A ZERO-ONE LAW FOR RECURRENCE AND TRANSIENCE OF FROG PROCESSES

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Abstract. We provide sufficient conditions for the validity of a dichotomy, i.e. zero-one law, between recurrence and transience of general frog models. In particular, the results cover frog models with i.i.d. numbers of frogs per site where the frog dynamics are given by quasi-transitive Markov chains or by random walks in a common random environment including super-critical percolation clusters on $\mathbb{Z}^d$. We also give a sufficient and almost sharp condition for recurrence of uniformly elliptic frog processes on $\mathbb{Z}^d$. Its proof uses the general zero-one law.

1. Introduction

Frog models are interacting particle systems which loosely can be described as follows. The system starts with a set of inactive particles located at points of a countably infinite state space. At time 0 all particles at a certain initial site get activated. Each active particle moves from site to site of the state space; when it visits a site containing inactive particles, it wakes them all up. The newly activated particles start moving in a similar fashion.

The first results for this type of model (under the name "egg model") were published in 1999, [TW99, Section 2.4]. The vivid term "frog model", where each particle is called a frog, was coined by R. Durrett ([Pop03, p. 278]) and became standard. For a review of some frog models and results up to 2003 see [Pop03]. Since then, a number of papers have appeared concerning recurrence and transience of frog models, see [GS09], [DP14], [HJJ14], [HJJ15]. For other aspects and generalizations of the model we refer to [LMP05], [KS08], [HW15], [GNR15], and the references therein.

The current paper addresses zero-one laws for recurrence and transience of frog models in a very general setting. A site is said to be recurrent (for a precise formulation see Definition 7) if activating all frogs at this site causes this site to be visited by infinitely many frogs originating at distinct sites. Otherwise it is said to be transient. Note that according to our definition sites without sleeping frogs, i.e. inactive particles, are transient. In the study of recurrence (resp. transience) of the frog model one often constructs an event on which a given site is recurrent (resp. transient) with positive probability. A zero-one law allows one to immediately

2010 Mathematics Subject Classification. 60K35, 60J10 .

Key words: dichotomy, egg model, frog model, infinite cluster, random conductances, random environment, random walk, recurrence, transience, transitive Markov chain, zero-one law.

conclude that a site is then recurrent (resp. transient) with probability 1, and that, moreover, under natural conditions, all sites are recurrent (resp. transient) with probability 1. The proof of Theorem 5 below illustrates such an approach. For recent results about zero-one laws for frog models and their applications we refer to [GS09], [HJJ14], [HJJ15].

We hope that our general result Theorem 13 covers a sufficiently wide range of frog models to be useful.[1] Moreover, in spite of somewhat bulky notation caused by generality, the idea of the proof is simple.

To illustrate the range of applicability of our main theorem, Theorem 13, we present two of its corollaries, see Section 4 for additional examples. Throughout we denote by $\eta(x)$ the number of frogs which are initially sleeping at site x of a countably infinite state space V. We let $S_j(x,i)$ be the position after $j \geq 0$ steps of the i-th frog, $i \geq 1$, which was sleeping at site $x \in V$ if it is woken up. In particular, $S_0(x,i) = x$ for all $x \in V$ and $i \geq 1$. For the first result, recall that a stochastic matrix K with state space V is called *transitive*[2] if for any $x, y \in V$ there is a permutation φ of V such that $\varphi(x) = y$ and $K(\varphi(u), \varphi(v)) = K(u,v)$ for all $u, v \in V$.

Theorem 1 (**Transitive Markov chains).** *Let the frog trajectories $(S_j(x,i))_{j\geq 0}$, $x \in V, i \geq 1$, be Markov chains with a common transitive and irreducible transition matrix on a countably infinite state space V. Let the numbers $\eta(x)$, $x \in V$, of sleeping frogs be identically distributed and assume that the quantities $\eta(x), (S_j(x,i))_{j\geq 0}$, $x \in V, i \geq 1$, are independent. Then either with probability 1 every $x \in V$ is transient or with probability 1 every $x \in V$ with $\eta(x) \geq 1$ is recurrent.*

Theorem 1, in particular, implies the validity of Conjecture 2 in [GS09, Section 3]. Since our setting is slightly different from the one in [GS09], we provide a detailed discussion in Appendix A.

The proof of Theorem 1 is particularly simple in the case considered in [GS09, Theorem 2.3], where the frogs perform independent homogeneous nearest-neighbor random walks on $\mathbb{Z}$ with drift to the right. It goes as follows. At the beginning we assign at random to each frog a trajectory which the frog will follow once it has been woken up. For $x \in \mathbb{Z}$ denote by R_x the event that waking up the frogs at x, when everybody else is still asleep, results in infinitely many frogs visiting x. Since $(\eta(x))_{x\in\mathbb{Z}}$ is i.i.d. and the frogs move independently of each other, the sequence $(\mathbf{1}_{R_x})_{x\in\mathbb{Z}}$ is stationary and ergodic with respect to (w.r.t.) the shifts on $\mathbb{Z}$. Therefore, this sequence is either a.s. identically equal to 0, in which case $P[R_x] = 0$ for all $x \in \mathbb{Z}$, or a.s. $\mathbf{1}_{R_x} = 1$ for infinitely many $x \geq 1$. In the latter case, if we wake up the frogs at a site $v \in \mathbb{Z}$ with $\eta(v) \geq 1$, then these frogs will a.s. be transient to the right and will therefore also visit a site x for which R_x occurs and wake up the

[1]However, our results do not cover models where the numbers of frogs per site are inhomogeneous such as, for example, in [Pop01]. See also Counterexample 21 below.

[2]The notion of transitive Markov chains can be found, e.g., in [Woe00, p. 13], [LPW09, Section 2.6.2], [LP, Definition 10.22], and already in [D65, Chapter X, §6].

frogs at x. This will trigger infinitely many frogs to visit x and therefore also v so that R_v occurs as well.

On $\mathbb{Z}^d$, $d \geq 2$, or in a more general setting the proof is not that simple since it is not obvious that any woken up frog will a.s. hit a site x for which R_x occurs, if there are such sites. Since the occurrence of the event R_x may depend on trajectories of all frogs, we need to introduce an "extra frog" and use properties of the environment viewed from the extra frog to prove that the extra frog hits a site x for which R_x occurs. After that, we have to get rid of the extra frog.

In our second corollary the frogs are not independent under P. We consider random walks in a common environment of random conductances, see [Bis11] for a survey of this model. Let $d \geq 2$ and assign to each undirected edge $\{x, y\}$ between nearest neighbors $x, y \in \mathbb{Z}^d$ a non-negative random variable $c(\{x, y\})$, called the *conductance* of this edge. Let $q(x) := \sum_{y:\|y-x\|_1=1} c(\{x, y\})$ and set $\varkappa(x, y) := c(\{x, y\})/q(x)$ for nearest neighbors x and y if $q(x) \neq 0$, $\varkappa(x, x) = 1$ if $q(x) = 0$, and $\varkappa(x, y) := 0$ in all other cases. Then $\varkappa$ is a random stochastic matrix. We say that x and y are *connected* in environment c if there is a nearest neighbor path connecting x and y along which all conductances are strictly positive. Let $\mathcal{C}(x)$ be the (random) cluster consisting of x and all points of V which are connected to x in the environment c. Note that due to our definition of recurrence only points in an infinite cluster can be recurrent.

