It is easy to see that every $(\mathcal{N}, h)$-adapted decomposition of $\mathbb{E}$ gives rise to the complementary subspace $\mathcal{N}_2 := \{n \in \mathcal{N} : n(\mathbb{E}_1) \subset \mathbb{E}_2\}$, i.e., $\mathcal{N}_2 \oplus \mathcal{A} = \mathcal{N}$. It turns out to be more subtle to prove the opposite statement as it involves the solution of certain quadratic equations in $\mathbb{E}$.

9.17 Proposition. Let $\mathcal{N} \subset \mathrm{S}(\mathbb{E}, h)$ be a maximal nilpotent commutative subalgebra. Then:

(i) For every linear subspace $\mathcal{N}_2 \subset \mathcal{N}$ satisfying $\mathcal{N} = \mathcal{N}_2 \oplus \mathcal{A}$, there exists an $(\mathcal{N}, h)$-adapted decomposition $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3$ with

$$\mathcal{N}_2 = \{n \in \mathcal{N} : n(\mathbb{E}_1) \subset \mathbb{E}_2\}\,.$$

(ii) For $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3$ as in (i) choose generators $e_1 \in \mathbb{E}_1$, $e_3 \in \mathbb{E}_3$ with $h(e_1, e_3) = 1$ and define $\kappa : \mathcal{A} \to \mathbb{F}$ by $n(e_1) = \kappa(n)e_3$ for all $n \in \mathcal{A}$. Let $\pi : \mathcal{N}^0 \to \mathcal{A}$ the projection corresponding with kernel $\mathcal{N}_1 \oplus \mathcal{N}$ and $\kappa \circ b_{\pi} : \mathcal{N}^0 \times \mathcal{N}^0 \to \mathbb{F}$ the corresponding nondegenerate symmetric 2-form. Then the map

$$\mathcal{N}^0 \to \mathbb{E}, \quad m \mapsto m(e_1)$$

is an isometry between $(\mathcal{N}^0, \kappa \circ b_{\pi})$ and $(\mathbb{E}, h)$ which respects the decompositions $\mathcal{N}^0 = \mathbb{F}\cdot\mathbb{1} \oplus \mathcal{N}_2 \oplus \mathcal{A}$ and $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3$

Proposition 9.17 is the main ingredient in the proof of the following theorem, which can be considered as a symmetric version of Proposition 9.10:

9.18 Theorem. Let h and h' be two non-degenerate symmetric forms on $\mathbb{E}$ and $\mathbb{E}'$ respectively over $\mathbb{F}$, and let $\mathcal{N} \subset \mathrm{End}(\mathbb{E})$, $\mathcal{N}' \subset \mathrm{End}(\mathbb{E}')$ be two maximal nilpotent and commutative subalgebras.

(i) Assume, that in addition $\mathcal{N} \subset \mathrm{S}(\mathbb{E}, h)$ and $\mathcal{N}' \subset \mathrm{S}(\mathbb{E}', h')$ holds. Then $\mathcal{N}$ and $\mathcal{N}'$ are isomorphic as $\mathbb{F}$-algebras if and only if there exists an isometry $\Psi : (\mathbb{E}, h) \to (\mathbb{E}', h')$ with $\mathcal{N}' = \Psi \circ \mathcal{N} \circ \Psi^{-1}$.

(ii) In particular, if $\mathbb{E} = \mathbb{E}'$ and $h = h'$, two such subalgebras $\mathcal{N}, \mathcal{N}'$ contained in $\mathrm{S}(\mathbb{E}, h)$ are isomorphic if and only if they are conjugate by an element in $\mathrm{SO}(\mathbb{E}, h)$.

In case $\mathbb{F} = \mathbb{R}, \mathbb{C}$ the above theorem has the following application for the classification of maximal abelian subalgebras of $\mathfrak{su}(p,q)$ and $\mathfrak{sl}(m,\mathbb{C})$. Let $\mathfrak{su}(p,q) \cong \mathfrak{su}(\mathbb{E}, h)$ and $\tau : \mathfrak{su}(p,q) \to \mathfrak{su}(p,q)$ be as in 3.4, induced by a conjugation $\tau : \mathbb{E} \to \mathbb{E}$. Recall that we write $\mathbb{V} = \mathbb{E}^\tau$ for the real points with respect to τ. Note that $\mathfrak{su}(p,q)^\tau \cong \mathfrak{so}(p,q)$ and $\mathrm{SU}(p,q)^\tau \cong \mathrm{SO}(p,q) \cong \mathrm{SO}(\mathbb{V}, h|_{\mathbb{V}})$. Further $\mathfrak{sl}(m,\mathbb{C})^\tau \cong \mathfrak{so}(m,\mathbb{C})$, i.e., $\tau : \mathfrak{sl}(m,\mathbb{C}) \to \mathfrak{sl}(m,\mathbb{C})$ is induced by a symmetric non-degenerate 2-form h_τ on $\mathbb{E}$.

9.19 Corollary. Two maximal abelian Lie subalgebras $\mathfrak{v}_1$, $\mathfrak{v}_2$ in $\mathfrak{su}(\mathbb{E}, h)$ respectively $\mathfrak{sl}(\mathbb{E})$ consisting of nilpotent elements and contained in $\mathfrak{su}(\mathbb{E}, h)^{-\tau} \cong i\mathrm{S}(\mathbb{V}, h)_0$ respectively $\mathfrak{sl}(\mathbb{E})^{-\tau} \cong \mathrm{S}(\mathbb{E}, h_\tau)_0$ are conjugate under $\mathrm{SO}(\mathbb{V}, h)$ respectively under $\mathrm{SO}(\mathbb{E}, h_\tau)$ if and only if the corresponding associative algebras $i\mathfrak{v}_1, i\mathfrak{v}_2$ in $\mathrm{End}(\mathbb{V})$ (respectively $\mathfrak{v}_1$ and $\mathfrak{v}_2$ in $\mathrm{End}(\mathbb{E})$) are isomorphic as $\mathbb{F}$-algebras.

Some affinely homogeneous surfaces

Let $\mathcal{N} \neq 0$ be a commutative associative nilpotent algebra over $\mathbb{F}$ of finite dimension with nil-index ν. For every $k \geq 1$ choose a linear subspace $V_k \subset \mathcal{N}^k$ with $\mathcal{N}^k = V_k \oplus \mathcal{N}^{k+1}$. Then $\mathcal{N} = \bigoplus_{k\geq 1} V_k$ with $V_\nu = \mathcal{N}^\nu$, and we write every $x \in \mathcal{N}$ in the form $x = \sum_{k=1}^{\nu} x_k$ with $x_k \in V_k$. With $\pi : \mathcal{N} \to \mathcal{N}^\nu$ we denote the canonical projection $x \mapsto x_\nu$. As before we denote by $\mathcal{N}^0 = \mathbb{F}\cdot\mathbb{1} \oplus \mathcal{N}$ the unital extension of $\mathcal{N}$ and extend π linearly to $\mathcal{N}^0$ by requiring $\pi(\mathbb{1}) = 0$.