Theorem 2 (**Random walks among random conductances**). *Assume that the family $(c(\{x, y\}))_{x,y\in\mathbb{Z}^d}$ is stationary and ergodic w.r.t. the shifts on $\mathbb{Z}^d$, $E[q(0)] < \infty$, and that there is a.s. at most one infinite cluster.*[3] *Given $(c(\{x, y\}))_{x,y}$ let the frog trajectories $(S_j(x, i))_{j\geq 0}$, $x \in \mathbb{Z}^d$, $i \geq 1$, be independent nearest-neighbor Markov chains with transition matrix $\varkappa$. Let the numbers $\eta(x)$, $x \in V$, of sleeping frogs be i.i.d. and independent of the conductances and the frog trajectories. Then either with probability 1 every $x \in V$ is transient or with probability 1 every $x \in V$ with $\eta(x) \geq 1$ and $\#\mathcal{C}(x) = \infty$ is recurrent.*

Remark 3. The recurrence question for a frog model is non-trivial only when the individual frogs in Theorem 2 are transient. For example, the simple random walk on the infinite Bernoulli percolation cluster is transient starting with $d \geq 3$ ([GKZ93]). In the case when conductances are i.i.d., the individual frogs in Theorem 2 are transient as soon as the simple random walk on the infinite cluster is transient ([PP96]). □

Among the first results on the frog model was [TW99, Theorem 5], which states that the frog process on $\mathbb{Z}^d$ is recurrent regardless of the dimension d if there is initially one frog per site and the frogs perform independent simple symmetric random walks. Since this fits into the setting of Theorem 2 (with $c(\{x, y\}) = 1$ and $\eta(x) = 1$) it raises the following question:

[3] The last condition is satisfied, for instance, if $(\mathbf{1}_{c(\{x,y\})>0})_{x,y}$ is i.i.d..

Problem 4. Is the model in Theorem 2 always recurrent, i.e. is it always true that $P[R_0] = P[\eta(0) \geq 1, \#\mathcal{C}(0) = \infty]$?

Finally, we obtain a sufficient recurrence condition for a large class of models on $\mathbb{Z}^d$ with independent frogs. The proof uses our general zero-one law, Theorem 13.

Theorem 5. *Let $d \geq 1$ and $\varepsilon \in (0, 1/(2d)]$ be fixed. Suppose that the frog trajectories $(S_j(x,i))_{j\geq 0}$, $x \in \mathbb{Z}^d, i \geq 1$, satisfy*

$$P[S_{j+1}(x,i) = S_j(x,i) + e \mid (S_k(x,i))_{0\leq k\leq j}] \geq \varepsilon$$

for all $j \geq 0$ and unit vectors $e \in \mathbb{Z}^d$. Let the numbers $\eta(x)$, $x \in V$, of sleeping frogs be identically distributed and assume that the quantities $\eta(x), (S_j(x,i))_{j\geq 0}$, $x \in V, i \geq 1$, are independent. Then there is a constant c_1 which depends only on d and ε such that if

$$P[\eta(0) \geq t] \geq \frac{c_1}{(\log t)^d} \tag{1}$$

for all large t then with probability 1 every $x \in \mathbb{Z}^d$ with $\eta(x) \geq 1$ is recurrent. For $d = 1$ the tail condition (1) can be replaced by the weaker moment assumption $E\left[\log_+ \eta(0)\right] = \infty$.

In Proposition 26 below we show that in the case when the frogs have a drift away from the origin the condition

$$E\left[(\log_+ \eta(0))^d\right] = \infty \tag{2}$$

is necessary for recurrence. Since (1) is only slightly stronger than (2), the condition (1) is close to being sharp, see also Problem 25.

Remark 6. In the case when $S(x,i)$ is a nearest neighbor random walk with a non-zero drift in the first coordinate direction, recurrence results were previously obtained in [GS09] ($d = 1$) and [DP14] ($d \geq 1$). Theorem 2.1 of [DP14] states that if

$$E\left[(\log_+ \eta(0))^{\frac{d+1}{2}}\right] = \infty \tag{3}$$

then the frog model is recurrent. It was proved previously in [GS09, Theorem 2.2] that for $d = 1$ the condition (3) is necessary and sufficient for recurrence. Theorem 5 shows that the frog model can be recurrent even if the expectation in (3) is finite, demonstrating that (3) is not necessary for recurrence when $d \geq 2$. $\square$

Organization of the paper. In Section 2 we introduce the general frog model and state our main result Theorem 13. The proof of this theorem is given in Section 3. Section 4 discusses a number of applications of Theorem 13, in particular, it contains the proofs of Theorems 1 and 2. Theorem 5 and its partial converse are proven in Section 5. Appendix A discusses the above mentioned conjecture from [GS09]. Technical results are collected in Appendices B and C.

2. The frog model and the main result

Let V be a countably infinite set. A frog configuration (n, s) on V consists of

- $n = (n(x))_{x\in V}$, where $n(x) \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ represents the number of frogs at site x indexed by $i = 1, 2, \dots, n(x)$ which can be woken up and will be called *initial frogs*;
- a collection of paths $s = (s_j(x, i))_{j\geq 0, x\in V, i\geq 1}$, where $s_j(x, i) \in V$ denotes the position of the i-th frog originating at x after j steps. We assume that $s_0(x, i) = x$.

The set of all frog configurations is denoted by $\mathbb{F} \subseteq \mathbb{N}_0^V \times V^{\mathbb{N}_0\times V\times\mathbb{N}}$. Next we define recurrence of a site $v \in V$ for a given configuration $(n, s) \in \mathbb{F}$. Set $W_0^v(n, s) := \{v\}$ and define recursively for $j \geq 1$,

$$W_j^v(n,s) := \Big\{x \in V \setminus \bigcup_{k=0}^{j-1} W_k^v(n,s) \,\Big|\, \exists k < j,\ y \in W_k^v(n,s),\ 1 \leq i \leq n(y) : \\ s_{j-k}(y,i) = x\Big\}$$

and let $W^v(n, s) := \bigcup_{j\geq 0} W_j^v(n, s)$. Note that if at time 0 we wake up the initial frogs at site v then $W_j^v(n, s)$ is the set of sites visited at time j for the first time by an active frog and $W^v(n, s)$ is the set of sites ever visited by an active frog provided that $n(v) \geq 1$.

Definition 7. *A site $v \in V$ is said to be **recurrent** for a frog configuration $(n, s) \in \mathbb{F}$ if there are infinitely many distinct $x \in W^v(n, s)$ for which there are $i \in \{1, \dots, n(x)\}$ and $j \geq 0$ such that $s_j(x, i) = v$. Otherwise v is called **transient** for (n, s).*

Note that if $n(v) = 0$ then v by the definition is transient for (n, s).

Remark 8. There are several possible ways to define recurrence of a site v w.r.t. a frog configuration. Our definition appears to be the most restrictive, since we take into account only visits by active frogs originating at distinct sites. It might seem more natural to count the number of times when v is occupied by any active frog. Our choice is based on two reasons: (i) in essentially all cases of interest our definition is equivalent to less restrictive ones (see, for example, the discussion in Appendix A); (ii) it allows us to consider stopped frog trajectories (see (4)) on the same state space without deactivating or removing the "stopped frogs" from the system. □

We consider random frog configurations. Equip $\mathbb{F}$ with its standard σ-field. Let $(\Omega, \mathcal{F}, P)$ be a probability space and (η, S) be an $\mathbb{F}$-valued random variable. We write $\eta = (\eta(x))_{x\in V}$ and $S = (S_j(x, i))_{j\geq 0, x\in V, i\geq 1}$, where $\eta(x)$ takes values in $\mathbb{N}_0$

and $S_j(x,i)$ in V with $S_0(x,i)=x$. For our main result Theorem 13 we shall need several auxiliary random variables[4] on $(\Omega,\mathcal{F},P)$, namely,