Denote by P the space of all polynomial maps $p : \mathcal{N} \to \mathcal{N}$ of the form

$$x \longmapsto \sum p_{i_1 i_2 \dots i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \,, \tag{9.20}$$

where the integers $r \geq 1$ and $1 \leq i_1 \leq \ldots \leq i_r$ satisfy $\sum_{j=1}^{r} i_j \leq \nu$ and the coefficients $p_{i_1 i_2 \dots i_r}$ are from $\mathcal{N}^0$. It is clear that with respect to composition P is a unital algebra over $\mathcal{N}^0$. We are mainly interested in polynomials $p \in P$ that are invertible in P, that is, where for every $1 \leq j \leq \nu$ the coefficient p_j in front of the linear monomial x_j is invertible in $\mathcal{N}^0$.

For every $p \in P$ the composition $f := \pi \circ p$ is a polynomial map $\mathcal{N} \to \mathcal{N}^\nu$ of degree $\leq \nu$. Furthermore, in case p is invertible, the algebraic subvariety

$$F := \{x \in \mathcal{N} : f(x) = 0\} \tag{9.21}$$

is smooth, in fact, is the graph of a polynomial map $\ker(\pi) \to \mathcal{N}^\nu$. Denote by $\mathrm{Aff}(\mathcal{N})$ the group of all affine automorphisms of $\mathcal{N}$.

9.22 Proposition. Suppose that $\mathcal{N}$ has nil-index $\nu \leq 4$ and that $p \in P$ is invertible. In addition assume that

(i) p_{12} is invertible in $\mathcal{N}^0$ if $\nu = 3$,

(ii) $V_j V_k \subset V_{j+k}$ for all j, k and that p_{112}, p_{13} are invertible in $\mathcal{N}^0$ if $\nu = 4$.

Then for $f := \pi \circ p$ the group $A := \{g \in \mathrm{Aff}(\mathcal{N}) : f \circ g = f\}$ acts transitively on every translated subvariety $c + F = f^{-1}(c)$, $c \in \mathcal{N}^\nu$.

Proof. We assume $\nu = 4$, the cases $\nu < 4$ are similar but easier. For every $k = 2, 3, 4$ denote by $\mathcal{L}_k \subset \mathrm{End}(\mathcal{N})$ the subspace of nilpotent transformations $x \mapsto \alpha_1 x_1,\ x \mapsto \alpha_2 x_1 + \alpha_1 x_2$, $x \mapsto \alpha_3 x_1 + \alpha_2 x_2 + \alpha_1 x_3$ respectively with arbitrary coefficients $\alpha_j \in V_j$.
Now fix an arbitrary point $a \in F$ and denote by $\tau \in \mathrm{Aff}(\mathcal{N})$ the translation $x \mapsto x + a$. A simple computation shows

$$f \circ \tau(x) = f(x) + x_1^2 R_2(x) + x_1 R_3(x) + R_4(x)$$

for suitable $R_k \in \mathcal{L}_k$. Then $\rho := \mathrm{id} - p_{112}^{-1} R_2 \in \mathrm{GL}(\mathcal{N})$ is unipotent and satisfies

$$f \circ \tau \circ \rho\,(x) = f(x) + x_1 S_3(x) + S_4(x)$$

for suitable $S_k \in \mathcal{L}_k$. Further $\sigma := \mathrm{id} - p_{13}^{-1} S_3 \in \mathrm{GL}(\mathcal{N})$ satisfies

$$f \circ \tau \circ \rho \circ \sigma(x) = f(x) + T_4(x)$$

for a suitable $T_4 \in \mathcal{L}_4$. Finally, $g(x) = \tau \circ \rho \circ \sigma(x - p_4^{-1} T_4(x))$ defines an element $g \in A$ with $g(c) = c + a$ for all $c \in \mathcal{N}^\nu$. $\square$

9.23 Remark. The proof of Proposition 9.22 also works for fields $\mathbb{F}$ of arbitrary characteristic. Special polynomials $p \in P$ can be defined in the following way: Let

$$\Phi := \sum_{k=1}^{\infty} c_k T^k \ \in\ \mathbb{F}[[T]] \tag{9.24}$$

be an arbitrary formal power series over $\mathbb{F}$ with vanishing constant term. Since $\mathcal{N}$ is nilpotent $p(x) := \Phi(x) = \sum c_k x^k$ defines a polynomial map $p \in P$. Clearly, p is invertible if and only if $c_1 \neq 0$. Furthermore, if we assume that $\mathbb{F}$ has characteristic 0, then invertibility of p_{12} is equivalent to $c_2 \neq 0$ and invertibility of $p_{13} p_{112}$ is equivalent to $c_2 c_3 \neq 0$. For simplicity we may add a constant term c_0 to the formal power series Φ (which will not count) if we at the same time extend the projection π from $\mathcal{N}$ to its unital extension $\mathcal{N}^0 = \mathbb{F}{\cdot}\mathbb{1} \oplus \mathcal{N}$ by requiring $\pi(\mathbb{1}) = 0$. Later we are mainly interested in the case where $\Phi = \exp$ is the usual exponential series.

The conditions (i), (ii) in Proposition 9.22 cannot be omitted: As a simple example with $\nu = 3$ consider the 3-dimensional cyclic algebra $\mathcal{N}$ with basis e_1, $e_2 = e_1^2$, $e_3 = e_1^3$ satisfying $e_1^4 = 0$. Then, identifying $\mathcal{N}$ with $\mathbb{F}^3$ in the obvious way, we get for $\Phi = T + T^3$ in (9.24) that $f(x) = x_3 + x_1^3$ on $\mathbb{F}^3$. In this case, the group A does not act transitively on $F = f^{-1}(0)$ in general. In fact, in case $\mathbb{F} = \mathbb{R}$ the affine group $\mathrm{Aff}(F) = \{g \in \mathrm{Aff}(\mathcal{N}) : g(F) = F\}$ has precisely two orbits in F - the line $\mathbb{R}e_2 \subset F$ and its complement in F. Indeed, the subgroup of all $(x_1, x_2, x_3) \mapsto (tx_1, x_2 + s, t^3 x_3)$ with $s \in \mathbb{R}$, $t \in \mathbb{R}^*$ acts transitively on the complement.

Nil-polynomials

In this subsection let $\mathcal{N}$ be an arbitrary commutative associative nilpotent algebra of finite dimension over $\mathbb{F}$ with annihilator $\mathcal{A}$ of dimension 1 and nil-index ν. Let us call every linear form ω on $\mathcal{N}$ with $\omega(\mathcal{A}) = \mathbb{F}$ a pointing of $\mathcal{N}$. Also, $\mathcal{N}$ with a fixed pointing is called a pointed algebra, a PANA for short. Two PANAs $(\mathcal{N}, \omega)$, $(\widetilde{\mathcal{N}}, \widetilde{\omega})$ are called isomorphic if there is an algebra isomorphism $g : \widetilde{\mathcal{N}} \to \mathcal{N}$ with $\widetilde{\omega} = \omega \circ g$. As before with projections we always consider every pointing ω on $\mathcal{N}$ linearly extended to $\mathcal{N}^0$ by requiring $\omega(\mathbb{1}) = 0$.