- V-valued random variables $S_j(x,0)$, $j\ge 0, x\in V$, denoting the position at time j of the so-called *extra frog* starting at site $S_0(x,0)=x$. We denote by $S^0 := (S_j(x,i))_{j\ge 0, x\in V, i\ge 0}$ the collection of frog trajectories including those of the extra frogs. The random variable (η,S^0) takes values in $\mathbb{F}^0\subseteq \mathbb{N}_0^V\times V^{\mathbb{N}_0\times V\times\mathbb{N}_0}$.
- a family $\varkappa=(\varkappa(x,y))_{x,y\in V}$ of $[0,1]$-valued random variables, called *ellipticity variables*, which will provide bounds on transition probabilities (see assumption (EL) below). We say that $y\in V$ *can be reached from* $x\in V$ if there exist an $m\ge 0$ and a sequence $(x=x_0,x_1,\dots,x_m=y)\in V^{m+1}$ such that $\varkappa(x_{i-1},x_i)>0$ for all $i=1,\dots,m$. We call $x,y\in V$ *equivalent w.r.t.* $\varkappa$ if y can be reached from x and x from y. The equivalence class of $v\in V$ w.r.t. $\varkappa$ is denoted by $\mathcal{C}(v)$ and is called the *cluster* of v.
- a family $T=(T(x,i))_{x\in V,i\ge 1}$ of $\mathbb{N}_0\cup\{\infty\}$-valued random variables. We think of $T(x,i)$ as the time at which the i-th frog originating at x is stopped. However, we do not require $T(x,i)$ to be a stopping time.
- a family $X=(X_v)_{v\in V}$ of V-valued random variables representing the choice of the extra frog which will be used to examine the recurrence/transience of the site v.

For $i,j\ge 0$ and $v\in V$ define the σ-field

$$\begin{aligned}\mathcal{F}_j(v,i) := \sigma\Big(&\varkappa, T,\eta, S_m(x,k);\\ &(m,(x,k))\in\Big(\mathbb{N}_0\times((V\times\mathbb{N})\backslash\{(v,i)\})\Big)\cup\Big(\{0,\dots,j\}\times\{(v,i)\}\Big)\Big).\end{aligned}$$

If $i\ge 1$ then $\mathcal{F}_j(v,i)$ contains all information about $\varkappa$, T, and η, as well as the information about the non-extra frogs with the exception of the steps past j of the i-th frog at v. Note that if $i=0$ then in addition to all information about $\varkappa$, T, η, and all non-extra frogs the σ-field $\mathcal{F}_j(v,i)$ contains the information about the first j steps of the extra frog at v.

Next we state several conditions on the distribution of $(\varkappa,T,X,\eta,S^0)$ under P. Our first assumption is a mild ellipticity condition. It relates the non-extra frog trajectories and the ellipticity variables.

(EL) For all $v,y\in V,\ i\ge 1$, and $j\ge 0$, P-a.s.
$$\varkappa(S_j(v,i),y)\le P\left[S_{j+1}(v,i)=y\,|\,\mathcal{F}_j(v,i)\right]\le \mathbf{1}_{\varkappa(S_j(v,i),y)>0}.$$

Lemma 9. *For $v\in V$ let R_v be the event that v is recurrent for (η,S). If* (EL) *holds then* $P[R_v\backslash\{\eta(v)\ge 1,\ \#\mathcal{C}(v)=\infty\}]=0$.

[4] In many applications some of these variables are trivial, i.e. $\mathcal{C}(v)=V$, $T(v,i)=\infty$, or $X_v=v$ a.s. for all $v\in V$, $i\ge 1$. But see Theorem 2 and the proofs of Theorem 5 and Theorem 16 where at least one set of these extra variables is non-trivially defined and plays an important role.

Proof. As already noted after Definition 7, $R_v \subseteq \{\eta(v) \geq 1\}$. Moreover, by the upper bound in (EL), a.s. every point in $W^v(\eta, S)$ can be reached from v and if R_v occurs then, by Definition 7, v can be reached from infinitely many points of $W^v(\eta, S)$. Hence R_v is a.s. contained in $\{\#\mathcal{C}(v) = \infty\}$. □

Definition 10. *We say that the frog process satisfies the* ***zero-one law for recurrence and transience*** *if* $P[R_v] = 0$ *for all* $v \in V$ *or* $P[R_v] = P[\eta(v) \geq 1,\ \#\mathcal{C}(v) = \infty]$ *for all* $v \in V$.

Remark 11. By Lemma 9 the zero-one law holds iff either for all $v \in V$, $P[R_v] = 0$ or for all $v \in V$ for which $P[\eta(v) \geq 1,\ \#\mathcal{C}(v) = \infty] > 0$ we have $P[R_v \,|\, \eta(v) \geq 1,\ \#\mathcal{C}(v) = \infty] = 1$. This justifies the name zero-one law. □

Our next assumption is the uniqueness of the infinite cluster. Together with (EL) it can be interpreted as an irreducibility assumption.

(UC) There is P-a.s. at most one infinite equivalence class w.r.t. $\varkappa$.

Stopping the frog trajectories $S_\cdot(x, i)$ at the respective time $T(x, i)$ we obtain

$$\bar{S} := (\bar{S}_j(x,i))_{j\geq 0, x\in V, i\geq 1} := (S_{j\wedge T(x,i)}(x,i))_{j\geq 0, x\in V, i\geq 1}. \tag{4}$$

Denote by $\bar{R}_v$ the event that $v \in V$ is recurrent for $(\eta, \bar{S})$. Obviously, $\bar{R}_v \subseteq R_v$. The next assumption has to do with a partial converse.

(T) For all $v \in V$, $P[R_v] > 0$ implies $P[\bar{R}_v] > 0$.

Note that (T) is void if $T \equiv \infty$.

To introduce our somewhat non-standard ergodicity assumption define for each $\varphi \in \mathrm{Sym}(V)$, i.e. permutation φ of V, the function $\theta_\varphi : \mathbb{F} \to \mathbb{F}$ by

$$\theta_\varphi(n, s) := \Big((n(\varphi(x)))_{x\in V}\,, \big(\varphi^{-1}(s_j(\varphi(x), i))\big)_{j\geq 0, x\in V, i\geq 1} \Big) \in \mathbb{F}. \tag{5}$$

Denote by $\mathcal{I}$ the σ-field of all $A \in \mathcal{F}$ for which there is a measurable set B such that $A \overset{P}{=} \{\theta_\varphi(\eta, \bar{S}) \in B\}$ for all $\varphi \in \mathrm{Sym}(V)$, where $A \overset{P}{=} C$ means that $P[A \triangle C] = 0$. Such A is called almost invariant (w.r.t. $\mathrm{Sym}(V)$). Then the ergodicity assumption[5] reads as follows:

(ERG) $P[A] \in \{0, 1\}$ for all $A \in \mathcal{I}$.

So far the distribution of the extra frogs has not played any role yet. The following assumption relates the extra frog and the first frog at $v \in V$.

(EX) For all $v \in V$ the paths $(S_j(v, 0))_{j\geq 0}$ and $(S_j(v, 1))_{j\geq 0}$ are i.i.d. given $\mathcal{F}_0(v, 1)$.

Remark 12. Note that (EL), (UC), (T), and (ERG) are conditions on the distribution of $(\varkappa, T, \eta, S)$. We claim that one can construct a probability space with random variables $\varkappa, T, \eta$ and S^0 on which $(\varkappa, T, \eta, S)$ has the same distribution as in

[5]Note that we do not require the distribution of $(\eta, \bar{S})$ to be invariant w.r.t. θ_φ for $\varphi \in \mathrm{Sym}(V)$.

the original model and, in addition, (EX) holds. In this sense (EX) can be assumed without loss of generality. We shall prove this claim in Appendix C. □

The next assumption is obviously satisfied in the most common case $X_v \equiv v$.