For every vector space V of finite dimension we denote by $\mathbb{F}[V]$ the algebra of all ($\mathbb{F}$-valued) polynomials on V.

9.25 Definition. $f \in \mathbb{F}[W]$ is called a nil-polynomial on W if there exists a PANA $(\mathcal{N}, \omega)$ and a linear isomorphism $\varphi : W \to \ker(\omega) \subset \mathcal{N}$ such that $f = \omega \circ \exp \circ \varphi$. Two nil-polynomials $f \in \mathbb{F}[W]$, $\widetilde{f} \in \mathbb{F}[\widetilde{W}]$ are called equivalent if there exists $t \in \mathrm{GL}(\mathbb{F}) \cong \mathbb{F}^*$ and a linear isomorphism $g : W \to \widetilde{W}$ with $\widetilde{f} = t \circ f \circ g^{-1}$.

9.26 Definition. In case $V \neq 0$ we call $f \in \mathbb{F}[V]$ an extended nil-polynomial on V if there exists a PANA $(\mathcal{N}, \omega)$ and a linear isomorphism $\varphi : V \to \mathcal{N}$ with $f = \omega \circ \exp \circ \varphi$. In case $V = 0$ every constant in $\mathbb{F}^*$ is called an extended nil-polynomial on V.

Nil-polynomials on vector spaces W of dimension n and extended nil-polynomials on vector spaces V of dimension $n+1$ correspond to each other. Indeed, every extended nil-polynomial on V is a sum $f = \sum_{k>0} f_{[k]}$ of homogeneous parts $f_{[k]}$ of degree k. Furthermore, $V = W \oplus A$ with $W = \ker f_{[1]}$ and $A = \{y \in V : f_{[2]}(x+y) = 0 \ \forall x \in V\} \cong \mathbb{F}$. Then the restriction of f to W is a nil-polynomial on W and every nil-polynomial on W occurs this way. For our applications in Section 7 we need extended nil-polynomials. In the following we consider only nil-polynomials for simplicity.

By definition, every equivalence class of nil-polynomials in $\mathbb{F}[W]$ is an orbit of the group $\mathrm{GL}(\mathbb{F}) \times \mathrm{GL}(W)$ acting in the obvious way on $\mathbb{F}[W]$. For every pair of nil-polynomials $P \in \mathbb{F}[W]$, $\widetilde{P} \in \mathbb{F}[\widetilde{W}]$ we get a new nil-polynomial $P \oplus \widetilde{P} \in \mathbb{F}[W \oplus \widetilde{W}]$ by setting $(P \oplus \widetilde{P})(x, \widetilde{x}) := P(x) + \widetilde{P}(\widetilde{x})$ for all $x \in W$ and $x\widetilde{\in}\widetilde{W}$.

Fix a nil-polynomial $f \in \mathbb{F}[W]$ in the following. Then we have the expansion $f = \sum_{k \geq 2} f_{[k]}$ into homogeneous parts. Notice that $f_{[2]}$ is a non-degenerate quadratic form on W. For every $k \geq 2$ there is a unique symmetric k-form ω_k on W with

$$\omega_k(x, \ldots, x) = k!\, f_{[k]}(x) \tag{9.27}$$

for all $x \in W$. Using $f_{[2]}$ and $f_{[3]}$ we define a commutative (not necessarily associative) product $(x, y) \mapsto x{\cdot}y$ on W by

$$\omega_2(x{\cdot}y, z) = \omega_3(x, y, z) \ \text{ for all } \ z \in W \tag{9.28}$$

and also a commutative product on $W \oplus \mathbb{F}$ by

$$(x, s)(y, t) := (x{\cdot}y, \omega_2(x, y))\,. \tag{9.29}$$

Then, if $f = \omega \circ \exp \circ \varphi$ for a PANA $(\mathcal{N}, \omega)$ with kernel $\mathcal{K} = \ker(\omega)$ and linear isomorphism $\varphi : W \to \mathcal{K}$ we have $\omega_k(x_1, \ldots, x_k) = \omega((\varphi x_1)(\varphi x_2) \cdots (\varphi x_k))$ for all $k \geq 2$ and $x_1, \ldots, x_k \in W$. For the annihilator $\mathcal{A}$ of $\mathcal{N}$ there is a unique linear isomorphism $\psi : \mathbb{F} \to \mathcal{A}$ such that $\pi = \psi \circ \omega$ is the canonical projection $\mathcal{K} \oplus \mathcal{A} \to \mathcal{A}$. With these ingredients we have

9.30 Lemma. With respect to the product (9.29) the linear map

$$W \oplus \mathbb{F} \to \mathcal{N}\,, \quad (x, s) \mapsto \varphi(x) + \psi(s)\,,$$

is an isomorphism of algebras. In particular, W with product $x{\cdot}y$ is isomorphic to the nilpotent algebra $\mathcal{N}/\mathcal{A}$ and has nil-index $\nu{-}1$.

Proof. For all $x, y \in W$ we have

$$(\varphi(x) + \psi(s))(\varphi(y) + \psi(t)) = (N - A) + A \text{ with}$$
$$N := \varphi(x)\varphi(y) \in \mathcal{N} \ \text{ and } \ A := \pi(\varphi(x)\varphi(y)) = \psi(\omega_2(x, y)) \in \mathcal{A}.$$

It remains to show $N - A = \varphi(x{\cdot}y)$. But this follows from

$$N - A \in \mathcal{K} \quad \text{and} \quad \omega\big(\varphi(x{\cdot}y)\varphi(z)\big) = \omega\big(\varphi(x)\varphi(y)\varphi(z)\big) = \omega\big((N - A)\varphi(z)\big)$$

for all $z \in W$. □

9.31 Corollary. Every nil-polynomial f on W is uniquely determined by its quadratic and cubic term, $f_{[2]}$ and $f_{[3]}$. In fact, the other $f_{[k]}$ are recursively determined by

$$\omega_{k+1}(x_0, x_1, \dots, x_k) = \omega_k(x_0{\cdot}x_1, x_2, \dots, x_k) \tag{9.32}$$

for all $k \geq 2$ and all $x_0, \dots, x_k \in W$, where the symmetric ω_k are determined by (9.27).

Another application of (9.30) is the following

9.33 Proposition. Let $(\mathcal{N}, \omega)$, $(\widetilde{\mathcal{N}}, \widetilde{\omega})$ be arbitrary PANAs and let $f \in \mathbb{F}[W]$, $\widetilde{f} \in \mathbb{F}[\widetilde{W}]$ be associated nil-polynomials respectively. Then, if f and $\widetilde{f}$ are equivalent as nil-polynomials, also $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ are isomorphic as algebras.

Proof. Write $\widetilde{f} = t \circ f \circ g^{-1}$ as in Definition 9.25 and define the products $\cdot$ and $\tilde{\cdot}$ on W and $\widetilde{W}$ as in (9.28). With respect to these products $g : W \to \widetilde{W}$ is an algebra isomorphism. As in (9.29) the products $\cdot$ and $\tilde{\cdot}$ extend to the algebras $W \oplus \mathbb{F} \cong \mathcal{N}$ and $\widetilde{W} \oplus \mathbb{F} \cong \widetilde{\mathcal{N}}$. Finally $g \oplus \mathrm{id}$ gives an algebra isomorphism between them. □

9.34 Lemma. For every nil-polynomial f on W the cubic term $\mathsf{c} := f_{[3]}$ is trace-free with respect to the quadratic term $\mathsf{q} := f_{[2]}$, see [7] p. 20 for this notion of trace,

Proof. Let $f \in \mathbb{F}[W]$ be given by the PANA $(\mathcal{N}, \omega)$ with nil-index ν. Without loss of generality we assume that $W = \ker(\omega)$. Choose a basis $e_1, \dots, e_n$ of W and a mapping $\alpha : \{1, \dots, n\} \to \{1, \dots, \nu{-}1\}$ such that $\{e_i : \alpha(i) = \ell\}$ is a basis of $\mathcal{N}^\ell / \mathcal{N}^{\ell+1}$ for $\ell = 1, \dots, \nu{-}1$. With respect to this basis the forms q, c are given by the tensors $g_{ij} = \omega_2(e_i, e_j)$ and $h_{ijk} = \omega_3(e_i, e_j, e_k)$. It is clear that $g_{ij} = 0$ holds if $\alpha(i) + \alpha(j) > \nu$. Since α is surjective, this implies $g^{ij} = 0$ if $\alpha(i) + \alpha(j) < \nu$, where $(g^{ij}) = (g_{ij})^{-1}$. On the other hand, $h_{ijk} = 0$ if $\alpha(i) + \alpha(j) \geq \nu$, proving the claim. □

Corollary 9.31 suggests the following question: Given a non-degenerate quadratic form q and a cubic form c on W. When does there exist a nil-polynomial $f \in \mathbb{F}[W]$ with $f_{[2]} = \mathsf{q}$ and $f_{[3]} = \mathsf{c}$? Using q, c we can define as above for $k = 2, 3$ the symmetric k-linear form ω_k on W and with it the commutative product $x{\cdot}y$ on W. A necessary and sufficient condition for a positive answer is that W with this product is a nilpotent and associative algebra. As a consequence we get for every fixed non-degenerate quadratic form q on W the following structural information on the space of all nil-polynomials f on W with $f_{[2]} = \mathsf{q}$: Denote by C the set of all cubic forms on W. Then C is a linear space of dimension $\binom{n+2}{3}$, $n = \dim W$, and

$$\mathsf{C}_\mathsf{q} := \{\mathsf{c} \in \mathsf{C} : \exists \text{ nil-polynomial } f \text{ on } W \text{ with } f_{[2]} = \mathsf{q}, f_{[3]} = \mathsf{c}\} \tag{9.35}$$

is an algebraic subset. The orthogonal group $\mathrm{O}(\mathsf{q}) = \{g \in \mathrm{GL}(W) : \mathsf{q} \circ g = \mathsf{q}\}$ acts from the right on C_q. The $\mathrm{O}(\mathsf{q})$-orbits in C_q are in 1-1-correspondence to the equivalence classes of nil-polynomials f on W with $f_{[2]} = \mathsf{q}$.

Examples of nil-polynomials of degree 3 can be constructed in the following way.

9.36 Proposition. Let W be an $\mathbb{F}$-vector space of finite dimension and q a non-degenerate quadratic form on W. Suppose furthermore that $W = W_1 \oplus W_2$ for totally isotropic linear subspaces W_k and that c is a cubic form on W_1. Then, if we extend c to W by $\mathsf{c}(x + y) = \mathsf{c}(x)$ for all $x \in W_1$, $y \in W_2$, the sum $f := \mathsf{q} + \mathsf{c}$ is a nil-polynomial on W. In particular, $\mathsf{c} \in \mathsf{C}_\mathsf{q}$.
Every $g \in \mathrm{GL}(W_1)$ extends to an $h \in \mathrm{O}(\mathsf{q}) \subset \mathrm{GL}(W)$ in such a way that with $\widetilde{\mathsf{c}} := \mathsf{c} \circ g$ also $\mathsf{q} + \widetilde{\mathsf{c}} = f \circ h$ is a nil-polynomial on W.

Proof. $\omega_3(x, y, t) = 0$ for all $t \in W_2$ implies $W_1{\cdot}W_1 \subset W_2$ and $W_1{\cdot}W_2 = 0$, that is, $(x{\cdot}y){\cdot}z = 0$ for all $x, y, z \in W$.

Now fix $g \in \mathrm{GL}(W_1)$. There exists a unique $g^\sharp \in \mathrm{GL}(W_2)$ with $\omega_2(gx, y) = \omega_2(x, g^\sharp y)$ for all $x \in W_1$ and $y \in W_2$. But then $h := g \times (g^\sharp)^{-1} \in \mathrm{O}(q)$ does the job. □

Let $\mathbb{F} \subset \mathbb{K}$ be a field extension and consider every polynomial f on V in the canonical way as polynomial $\widetilde{f}$ on $V \otimes_{\mathbb{F}} \mathbb{K}$. Then with f also $\widetilde{f}$ is a nil-polynomial. In general, for non-equivalent nil-polynomials f, g on V the nil-polynomials $\widetilde{f}, \widetilde{g}$ may be equivalent. We use these extensions in case $\mathbb{R} \subset \mathbb{C}$.

Graded PANAs

Let $(\mathcal{N}, \omega)$ with annihilator $\mathcal{A}$ be a PANA in the following. A grading then is a decomposition

$$\mathcal{N} = \bigoplus_{k>0} \mathcal{N}_k , \qquad \mathcal{N}_j \mathcal{N}_k \subset \mathcal{N}_{j+k} . \tag{9.37}$$

Clearly $\mathcal{A} = \mathcal{N}_d$ for $d := \max\{k : \mathcal{N}_k \neq 0\}$. Without loss of generality we assume that $W := \bigoplus_{k<d} \mathcal{N}_k$ is the kernel of ω.

For the corresponding nil-polynomial $f = \omega \circ \exp \in \mathbb{F}[W]$, $\ell := d - 1$ and ν the nil-index of $\mathcal{N}$ we then have

$$f = \sum_{k=2}^{\nu} f_{[k]} \quad \text{with} \quad f_{[k]}(x) = \frac{1}{k!}\Big(\sum_{j_1+\ldots+j_k=d} x_{j_1} x_{j_2} \cdots x_{j_k} \Big) \tag{9.38}$$

for all $x = (x_1, \ldots, x_\ell) \in \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_\ell = W \subset \mathcal{N}$, where every index j_ℓ in (9.38) is positive. If we put, using (9.1),

$$\|\mu\| = \mu_1 + 2\mu_2 + \ldots + \ell\mu_\ell \qquad \text{and} \qquad x^{(\mu)} = x_1^{(\mu_1)} x_2^{(\mu_2)} \cdots x_\ell^{(\mu_\ell)}$$

for every multi-index $\mu \in \mathbb{N}^\ell$ and every $x = (x_1, \ldots, x_\ell)$, we can rewrite (9.38) as

$$f(x) = \sum_{\|\mu\|=d} x^{(\mu)} . \tag{9.39}$$

We consider an example.