(Xv) For all $v \in V$, X_v and $(\varkappa, T, \eta, S^0)$ are independent and $P[X_v = v] > 0$.

Now we state the final and crucial assumption. Let $o : V \to V$ be such that $o \circ o = o$. We call $o(x)$ the representative of $x \in V$.[6] For each $x \in V$ fix some $\varphi_x \in \mathrm{Sym}(V)$ such that $\varphi_x(o(x)) = x$. We call

$$Z(x) := \left(\left(\mathbf{1}_{\bar{R}_{\varphi_x(y)}} \right)_{y \in V}, \left(\varkappa\left(\varphi_x(y), \varphi_x(z)\right)\right)_{y,z \in V}, o(x) \right) \tag{6}$$

the *environment viewed from the vertex* $x \in V$. We assume the existence of equivalent measures under which the environment viewed from the extra frogs always looks the same[7].

(ID) For each $v \in V$ with $P[\#\mathcal{C}(X_v) = \infty] > 0$ there is a probability measure P_v which is equivalent to (i.e. mutually absolutely continuous with) $P[\,\cdot\,|\,\#\mathcal{C}(X_v) = \infty]$ such that $Z\left(S_j(X_v, 0)\right), j \geq 0$, is identically distributed under P_v.

Our main result is the following.

Theorem 13 (**Zero-one law**). *Assume* (EL), (UC), (T), (ERG), (EX), (Xv), *and* (ID). *Then the zero-one law for recurrence and transience holds.*

3. Proof of Theorem 13

Lemma 14. *Let $\varphi \in \mathrm{Sym}(V)$, $v \in V$, and $(n, s) \in \mathbb{F}$. Then $\varphi(v)$ is recurrent for (n, s) iff v is recurrent for $\theta_\varphi(n, s)$.*

Proof. For $x, y \in V$ and $(\widetilde{n}, \widetilde{s}) \in \mathbb{F}$ let $I_y^x(\widetilde{n}, \widetilde{s}) := 1$ if there exist $j \geq 0$ and $1 \leq i \leq \widetilde{n}(y)$ such that $\widetilde{s}_j(y, i) = x$. Otherwise, set $I_y^x(\widetilde{n}, \widetilde{s}) := 0$. Then by Definition 7,

$$x \text{ is recurrent for } (\widetilde{n}, \widetilde{s}) \quad \text{iff} \sum_{y \in W^x(\widetilde{n}, \widetilde{s})} I_y^x(\widetilde{n}, \widetilde{s}) = \infty. \tag{7}$$

[6] For a frog model on a transitive graph, $o(x) \equiv$ const representing a reference point such as 0 for $\mathbb{Z}^d$ or just an arbitrary fixed vertex. In general, when V naturally splits into orbits under the action of some subgroup of $\mathrm{Sym}(V)$, $o(x)$ designates a reference point for each orbit. See Theorem 16 and Example 18.

[7] Note that (ID) is weaker than the more common assumption that the environment viewed from the particle (here the extra frog) is stationary. See the proof of Theorem 5 where we have an example where the latter assumption is not fulfilled but (ID) is. Counterexamples 19 and 24 show that in Theorem 13 the assumption (ID) cannot be replaced by the assumption that the environment is stationary w.r.t. to the canonical spatial shifts of the state space V.

By induction over j we see that $W_j^{\varphi(v)}(n,s) = \varphi\left(W_j^v\left(\theta_\varphi(n,s)\right)\right)$ for all $j \geq 0$ and, consequently, $W^{\varphi(v)}(n,s) = \varphi\left(W^v\left(\theta_\varphi(n,s)\right)\right)$. Moreover, $I_y^{\varphi(v)}(n,s) = I^v_{\varphi^{-1}(y)}(\theta_\varphi(n,s))$. Thus, by (7) the statement that $\varphi(v)$ is recurrent for (n,s) is equivalent to

$$\sum_{\varphi^{-1}(y)\in W^v(\theta_\varphi(n,s))} I^v_{\varphi^{-1}(y)}(\theta_\varphi(n,s)) = \infty.$$

However, this is equivalent, again due to (7), to the recurrence of v for $\theta_\varphi(n,s)$. □

Lemma 15. *If* (EL) *and* (EX) *hold then for all* $v, y \in V$ *and* $j \geq 0$,

$$P\left[S_{j+1}(v,0) = y \,|\, \mathcal{F}_j(v,0)\right] \geq \varkappa(S_j(v,0), y) \quad P\text{-a.s.}.$$

We postpone the proof of Lemma 15 to Appendix C.

Proof of Theorem 13. We assume that there is a $u \in V$ such that $P[R_u] > 0$ since otherwise there is nothing to prove. We need to show that for all $v \in V$,

$$P[R_v] = P[\eta(v) \geq 1,\, \#\mathcal{C}(v) = \infty]. \tag{8}$$

Fix $v \in V$. Let

$$P[\#\mathcal{C}(v) = \infty] > 0 \tag{9}$$

since (8) is trivial otherwise due to Lemma 9. We shall prove (8) in five steps.

Step 1: For a.e. realization, the probability to reach a recurrent site from a fixed site x of the infinite cluster is positive. Define for all $x \in V$ the random variable

$$D(x) := \sup\left\{\frac{\mathbf{1}_{\bar{R}_{x_m}}}{m+1}\prod_{i=1}^m \varkappa(x_{i-1}, x_i) \mid m \geq 0, x = x_0, x_1, \ldots, x_m \in V\right\},$$

where $\sup\emptyset := 0$ and $\prod_{i=1}^0 a_i := 1$. Note that $D(x) = 1$ iff the supremum in the definition above is attained for $m = 0$, i.e. iff x is recurrent for $(\eta, \bar{S})$. The random variable $D(x)$ will serve as a lower bound on the probability that an extra frog currently located at x will ever reach a site y which is recurrent for $(\eta, \bar{S})$. We shall show that for all $x \in V$ such that $P[\#\mathcal{C}(x) = \infty] > 0$,

$$P[D(x) > 0 \,|\, \#\mathcal{C}(x) = \infty] = 1. \tag{10}$$

Since $P[R_u] > 0$ we have by assumption (T) that $P[\bar{R}_u] > 0$. Therefore, also $P[\bar{R}] > 0$, where $\bar{R} := \bigcup_{w\in V}\bar{R}_w$. Due to Lemma 14, $\bar{R} \in \mathcal{I}$. Consequently, by (ERG), $P[\bar{R}] = 1$. Hence there is a.s. a $w \in V$ for which $\bar{R}_w$ occurs and for which by Lemma 9, $\mathcal{C}(w)$ is infinite. If $\mathcal{C}(x)$ is infinite then by (UC) a.s. $\mathcal{C}(x) = \mathcal{C}(w)$ and hence there is a path connecting x to w along which all ellipticity variables $\varkappa$ are strictly positive. Multiplying these variables yields (10).