9.40 Cyclic PANAs For fixed integer $\nu \geq 1$ let $\mathcal{N}$ be the cyclic algebra of nil-index ν over $\mathbb{F}$, that is, there is an element $\xi \in \mathcal{N}$ such that the powers ξ^k, $1 \leq k \leq \nu$, form a basis of $\mathcal{N}$ and $\xi^{\nu+1} = 0$. Then $\mathcal{N}$ is a graded algebra with respect to $\mathcal{N}_k := \mathbb{F}\,\xi^k$ for all $k > 0$ in (9.37) and becomes a PANA with respect to the pointing ω uniquely determined by $\omega(\xi^k) = \delta_{k,\nu}$ for all k. For $n := \nu - 1$ we identify $\mathbb{F}^n$ and $\ker(\omega)$ via $(x_1, \ldots, x_n) = \sum x_k \xi^k$. The corresponding nil-polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$ will then be called a cyclic nil-polynomial, see also Table 1. In case $\mathbb{F} = \mathbb{R}$ the quadratic form $f_{[2]}$ has type $(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor)$.

$f_{[2]}$	$f_{[3]}$	$f_{[4]}$	$f_{[5]}$	$f_{[6]}$
0	0	0	0	0
$x_1^{(2)}$	0	0	0	0
$x_1 x_2$	$x_1^{(3)}$	0	0	0
$x_1 x_3 + x_2^{(2)}$	$x_1^{(2)} x_2$	$x_1^{(4)}$	0	0
$x_1 x_4 + x_2 x_3$	$x_1 x_2^{(2)} + x_1^{(2)} x_3$	$x_1^{(3)} x_2$	$x_1^{(5)}$	0
$x_1 x_5 + x_2 x_4 + x_3^{(2)}$	$x_1 x_2 x_3 + x_1^{(2)} x_4 + x_2^{(3)}$	$x_1^{(2)} x_2^{(2)} + x_1^{(3)} x_3$	$x_1^{(4)} x_2$	$x_1^{(6)}$

Table 1: Cyclic nil-polynomials $f \in \mathbb{F}[x_1, \ldots, x_{n-1}]$ for $1 \leq n \leq 6$, where $y^{(k)} := y^k/(k!)$

For graded PANAs $\mathcal{N} = \bigoplus_{k>0}\mathcal{N}_k$ with annihilator $\mathcal{A} = \mathcal{N}_d$ we have for every $s \in \mathbb{F}^*$ the algebra automorphism

$$\theta_s := \bigoplus_{k>0} s^k \operatorname{id}_{|\mathcal{N}_k} \in \operatorname{Aut}(\mathcal{N})\,. \tag{9.41}$$

As a consequence, if $t \in \mathbb{F}^*$ admits a d^{th} root in $\mathbb{F}$, the pointings ω and $t\omega$ differ by an automorphism of $\mathcal{N}$.

We mention that the PANA $\mathcal{N}$ associated with the nil-polynomial $f = \omega + \mathsf{q} + \mathsf{c}$ considered in Proposition 9.36 also has a grading: Indeed, put $\mathcal{N}_1 := W_1$, $\mathcal{N}_2 := W_2$ and $\mathcal{A} := \mathcal{N}_3 := \mathbb{F}$ with products given by $xy := x{\cdot}y$ if $x, y \in W_1$ and $xy := \omega_2(x, y)$ if $x \in W_1$, $y \in W_2$. Using this we can improve the second statement in 9.36.

PANAs $\mathcal{N}$ admitting a grading enjoy a special property: It is easy to see as a consequence of (9.39) that for every associated nil-polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$ there exist linear forms $\lambda_1, \ldots, \lambda_n$ on $\mathbb{F}^n$ with

$$f = \sum_{k=1}^{n} \lambda_k\, {}^{\partial f}/_{\partial x_k}\,. \tag{9.42}$$

9.43 Proposition. With the notation of Proposition 9.36 assume that the cubic form c on W_1 has the following property:

$(*)$ $\quad z = 0$ is the only element $z \in W_1$ with $\mathsf{c}(x + z) = \mathsf{c}(x)$ for all $x \in W_1$.

The graded PANA $\mathcal{N} = W_1 \oplus W_2 \oplus \mathbb{F}$ with product $(x, y) \mapsto xy$ corresponding to the nil-polynomial $f = \mathsf{q} + \mathsf{c}$ on W then satisfies $\mathcal{N}^2 = W_2 \oplus \mathbb{F}$ as a consequence of $(*)$. Furthermore, if $\widetilde{\mathsf{c}}$ is a second cubic form on W_1 with nil-polynomial $\widetilde{f} = \mathsf{q} + \widetilde{\mathsf{c}}$ and PANA $\widetilde{\mathcal{N}} = W_1 \oplus W_2 \oplus \mathbb{F}$ with appropriate product, the following conditions are equivalent.

(i) The nil-polynomials $f, \widetilde{f}$ are equivalent.
(ii) The algebras $\mathcal{N}, \widetilde{\mathcal{N}}$ are isomorphic as abstract algebras.
(iii) $\widetilde{f} = f \circ g$ for some $g \in \mathrm{GL}(W_1)$.

Proof. Let $\omega : W \oplus \mathbb{F} \to \mathbb{F}$ be the canonical projection. Then ω is a pointing for $\mathcal{N}$. As in (9.27) define the ω_k for the nil-polynomial $f = f_{[2]} + f_{[3]}$ on W. We have to show that $\{x{\cdot}y : x, y \in W_1\}$ spans W_2. If not, there exists a vector $z \neq 0$ in W_1 with $\omega_2(x{\cdot}y, z) = 0$ for all $x, y \in W_1$. But then $\omega_3(x, y, z) = 0$ for all $x, y \in W_1$ implies $\mathsf{c}(x + z) = \mathsf{c}(x)$ for all $x \in W_1$, a contradiction.

(i) $\Longleftrightarrow$ (ii) This follows immediately from Definition 9.26.

(iii) $\Longrightarrow$ (i) This follows immediately from the second claim in Proposition 9.36.

(ii) $\Longrightarrow$ (iii) Let $h : \mathcal{N} \to \widetilde{\mathcal{N}}$ be an algebra isomorphism. Then $h(W_2 \oplus \mathbb{F}) \subset \widetilde{\mathcal{N}}^2 \subset (W_2 \oplus \mathbb{F})$ as a consequence of $(*)$, that is, there is a $g \in \mathrm{GL}(W_1)$ with $h(x) \equiv g(x) \bmod \mathcal{N}^2$ for all $x \in W_1$. By the second statement in Proposition 9.36 we may assume $g = \mathrm{id}$ without loss of generality. But then $\widetilde{\mathsf{c}}(x) = \mathsf{c}(h(x)) = \mathsf{c}(x)$ for all $x \in W_1$ implies $\widetilde{f} = f$. $\square$

Suppose that $W_1 \cong \mathbb{F}^m$ with coordinates $(x_1, \ldots, x_m)$ has dimension $m > 0$ in Proposition 9.43. Then $W \cong \mathbb{F}^{2m}$ with coordinates $(x_1, \ldots, x_m, y_1, \ldots, y_m)$ and we may assume $\mathsf{q}(x, y) = x_1y_1 + \ldots + x_my_m$. As already mentioned, the linear space C of all cubic forms c on W_1 has dimension $\binom{m+2}{3}$. The subset C^* of all $\mathsf{c} \in \mathsf{C}$ satisfying the condition $(*)$ in Proposition 9.43 is Zariski open and dense in C. The group $\mathrm{GL}(W_1)$ acts on C^* from the right and has dimension m^2 over $\mathbb{F}$. The difference of dimensions is $\binom{m}{3}$. But this number is also the cardinality of the subset $J \subset \mathbb{N}^3$, consisting of all triples $j = (j_1, j_2, j_3)$ with $1 \le j_1 < j_2 < j_3 \le m$. Consider the affine map

$$\alpha : \mathbb{F}^J \to \mathsf{C}\,, \quad (t_j) \mapsto \mathsf{c}_0 + \sum_{j \in J} t_j \mathsf{c}_j\,, \tag{9.44}$$

where $c_0 := x_1^3 + \ldots + x_m^3 \in C^*$ and $c_j := x_{j_1} x_{j_2} x_{j_3} \in C$ for all $j \in J$.

In case $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, for a suitable neighbourhood U of $0 \in \mathbb{F}^J$ the map $\alpha : U \to C^*$ intersects all $\mathrm{GL}(n, \mathbb{F})$-orbits in C^* transversally. Indeed, since all partial derivatives of c_0 are monomials containing a square, the tangent space at c_0 of its $\mathrm{GL}(n, \mathbb{F})$-orbit is transversal to the linear subspace $\langle c_j : j \in J \rangle$ of C. In particular, in case $m \geq 3$ there is a family of dimension $\binom{m}{3} \geq 1$ over $\mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$) of pairwise different $\mathrm{GL}(n, \mathbb{F})$-orbits and thus of non-equivalent nil-polynomials of degree 3 on W. Notice that in case $m = 3$ the mapping α in (9.44) reduces to

$$\alpha : \mathbb{F} \to C, \quad t \mapsto x_1^3 + x_2^3 + x_3^3 + t x_1 x_2 x_3 \,.$$

We apply Proposition 9.22 to the examples considered in Proposition 9.36. Here $V = W_1 \oplus W_2 \oplus \mathbb{F}$, q is a quadratic form on $W := W_1 \oplus W_2$ and c is a cubic form on W_1. With the extended nil-polynomial $f(x, y, t) = t + q(x + y) + c(x)$ on $W_1 \oplus W_2 \oplus \mathbb{F}$ consider the hypersurface

$$F := \{z \in V : f(z) = 0\}$$

and identify $\mathbb{F}$ with the line $\{0\} \oplus \mathbb{F}$ in $W \oplus \mathbb{F}$. Then the affine group $\mathrm{Aff}(F)$ is transitive on F by Proposition 9.22, and every orbit in V intersects the line $\mathbb{F}$. For every $s \in \mathbb{F}^*$ the transformation θ_s, see (9.41), satisfies $f \circ \theta_s = s^3 h$ for every $s \in \mathbb{F}^*$, that is, $\theta_s \in \mathrm{Aff}(F)$. As a consequence, the number of $\mathrm{Aff}(F)$-orbits in V is bounded by the number of $(\mathbb{F}^*)^3$-orbits in $\mathbb{F}$. In particular, if $(\mathbb{F}^*)^3 = \mathbb{F}^*$, then there are precisely two $\mathrm{Aff}(F)$-orbits in V, namely F and its complement. This situation occurs, for instance, for $\mathbb{F} = \mathbb{R}$ and also for $\mathbb{F} = \mathbb{C}$. In any case, F is the only Zariski closed $\mathrm{Aff}(F)$-orbit in V, and every other orbit is Zariski dense.

Nil-polynomials of degree 4

The method in Proposition 9.36 can be generalized to get nil-polynomials of higher degree, say of degree 4 for simplicity. Throughout the subsection we use the notation (9.1).

Let $W = W_1 \oplus W_2 \oplus W_3$ be a vector space with $W_1 = \mathbb{F}^n$, $W_2 = \mathbb{F}^m$ and let q be a fixed non-degenerate quadratic form on W in the following. Assume that W_1, W_3 are totally isotropic and that $W_1 \oplus W_3$, W_2 are orthogonal with respect to q. Then W has dimension $2n + m$, and without loss of generality we assume that

$$q(y) = \sum_{k=1}^{m} \varepsilon_k y_k^{(2)} \quad \text{for suitable} \quad \varepsilon_k \in \mathbb{F}^* \ \text{ and all } \ y \in W_2 \,.$$

As before let C be the space of all cubic forms on W. Our aim is to find cubic forms $c \in C_q$ that are the cubic part of a nil-polynomial of degree 4.

Denote by C' the space of all cubic forms c on $W_1 \oplus W_2$ such that $c(x + y)$ is quadratic in $x \in W_1$ and linear in $y \in W_2$, or equivalently, which are of the form

$$c(x + y) = \frac{1}{2} \sum_{k=1}^{m} \sum_{i,j=1}^{n} c_{ijk} x_i x_j y_k \quad \text{for all} \quad x \in W_1, y \in W_2$$

with suitable coefficients $c_{ijk} = c_{jik} \in \mathbb{F}$. Extending every $c \in C'$ trivially to a cubic form on W we consider C' as subset of C.

For fixed $c \in C'$ the symmetric 2- and 3-linear forms ω_2, ω_3 on W are defined by $\omega_2(x, x) = 2q(x)$ and $\omega_3(x, x, x) = 6c(x)$ for all $x \in W$. With the commutative product

$x{\cdot}y$ on W, see (9.28), define in addition also the k-linear forms ω_k by (9.32) for all $k \ge 4$. Then, for every $x, y \in W_1$ the identity $\omega_2(x{\cdot}y, t) = \omega_3(x, y, t) = 0$ for all $t \in W_1 \oplus W_3$ implies $x{\cdot}y \in W_2$, that is $W_1{\cdot}W_1 \subset W_2$. In the same way $\omega_2(x{\cdot}y, t) = 0$ for all $x \in W_1$, $y \in W_2$ and $t \in W_2 \oplus W_3$ implies $W_1{\cdot}W_2 \subset W_3$. Also $W_j{\cdot}W_k = 0$ follows for all j, k with $j + k \ge 4$. Therefore c belongs to C_q if and only if $(a{\cdot}b){\cdot}c = a{\cdot}(b{\cdot}c)$ for all $a, b, c \in W_1$.