Step 2: A modification of (10) "as seen from the extra frog" holds. More precisely, we shall show that

$$P\left[\limsup_{j\to\infty} D(S_j(v,0)) > 0 \;\middle|\; \#C(v) = \infty\right] = 1. \tag{11}$$

By (Xv), X_v and $(\#\mathcal{C}(x), D(x))_{x\in V}$ are independent and $P[X_v = v] > 0$. Hence, due to (9), $P[\#\mathcal{C}(X_v) = \infty] > 0$ and, from (10), $\widetilde{P}_v[D(X_v) > 0] = 1$, where $\widetilde{P}_v := P[\,\cdot \mid \#\mathcal{C}(X_v) = \infty]$. Therefore, since $P_v \ll \widetilde{P}_v$ by (ID),

$$P_v[D(X_v) > 0] = 1. \tag{12}$$

Define $f : \{0,1\}^V \times [0,1]^{V^2} \times V \to [0,1]$ by

$$f\left((r_x)_{x\in V}, (a_{x,y})_{x,y\in V}, y_0\right) := \sup\left\{ \frac{r_{y_m}}{m+1}\prod_{i=1}^m a_{y_{i-1},y_i} \;\Big|\; m \geq 0, y_1,\ldots,y_m \in V\right\}.$$

Then for all $x \in V$, by definition (6),

$$f(Z(x)) = \sup\left\{ \frac{\mathbf{1}_{\bar{R}_{\varphi_x(y_m)}}}{m+1}\prod_{i=1}^m \varkappa(\varphi_x(y_{i-1}),\varphi_x(y_i)) \;\Big|\; m \geq 0, o(x) = y_0, y_1,\ldots,y_m \in V\right\}.$$

Replacing $\varphi_x(y_i)$ with x_i and recalling that $\varphi_x(o(x)) = x$ we obtain $D(x) = f\left(Z(x)\right)$. Therefore, (ID) implies that $(D(S_j(X_v,0)))_{j\geq 0}$ is identically distributed under P_v. Consequently, by continuity,

$$\begin{aligned}
P_v\left[\limsup_{j\to\infty} D(S_j(X_v,0)) > 0\right] &= \lim_{K\to\infty}\lim_{J\to\infty} P_v\left[\sup_{j\geq J} D(S_j(X_v,0)) > 1/K\right] \\
\geq \lim_{K\to\infty}\limsup_{J\to\infty} P_v\left[D(S_J(X_v,0)) > 1/K\right] &= \lim_{K\to\infty} P_v\left[D(S_0(X_v,0)) > 1/K\right] = 1
\end{aligned}$$

due to $S_0(X_v,0) = X_v$ and (12). Therefore, $\widetilde{P}_v$-a.s. $\limsup_{j\to\infty} D(S_j(X_v,0)) > 0$ because $\widetilde{P}_v \ll P_v$. Due to (Xv) and $P[X_v = v] > 0$ this implies (11).

Step 3: The extra frog which starts in the infinite cluster will hit a recurrent site. We shall argue that

$$P[\exists j \geq 0\ D(S_j(v,0) = 1 \mid \#\mathcal{C}(v) = \infty] = 1. \tag{13}$$

Define for $K \geq 1$, $\tau_{0,K} := -K$ and then recursively for $m \geq 1$,

$$\tau_{m,K} \;:=\; \inf\{j \geq \tau_{m-1,K} + K \mid D(S_j(v,0)) > 1/K\}.$$

Note that $\tau_{m,K}$ is a stopping time w.r.t. the filtration $(\mathcal{F}_j(v,0))_{j\geq 0}$. Define the events

$$A_{m,K} := \bigcap_{j=0}^{K-1}\left\{D\left(S_{\tau_{m,K}+j}(v,0)\right) < 1, \tau_{m+1,K} < \infty\right\} \in \mathcal{F}_{\tau_{m+1,K}}(v,0).$$

Roughly speaking, on the event $A_{m,K}$ the extra frog from site v is at time $\tau_{m,K}$ not too far from a recurrent site, but still does not reach any recurrent site within the next $K-1$ steps. Then for all $M \geq 0$ and $K \geq 2$,

$$P\left[\bigcap_{m=1}^M A_{m,K}\right] = E\left[P\left[A_{M,K} \mid \mathcal{F}_{\tau_{M,K}}(v,0)\right]; \bigcap_{m=1}^{M-1} A_{m,K}\right]. \tag{14}$$

On the event $A_{M-1,K}$ we have $\tau_{M,K} < \infty$ and therefore $D\left(S_{\tau_{M,K}}(v,0)\right) > 1/K$. Consequently, there is a path of length less than K which starts at $S_{\tau_{M,K}}(v,0)$ and ends at a recurrent site and along which the product of the ellipticity variables is larger than $1/K$. By repeated application of Lemma 15 we conclude that the probability that the extra frog originating at v does not reach a recurrent site in the next $K-1$ steps after time $\tau_{M,K}$ is at most $1-1/K$. Hence, $P\left[A_{M,K}\,|\,\mathcal{F}_{\tau_{M,K}}(v,0)\right] \leq 1-1/K$. Induction over M then yields that the expression in (14) can be estimated from above by $(1-1/K)^M$. By letting $M \to \infty$ we obtain for all $K \geq 1$,

$$P\left[\bigcap_{m\geq 1} A_{m,K}\right] = 0. \tag{15}$$

By (11) and continuity,

$$\begin{aligned}
&P\left[\forall j \geq 0\ D(S_j(v,0)) < 1 \mid \#C(v) = \infty\right] \\
&\quad = \lim_{K\to\infty} P\left[\forall j \geq 0\ D(S_j(v,0)) < 1,\ \limsup_{k\to\infty} D(S_k(v,0)) > 1/K \;\middle|\; \#C(v) = \infty\right] \\
&\quad \leq \liminf_{K\to\infty} P\left[\bigcap_{m\geq 1} A_{m,K} \;\middle|\; \#C(v) = \infty\right] \overset{(15)}{=} 0.
\end{aligned}$$

This implies (13).

Step 4: Adding the extra frog to a site in the infinite cluster makes that site recurrent. By Step 3 we have that if $\mathcal{C}(v)$ is infinite then waking up the initial frogs at v and the extra frog at v results P-a.s. in waking up the frogs in at least one (random) site z which is recurrent for $(\eta, \bar{S})$ and hence also recurrent for (η, S). This in turn causes frogs from infinitely many distinct sites to visit z. Due to the lower bound in (EL) the same holds a.s. true for any other point in $\mathcal{C}(z)$ as well. The site v is such a point since $\mathcal{C}(z)$ is infinite by Lemma 9 and therefore identical to $\mathcal{C}(v)$ due to (UC).

To phrase this conclusion more precisely, consider for $(n, s^0) \in \mathbb{F}^0$, $n = (n(x))_{x\in V}$, $s^0 = (s_j(x,i))_{i,j\geq 0, x\in V}$, the configuration

$$a_v(n, s^0) := \left(\left(n(x) + \mathbf{1}_{x=v}\right)_{x\in V}, \left(s_j(x, i - \mathbf{1}_{x=v})\right)_{j\geq 0, x\in V, i\geq 1}\right) \in \mathbb{F},$$

which we get by adding the extra frog at v to the set of initial frogs. We have shown above that

$$P\left[v \text{ is recurrent for } a_v(\eta, S^0) \,|\, \#\mathcal{C}(v) = \infty\right] = 1. \tag{16}$$

Step 5: Removing the extra frog. Consider for $(n, s^0) \in \mathbb{F}^0$ also the configuration

$$t_v(n, s^0) := \left(n, \left(s_j(x, i - \mathbf{1}_{x=v,i=1})\right)_{j\geq 0, x\in V, i\geq 1}\right) \in \mathbb{F},$$

which we get by replacing the first frog at v by the extra frog at v. Observe that if $n(v) \geq 1$ then for all $j \geq 0$,

$$W_j\left(a_v(n, s^0)\right) \subseteq W_j(n, s) \cup W_j\left(t_v(n, s^0)\right), \tag{17}$$

where $s=(s_j(x,i))_{j\ge 0,x\in V,i\ge 1}$. Indeed, each frog $\mathfrak{f}$ at a site from the set $W_j\left(a_v(n,s^0)\right)$ was either activated at time 0 or was woken up by some frog, its "ancestor" (if $\mathfrak{f}$ was woken up by several frogs simultaneously then we pick any one as its "ancestor"). Following the ancestry line back to the time 0 we can identify which of the frogs woken up at time 0 started this ancestry line. If the line was started by the extra frog at v then $\mathfrak{f}$ is located at a site in $W_j\left(t_v(n,s^0)\right)$, if not then we can say that $\mathfrak{f}$ is at a site from $W_j\left(n,s\right)$.