In terms of the standard basis $e_1, \dots, e_m$ of $W_2 = \mathbb{F}^m$ we have

$$a{\cdot}b = \sum_{k=1}^{m} \Big(\sum_{i,j=1}^{n} \varepsilon_k^{-1} \mathsf{c}_{ijk} a_i b_j \Big) e_k \quad \text{for all} \quad a, b \in W_1$$

and thus with $\Theta_{i,j,r,s} := \sum_{k=1}^m \varepsilon_k^{-1} \mathsf{c}_{ijk} \mathsf{c}_{rsk}$ we get the identity

$$\begin{aligned} \omega_2\big((a{\cdot}b){\cdot}c, t\big) &= \omega_3(a{\cdot}b, c, t) = \omega_2(a{\cdot}b, c{\cdot}t) \\ &= \sum_{i,j,r,s=1}^{n} \Theta_{i,j,r,s}\, a_i b_j c_r t_s \quad \text{for all} \quad a, b, c, t \in W_1 \,. \end{aligned}$$

Therefore,

$$A := \mathsf{C}' \cap \mathsf{C}_q = \{\mathsf{c} \in \mathsf{C}' : \Theta_{i,j,r,s} \text{ is symmetric in } i, r\} \,. \tag{9.45}$$

Notice that the condition in (9.45) implies that $\Theta_{i,j,r,s}$ is symmetric in all indices. A is a rational subvariety of the linear space C', it consists of all those c for which the corresponding product $x{\cdot}y$ on W is associative.

For every $\mathsf{c} \in A$ the corresponding nil-polynomial f on W has the form

$$f = f_{[2]} + f_{[3]} + f_{[4]} \ \text{ with } \ f_{[2]} = q,\ f_{[3]} = \mathsf{c} \ \text{ and}$$

$$f_{[4]}(z) = \frac{1}{12} q(x{\cdot}x) \ \text{ for all } \ z = (x, y, t) \in W_1 \oplus W_2 \oplus W_3 \,.$$

The group $\Gamma := \mathrm{GL}(W_1) \times \mathrm{O}(q_{|W_2}) \ \subset \ \mathrm{GL}(W_1 \oplus W_2)$ acts on C' by $\mathsf{c} \mapsto \mathsf{c} \circ \gamma^{-1}$ for every $\gamma \in \Gamma$. Furthermore, $(g, h) \mapsto (g, h, (g^\sharp)^{-1})$ embeds Γ into $\mathrm{O}(q)$, compare the proof of Proposition 9.36. As a consequence, the subvariety $A \subset \mathsf{C}'$ is invariant under Γ.

For every $\mathsf{c} \in A$ the corresponding nil-polynomial comes from a graded PANA with nil-index 4, provided $\mathsf{c} \neq 0$. Indeed, put $W_4 := \mathbb{F}$ and endow $W \oplus W_4$ with the product (9.29). It is obvious that the linear span of $W_1{\cdot}W_1$ in W_2 has dimension $\le \binom{n+1}{2}$.

Let us consider the special case $n = 2$ with $m = \binom{n+1}{2} = 3$ in more detail. For simplicity we assume that for suitable coordinates (x_1, x_2) of W_1, (y_1, y_2, y_3) of W_2 and (z_1, z_2) of W_3 the quadratic form q is given by

$$q = x_1 z_1 + x_2 z_2 + y_1^{(2)} + y_2^{(2)} + \varepsilon y_3^{(2)} \quad \text{for fixed} \quad \varepsilon \in \mathbb{F}^* \tag{9.46}$$

(in case $\mathbb{F} = \mathbb{R}, \mathbb{C}$ this is not a real restriction). For every $t \in \mathbb{F}$ consider the cubic form

$$\mathsf{c}_t := (x_1^{(2)} + x_2^{(2)}) y_1 + x_1 x_2 y_2 + t x_2^{(2)} y_3$$

on $W_1 \oplus W_2$. A simple computation reveals that every c_t is contained in $A = \mathsf{C}' \cap \mathsf{C}_q$. The corresponding nil-polynomial (depending on the choice of ε) then is

$$f_t = q + \mathsf{c}_t + \mathsf{d}_t \quad \text{with} \quad \mathsf{d}_t := x_1^{(4)} + x_1^{(2)} x_2^{(2)} + (1 + \varepsilon^{-1} t^2) x_2^{(4)} \,.$$

In addition we put

$$f_\infty := q + x_2^{(2)} y_3 + \varepsilon^{-1} x_2^{(4)}$$

(a smash product with the cyclic nil-polynomial of degree 4, see Table 1), where ∞ in the projective line $\mathbb{P}_1(\mathbb{F}) = \mathbb{F} \cup \{\infty\}$ is the point at infinity. Notice that for $t \in \mathbb{F}^*$ the nil-polynomials f_t and $\widetilde{f_{1/t}} := q + t^{-1} c_t + t^{-2} d_t$ are equivalent.

It is obvious that f_t is equivalent to f_{-t} for every $t \in \mathbb{P}_1(\mathbb{F})$. Also, for every cubic term c_t with $t \in \mathbb{F}^*$ the set $W_1{\cdot}W_1$ spans W_2. For $t = 0, \infty$ the linear span of $W_1{\cdot}W_1$ in W_2 has dimension $2, 1$ respectively. For every $t \in \mathbb{F}^*$ an invariant of d_t is the number $\phi(t) := g_2(d_t)^3 / g_3(d_t)^2 = \varepsilon^2 t^{-4}(4+\varepsilon^{-1}t^2)^3 \in \mathbb{F}$, where g_2, g_3 are the classical invariants of binary quartics, compare [19] p. 27. Since every fiber of $\phi : \mathbb{F}^* \to \mathbb{F}$ contains at most 6 elements we conclude

9.47 Proposition. For every Field $\mathbb{F}$ and every fixed $\varepsilon \in \mathbb{F}^*$ the set of all equivalence classes given by all nil-polynomials f_t, $t \in \mathbb{F}$, has the same cardinality as $\mathbb{F}$ and, in particular, is infinite.

Remarks 1. In case $\mathbb{F} = \mathbb{Q}$ is the rational field there are infinitely many choices of $\varepsilon \in \mathbb{Q}^*$ leading to pairwise non-equivalent quadratic forms q in (9.46). For each such choice there is an infinite number of pairwise non-equivalent nil-polynomials f_t of degree 4 over $\mathbb{Q}$.

2. In case $\mathbb{F} = \mathbb{R}$ is the real field there are essentially the two choices $\varepsilon = \pm 1$. In case $\varepsilon = 1$ the form q has type $(5, 2)$ and all nil-polynomials f_t with $0 \le t \le \sqrt{8}$ are pairwise non-equivalent. In case $\varepsilon = -1$ the form q has type $(4, 3)$ and all f_t with $0 \le t \le \infty$ are pairwise non-equivalent.

3. Nil-polynomials of degree ≥ 5 can be constructed just as in the case of degrees 3 and 4 as before. As an example we briefly touch the case of degree 5: Fix a vector space W of finite dimension over $\mathbb{F}$ together with a non-degenerate quadratic form q on W. Assume furthermore that there is a direct sum decomposition $W = W_1 \oplus W_2 \oplus W_3 \oplus W_4$ into totally isotropic subspaces such that $W_1 \oplus W_4$ and $W_2 \oplus W_3$ are orthogonal. Then consider a cubic form c on W that can be written as a sum $c = c' + c''$ of cubic forms with the following properties: c' is a cubic form on $W_1 \oplus W_3$ (trivially extended to W) that is linear in the variables of W_3 while c'' is a cubic form on $W_1 \oplus W_2$ that is linear in the variables of W_1. Denote by $x{\cdot}y$ the commutative product on W determined by q and c. If we put $W_k := 0$ for all $k > 4$ we have $W_j{\cdot}W_k \subset W_{j+k}$ for all j, k. Therefore, $c \in C_q$ if and only if the product $x{\cdot}y$ on W is associative, see (9.35) for the notation. This is true without any assumption if $c = c'$ or $c = c''$. But $W_1{\cdot}W_1 = 0$ in the first and $W_2{\cdot}W_2 = 0$ in the latter case. On the other hand, the nil-polynomial associated to $c \in C_q$ has degree 5 if $W_2{\cdot}W_2$ spans W_4 and $W_1{\cdot}W_1 \ne 0$.

Affine homogeneity

In this subsection let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. Also let $\mathcal{N} \ne 0$ be an arbitrary commutative associative nilpotent algebra of finite dimension over $\mathbb{F}$. In addition we assume that there exists a $\mathbb{Z}$-gradation

$$\mathcal{N} = \bigoplus_{k>0} \mathcal{N}_k\,, \quad \mathcal{N}_j \mathcal{N}_k \subset \mathcal{N}_{j+k}\,.$$

Let $d := \max\{k : \mathcal{N}_k \ne 0\}$ and denote by $\pi : \mathcal{N} \to \mathcal{N}_d$ the canonical projection with kernel $\mathcal{K} := \bigoplus_{k<d} \mathcal{N}_k$. We do not require that $\mathcal{N}_k \ne 0$ for all $1 \le k \le d$ nor that $\mathcal{N}_d$ is the annihilator or has dimension 1. Extending π linearly to $\mathcal{N}^0$ by $\pi(\mathbb{1}) = 0$ we have the polynomial map $f := \pi \circ \exp : \mathcal{N} \to \mathcal{N}_d$. The submanifold $F := f^{-1}(0)$ then is the graph of a polynomial map $\mathcal{K} \to \mathcal{N}_d$ and $\mathcal{K} = T_0 F$ is the tangent space to F at the origin. We

are interested in the affine group $\mathrm{Aff}(F) = \{g \in \mathrm{Aff}(\mathcal{N}) : g(F) = F\}$ and its subgroup $A = A(f) := \{g \in \mathrm{Aff}(\mathcal{N}) : f \circ g = f\}$.

Every point $x \in \mathcal{N}$ has a unique representation $x = x_1 + \ldots + x_d$ with $x_k \in \mathcal{N}_k$. Consider on $\mathcal{N}$ the linear span $\mathfrak{a}$ of all nilpotent affine vector fields of the form

$$(d-j)\alpha_j {\partial}/{\partial x_j} - \sum_{k=1}^{d-j} k\alpha_j x_k \, {\partial}/{\partial x_{j+k}} \quad \text{with} \quad 1 \le j < d \ \text{ and } \ \alpha_j \in \mathcal{N}_j\,.$$

As an example, in case $\mathcal{N} \cong \mathbb{F}^4$ is the cyclic PANA of nil-index 4, see Table 1, we have $d = 4$, $f(x) = x_4 + x_1x_3 + x_2^{(2)} + x_1^{(2)}x_2 + x_1^{(4)}$, and $\mathfrak{a}$ is the linear span of the vector fields

$$\begin{aligned} &3\,{\partial}/{\partial x_1} - x_1\,{\partial}/{\partial x_2} - 2x_2\,{\partial}/{\partial x_3} - 3x_3\,{\partial}/{\partial x_4}\\ &\qquad\qquad 2\,{\partial}/{\partial x_2} - \ x_1\,{\partial}/{\partial x_3} - 2x_2\,{\partial}/{\partial x_4}\\ &\qquad\qquad\qquad\qquad {\partial}/{\partial x_3} - \ x_1\,{\partial}/{\partial x_4}\ . \end{aligned}$$

With some computation we get:

9.48 Lemma. $\mathfrak{a}$ is a nilpotent Lie algebra and the evaluation map $\varepsilon_a : \mathfrak{a} \to \mathcal{N},\ \xi \mapsto \xi_a$, is injective for every $a \in \mathcal{N}$. In particular, all orbits in $\mathcal{N}$ of the nilpotent subgroup $\exp(\mathfrak{a}) \subset \mathrm{Aff}(\mathcal{N})$ have the same dimension.

9.49 Proposition. $\mathfrak{a}f = 0$, that is, $\exp(\mathfrak{a}) \subset A(f)$. In particular, $A(f)$ acts transitively on every $c + F = f^{-1}(c),\ c \in \mathcal{N}_d$.

Proof. Put $\xi := (d-j)\alpha{\partial}/{\partial x_j} - \sum_{k=1}^{d-j} k\alpha x_k\,{\partial}/{\partial x_{j+k}}$ for fixed $1 \le j < d$ and $\alpha \in \mathcal{N}_j$. Then

$$\xi f = \sum c_\nu \alpha\, x_1^{(\nu_1)} x_2^{(\nu_2)} \cdots x_d^{(\nu_d)}\,,$$

where the sum is taken over all multi indices $\nu \in \mathbb{N}^d$ with $\nu_1 + 2\nu_2 + \ldots + d\nu_d = d - j$ and c_ν certain rational coefficients. Now fix such a multi index ν. For simpler notation we put $x^{(-1)} := 0$ for every $x \in \mathcal{N}$. Then we have

$$\begin{aligned} c_\nu \alpha\, x_1^{(\nu_1)} x_2^{(\nu_2)} \cdots x_d^{(\nu_d)} &= (d-j)\alpha{\partial}/{\partial x_j}\left(x_1^{(\nu_1)} \cdots x_j^{(\nu_j+1)} \cdots x_d^{(\nu_d)}\right) - \\ &\quad \sum_{k=1}^{d-j} k\alpha x_k\,{\partial}/{\partial x_{j+k}}\left(x_1^{(\nu_1)} \cdots x_k^{(\nu_k-1)} \cdots x_{j+k}^{(\nu_{j+k}+1)} \cdots x_d^{(\nu_d)}\right)\\ &= \Big(d - j - \sum_{k=1}^{d-j} k\nu_k\Big)\alpha\, x_1^{(\nu_1)} x_2^{(\nu_2)} \cdots x_d^{(\nu_d)} \ = \ 0 \end{aligned}$$

since $\nu_k = 0$ for $k > d - j$. □

Next we specialize to the case where $\mathcal{N}_d$ has dimension 1, that is, F is a hypersurface in $\mathcal{N}$ (we still do not require that $\mathcal{N}_d$ is the annihilator of $\mathcal{N}$ although contained in it). For every $s \in \mathbb{F}^*$ we have the semi-simple linear transformation

$$\theta_s := \bigoplus_{k>0} s^k\, \mathrm{id}_{|\mathcal{N}_k} \in \mathrm{GL}(F)$$

satisfying $f \circ \theta_s = s^d f$. As a consequence we have that the group $\mathrm{Aff}(F)$ has at most 3 orbits in $\mathcal{N}$. In case d odd or $\mathbb{F} = \mathbb{C}$ this group has only two orbits in $\mathcal{N}$, the closed hypersurface F and the open complement $\mathcal{N}\backslash F$. As in the subsection 8.3 we get in case $\mathbb{F} = \mathbb{C}$ further affinely homogeneous real surfaces.

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