Therefore, by (17), if v is recurrent for $a_v(n,s^0)$ and $n(v)\ge 1$, then it is recurrent for (n,s) or for $t_v(n,s)$. Hence,

$$\begin{aligned}
&P[\eta(v)=0 \text{ or } \#\mathcal{C}(v)<\infty] \\
&\quad\overset{(16)}{\ge}\quad P[\eta(v)=0 \text{ or } v \text{ is transient for } a_v(\eta,S^0)] \\
&\quad\ge\quad P\left[v \text{ is transient for } (\eta,S) \text{ and for } t_v(\eta,S^0)\right] \\
&\quad=\quad E\left[P\left[R_v^c\cap\{v \text{ is transient for } t_v(\eta,S^0)\}\mid\mathcal{F}_0(v,1)\right]\right] \\
&\quad\overset{\text{(EX)}}{=}\quad E\left[P\left[R_v^c\,|\,\mathcal{F}_0(v,1)\right]P\left[v \text{ is transient for } t_v(\eta,S^0)\,|\,\mathcal{F}_0(v,1)\right]\right] \\
&\quad\overset{\text{(EX)}}{=}\quad E\left[P\left[R_v^c\,|\,\mathcal{F}_0(v,1)\right]^2\right]\;\ge\; P[\eta(v)=0 \text{ or } \#\mathcal{C}(v)<\infty],
\end{aligned}\tag{18}$$

where the last inequality holds since by Lemma 9 a.s.

$$\begin{aligned}
P\left[R_v^c\,|\,\mathcal{F}_0(v,1)\right] &\ge P\left[\eta(v)=0 \text{ or } \#\mathcal{C}(v)<\infty\,|\,\mathcal{F}_0(v,1)\right] \\
&= \mathbf{1}_{\{\eta(v)=0 \text{ or } \#\mathcal{C}(v)<\infty\}}.
\end{aligned}\tag{19}$$

Therefore, the inequalities in (18) are, in fact, identities. Hence we also have P-a.s. equality in (19). Taking expectations yields (8). ☐

4. Examples illustrating the scope of Theorem 13

In this section we give examples of frog processes covered by our main result, Theorem 13, as well as some counterexamples which demonstrate that our assumptions are essential for the validity of the zero-one law. The examples are split into two groups. The first group concerns models where frog dynamics is a (quasi-) transitive Markov chain as well as an example where our general setting goes beyond the chain quasi-transitivity condition. The second group discusses models where the underlying frog dynamics is random walk in random environment so that under the averaged measure the processes are not markovian. For all examples in the first group $\mathcal{C}(v)=V$ for all $v\in V$, while the second group includes examples with degenerate environments such as random walks on the infinite percolation cluster.

4.1. **Transitive Markov chains and more.** In this subsection we assume that $K:V\times V\to[0,1]$ is a stochastic matrix. Let Φ be a subgroup of $\mathrm{Sym}(V)$ such that for all $\varphi\in\Phi$,

$$K\left(\varphi(u),\varphi(v)\right)=K(u,v)\quad\text{for all } u,v\in V.\tag{20}$$

For $x, y \in V$ we say that $x \sim y$ iff there is a $\varphi \in \Phi$ such that $\varphi(x) = y$. Since Φ is a group, $\sim$ is an equivalence relation. Let $[x] := \{\varphi(x) \,|\, \varphi \in \Phi\}$ be the equivalence class of $x \in V$, also called the *orbit* of x. If $(X_n)_{n\geq 0}$ is a Markov chain with transition matrix K then $([X_n])_{n\geq 0}$, the so-called *factor chain* (cf. [Woe00, (1.31)]), is a Markov chain with state space $\widetilde{V} := \{[x] \,|\, x \in V\}$ and transition matrix

$$\widetilde{K}([x], [y]) := \sum_{z\in[y]} K(x, z).$$

Theorem 16. *Let K be irreducible, all orbits $[x]$, $x \in V$, be infinite, and $\widetilde{K}$ have an invariant probability measure $\widetilde{\mu}$. Assume that the frog trajectories $(S_j(x, i))_{j\geq 0}$, $x \in V$, $i \geq 1$, are Markov chains with common transition matrix K. Suppose that $\eta(x)$ and $\eta(y)$ have the same distribution whenever $x \sim y$ and that the quantities $\eta(x), (S_j(x, i))_{j\geq 0}$, $x \in V$, $i \geq 1$, are independent. Then the zero-one law for recurrence and transience holds.*

Proof. We check the assumptions of Theorem 13. Assumption (EL) holds with $\varkappa(x, y) := K(x, y)$. Since K is irreducible, (UC) is satisfied as well. We set $T \equiv \infty$ so that (T) holds trivially.

For the proof of (ERG) we shall use Proposition 27 and consider the independent random variables $H(x) := \Big(\eta(x), (S_j(x, i))_{j\geq 0, i\geq 1}\Big)$, $x \in V$. For $\varphi \in \Phi$, $m \geq 0$, and $(s_j(i))_{j\geq 0, i\geq 1} \in V^{\mathbb{N}_0\times\mathbb{N}}$ define $g_\varphi(m, (s_j(i))_{j\geq 0, i\geq 1}) := (m, (\varphi^{-1}(s_j(i)))_{j\geq 0, i\geq 1})$. Since $\varphi \in \mathrm{Sym(V)}$, the random variables $H_\varphi(x) := g_\varphi(H(\varphi(x))$, $x \in V$, are independent as well. Moreover, for all $\varphi \in \Phi, x \in V$, and $i \geq 1$, $(\varphi^{-1}(S_j(\varphi(x), i)))_{j\geq 0}$ is a Markov chain starting at x with transition matrix K since for all $y \in V$,

$$P[\varphi^{-1}(S_j(\varphi(x), i)) = y] = P[S_j(\varphi(x), i)) = \varphi(y)] = K^j(\varphi(x), \varphi(y)) = K^j(x, y).$$

Therefore, due to independence,

$$(H_\varphi(x))_{x\in V} = \theta_\varphi(\eta, S) \overset{d}{=} (\eta, S) = (H(x))_{x\in V}, \tag{21}$$

and hence (35). By Proposition 27 we obtain (ERG).

To satisfy assumption (EX) let the extra frogs $(S_j(v, 0))_{j\geq 0}$, $v \in V$, be Markov chains with transition matrix K which are independent of (η, S).

For the remaining conditions let $\mathcal{O} \subseteq V$ be a complete set of representatives of $\sim$ and $o : V \to \mathcal{O}$ be a map which assigns to $x \in V$ its representative $o(x) \in [x]$. For all $x \in V$ choose $\varphi_x \in \Phi$ so that $\varphi_x(o(x)) = x$. We let X_v, $v \in V$, be independent of (η, S^0) so that $P[X_v = r] = \widetilde{\mu}([r])$ for all $r \in \{v\} \cup \mathcal{O}\backslash\{o(v)\}$. Note that the requirement $P[X_v = v] > 0$ is fulfilled since $\widetilde{K}$ is irreducible since K is so. Therefore (Xv) holds.

Finally, we shall check (ID). Fix $v \in V$ and set $P_v := P$. Then for all measurable $B \subseteq \{0,1\}^V, C \subseteq [0,1]^{V^2}$, $D \subseteq \mathcal{O}$, and all $j \geq 0$,

$$
\begin{aligned}
&P\left[Z\left(S_j(X_v,0)\right) \in B \times C \times D\right] \\
&\quad = \sum_{x \in V} P\left[\left(\mathbf{1}_{R_{\varphi_x(y)}}\right)_{y \in V} \in B, \left(\varkappa\left(\varphi_x(y), \varphi_x(z)\right)\right)_{y,z \in V} \in C, S_j(X_v,0) = x, o(x) \in D\right] \\
&\quad = \sum_{x \in V} P\left[\left(\mathbf{1}_{R_{\varphi_x(y)}}\right)_{y \in V} \in B\right] \mathbf{1}_C\left(\left(\varkappa(y,z)\right)_{y,z \in V}\right) P\left[S_j(X_v,0) = x, o(x) \in D\right]
\end{aligned}
$$

by (20) and independence of (η, S) from $(X_v, S_\cdot(\cdot,0))$. Due to Lemma 14 and (21), the above expression is equal to

$$
P\left[(\mathbf{1}_{R_v})_{v \in V} \in B\right] \mathbf{1}_C\left(\left(\varkappa(y,z)\right)_{y,z \in V}\right) P\left[o\left(S_j(X_v,0)\right) \in D\right].
$$

This does not depend on j since $(o(S_j(X_v,0)))_{j \geq 0}$ is a stationary Markov chain. Therefore, (ID) holds. Theorem 13 now yields the claim. □

Proof of Theorem 1. Since the frog trajectories are assumed to be transitive Markov chains, there is only one orbit. Thus Theorem 1 follows from Theorem 16. □

Example 17 (Quasi-transitive Markov chains). If in the setting of Theorem 16 there are only finitely many orbits, i.e. the factor chain is finite, then there is always an invariant probability measure $\widetilde{\mu}$. Thus the 0–1 law holds.

We remark that in this case the Markov chain (V, K) is called *quasi-transitive* under the action of the group Φ (see [Woe00, pp. 13,14]). As a representative example, consider a periodic model on $\mathbb{Z}^d$. Namely, set $V = \mathbb{Z}^d$, fix a period $L \in \mathbb{N}$, $L \geq 2$, and let Φ be the group of all shifts by vectors from $L\mathbb{Z}^d$, so that $x \sim y$ iff $y - x \in L\mathbb{Z}^d$. Then any frog model with "independent ingredients" such that its frog number distributions and Markovian dynamics are compatible with the periodic structure (as required by Theorem 16) satisfies the 0–1 law for recurrence and transience. □

Our next example shows that the applicability of Theorem 1 goes beyond the quasi-transitive setting.

Example 18 (Frogs on a comb I). Let $V := \mathbb{Z} \times \mathbb{N}_0$ and $p_1, p_2 > 0$ such that $p_1 + p_2 < 1$. Define for $x \in \mathbb{Z}$,

$$
\begin{aligned}
K((x,0),(x+1,0)) &:= p_1, \\
K((x,0),(x-1,0)) &:= 1 - p_1 - p_2, \\
K((x,y),(x,y+1)) &:= p_2 \quad \text{for } y \geq 0, \text{ and} \\
K((x,y),(x,y-1)) &:= 1 - p_2 \quad \text{for } y \geq 1.
\end{aligned}
$$

Let the random quantities $\eta(v), (S_j(v,i))_{j \geq 0}, v \in V,\ i \geq 1$, be independent and assume that for each $y \geq 0$ the random variables $\eta(x,y), x \in \mathbb{Z}$, are identically distributed. Let Φ be the group of shifts "along $\mathbb{Z}$" of the form $(x,y) \mapsto (x +$

$k, y), k \in \mathbb{Z}$. They satisfy (20). The orbits are $\mathbb{Z} \times \{y\}$, $y \geq 0$. The transition matrix for the factor chain is given by

$$\begin{aligned} \widetilde{K}(\mathbb{Z} \times \{y\}, \mathbb{Z} \times \{y+1\}) &= p_2 \qquad \text{and} \\ \widetilde{K}(\mathbb{Z} \times \{y\}, \mathbb{Z} \times \{(y-1) \vee 0\}) &= 1 - p_2. \end{aligned}$$

If $p_2 < 1/2$ then $\widetilde{K}$ admits an invariant probability measure $\widetilde{\mu}$ and Theorem 16 yields the 0–1 law for recurrence and transience. □

Counterexample 19 (Frogs on a comb II). If $p_2 \geq 1/2$ in the setting of Example 18 then there is no invariant probability measure $\widetilde{\mu}$ for $\widetilde{K}$. We shall show that for $p_2 > 1/2$ the 0–1 law can fail. Let $\eta(x, y)$, $x \in \mathbb{Z}$, $y \geq 1$, be identically distributed with $E[\log_+ \eta(x, y)] < \infty$ while $\eta(x, 0)$, $x \in \mathbb{Z}$, be super heavy-tailed so that

$$\liminf_{k \to \infty} k\, P[\log \eta(x, 0) \geq k] > -\log p_1. \tag{22}$$

Then, on the one hand, for each $x \in \mathbb{Z}$ the probability, given η, of the event that all frogs $S((x, y), i), y \geq 0,\ 1 \leq i \leq \eta(x, y)$, stay forever on the tooth $\{x\} \times \mathbb{N}_0$ is bounded from below by

$$\prod_{y \geq 0} (1 - a^{y+1})^{\eta(x,y)} \geq \exp\left(-c_2 \sum_{y \geq 0} \eta(x, y) a^y\right) \tag{23}$$

for suitable constants $0 < a < 1$ and $c_2 > 0$. By Lemma 29 ($d = 1$) the right hand side of (23) is strictly positive. Note that on this event no site of the tooth $\{x\} \times \mathbb{N}_0$ can be visited infinitely many times (otherwise, by the ellipticity, the process would not stay on the tooth). Since x was arbitrary, $P[R_v] < P[\eta(v) \geq 1]$ for all $v \in V$.

On the other hand, $P[R_v] > 0$ for all $v \in V$. To show this, let Z_n, $n \geq 0$, be the number of active or just woken up frogs at site $(n, 0)$ at time n if at time 0 we wake up all initial frogs at $(0, 0)$. As long as it is strictly positive, $(Z_n)_{n \geq 0}$ evolves like a branching process with offspring distribution Bernoulli(p_1) and $\eta(n, 0)$ immigrants at each time n. By [Bau13, Theorem 2.2] and (22) such Markov chain is transient. Hence $P[\forall n \geq 0 : Z_n > 0] > 0$. Consequently, waking up the frogs at $(0, 0)$ results with positive probability in waking up the frogs at all sites $(n, 0), n \geq 0$. Any frog originating at $(n, 0), n \geq 0$, visits $(0, 0)$ after n steps with probability $(1 - p_1 - p_2)^n$. Moreover, since (22) implies $E[\log_+ \eta(x, y)] = \infty$ we obtain from Lemma 29 ($d = 1$) that P-a.s. $\sum_n (1 - p_1 - p_2)^n \eta(n, 0)$ diverges. Therefore, by the second part of the Borel-Cantelli lemma $P[\cdot \,|\, \eta]$-a.s. (and hence also P-a.s.) infinitely many frogs originating at $(n, 0), n \geq 0$, would visit $(0, 0)$ in the shortest possible time, if woken up. We conclude that $P[R_{(0,0)}] > 0$ and therefore, by ellipticity, $P[R_v] > 0$ for all $v \in V$. □

Problem 20. Study the recurrence and transience of the above frog model on the comb for $p_2 \leq 1/2$.

Counterexample 21 (Not identically distributed frog numbers on $\mathbb{Z}^d$). [Pop01, Theorem 1.3 (ii)] provides examples where the frogs perform independent simple symmetric random walks on $V = \mathbb{Z}^d$, $d \geq 3$, and the 0–1 law fails. In these examples, all the assumptions of Theorem 1 are satisfied except that $(\eta(x))_{x\in\mathbb{Z}^d}$ is not identically distributed.

4.2. **Random walk in random environment (RWRE).** In this subsection we consider frogs which jump as independent random walks in a common random environment on $V = \mathbb{Z}^d$ for some $d \geq 1$. Fix $d \geq 1$ and let

$$\Pi := \Big\{(\pi(x,y))_{x,y\in\mathbb{Z}^d} \in [0,1]^{\mathbb{Z}^d\times\mathbb{Z}^d} \mid \forall x \in \mathbb{Z}^d : \sum_{y\in\mathbb{Z}^d} \pi(x,y) = 1\Big\}$$

be the set of all stochastic matrices on $\mathbb{Z}^d$. We endow Π with the standard Borel σ-field $\mathcal{B}(\Pi)$. An element $\pi \in \Pi$ is called a *random walk environment*. A time homogeneous Markov chain on $\mathbb{Z}^d$ with transition matrix π is called a *random walk in the environment* π. For the following we need some more notation.

For all $z \in \mathbb{Z}^d$ we define by $\varphi_z(x) := x + z$ the shift φ_z by z on $\mathbb{Z}^d$. This shift can also be applied to families of the form $f = (f(x))_{x\in\mathbb{Z}^d}$ and $g = (g(x,y))_{x,y\in\mathbb{Z}^d}$ by $\varphi_z(f) := (f(x+z))_{x\in\mathbb{Z}^d}$ and $\varphi_z(g) := (g(x+z,y+z))_{x,y\in\mathbb{Z}^d}$, respectively. It can also be applied to finite sequences $F = (f_1,\dots,f_n)$ of such families by setting $\varphi_z(F) := (\varphi_z(f_1),\dots,\varphi_z(f_n))$. Moreover, such F is called *ergodic w.r.t. the shifts on* $\mathbb{Z}^d$ if $P[A] \in \{0,1\}$ for all events A for which there is a measurable set B such that $A \overset{P}{=} \{\varphi_z(F) \in B\}$ for all $z \in \mathbb{Z}^d$.

The *random walk environment viewed from the particle* is a Markov chain with state space Π and transition kernel K defined by

$$K(\pi,B) := \sum_{z\in\mathbb{Z}^d} \pi(0,z)\mathbf{1}_B\left(\varphi_z(\pi)\right), \qquad \pi \in \Pi, B \in \mathcal{B}(\Pi). \tag{24}$$

Note that if $(S_j)_{j\geq 0}$ is a random walk in the environment π starting at 0, then $(\varphi_{S_j}(\pi))_{j\geq 0}$ is a Markov chain on Π with kernel K starting from π.

For our general result below we need to augment the random walk environment with the numbers of initial frogs at each site. Therefore, we consider $\bar\Pi := \mathbb{N}_0^{\mathbb{Z}^d} \times \Pi$, endow it with its standard Borel σ-field $\mathcal{B}(\bar\Pi)$ and call its elements $\bar\pi = (n,\pi) = \big((n(x))_{x\in\mathbb{Z}^d}, (\pi(x,y))_{x,y\in\mathbb{Z}^d}\big)$ *augmented environments*. The *augmented environment viewed from the particle* is a Markov chain with state space $\bar\Pi$ and transition kernel $\bar K$ defined by

$$\bar K(\bar\pi,\bar B) := \sum_{z\in\mathbb{Z}^d} \pi(0,z)\mathbf{1}_{\bar B}\left(\varphi_z(\bar\pi)\right), \qquad \bar\pi = (n,\pi) \in \bar\Pi, \bar B \in \mathcal{B}(\bar\Pi).$$

As above, if $\bar\pi = (n,\pi) \in \bar\Pi$ and $(S_j)_{j\geq 0}$ is a random walk in the environment π starting at 0, then

$$(\varphi_{S_j}(\bar\pi))_{j\geq 0} \quad \text{is a Markov chain on } \bar\Pi \text{ with kernel } \bar K \text{ starting from } \bar\pi. \tag{25}$$

Theorem 22 (RWRE). *Suppose that the family $\varkappa = (\varkappa(x,y))_{x,y\in\mathbb{Z}^d}$ of ellipticity variables takes values in Π and satisfies* (UC) *with $P[\#\mathcal{C}(0) = \infty] > 0$. Assume that $\bar\varkappa := ((\eta(x))_{x\in\mathbb{Z}^d}, \varkappa)$ is stationary and ergodic w.r.t. the shifts on $\mathbb{Z}^d$. Suppose* *that given $\bar{\varkappa}$, the sequences $(S_j(x,i))_{j\geq 0}, x \in \mathbb{Z}^d,\ i \geq 1$, are independent Markov chains with common transition matrix $\varkappa$. Finally, assume that there is a probability measure $\bar{P}$ on $\bar{\Pi}$ which is invariant w.r.t. $\bar{K}$ and equivalent to the distribution of $\bar{\varkappa}$ under $P[\,\cdot\,|\,\#\mathcal{C}(0) = \infty]$. Then the zero-one law for recurrence and transience holds.*

Proof. We check the assumptions of Theorem 13. (EL) is fulfilled by construction. We let $T \equiv \infty$ and $X_v := v$ for all $v \in \mathbb{Z}^d$ so that (T) and (Xv) trivially hold. Denote by $\mathcal{M}(\mathbb{Z}^d)$ the set of all probability measures on $\mathbb{Z}^d$. Choose $f : \mathcal{M}(\mathbb{Z}^d) \times [0,1] \to \mathbb{Z}^d$ such that for all $\mu \in \mathcal{M}(\mathbb{Z}^d)$ and all random variables X which are uniformly distributed on $[0,1]$, $f(\mu, X)$ has distribution μ. Then we may assume without loss of generality that for all $i, j \geq 0, x \in \mathbb{Z}^d$,

$$S_{j+1}(x,i) = S_j(x,i) + f\left(\left(\varkappa\left(S_j(x,i), S_j(x,i)+y\right)\right)_{y\in\mathbb{Z}^d}, U_j(x,i)\right), \tag{26}$$

where $U_j(x,i),\ i,j \geq 0,\ x \in \mathbb{Z}^d$, are independent and uniformly distributed on $[0,1]$ and independent of $\bar{\varkappa}$. Note that (EX) holds.

We are left to check (ERG) and (ID). For $x \in \mathbb{Z}^d$ let $o(x) := 0 \in \mathbb{Z}^d$ and $U(x) := (U_j(x,i))_{j\geq 0, i\geq 1}$. Set $U := (U(x))_{x\in\mathbb{Z}^d}$. Since U is i.i.d. and independent of $\bar{\varkappa}$, $(U, \bar{\varkappa})$ is stationary and ergodic[8] w.r.t. the shifts on $\mathbb{Z}^d$. Moreover, there is a deterministic measurable function g_1 such that for all $z \in \mathbb{Z}^d$,

$$\theta_{\varphi_z}(\eta, S) \overset{(5)}{=} \left(\varphi_z(\eta), (S_j(x+z,i) - z)_{j\geq 0, x\in\mathbb{Z}^d, i\geq 1}\right) = g_1\left(\varphi_z(U, \bar{\varkappa})\right). \tag{27}$$

Indeed, fix $i \geq 1$ and $x, z \in \mathbb{Z}^d$ and abbreviate $\Delta_j := S_j(x+z,i) - z$ for all $j \geq 0$. Then due to (26) for all $j \geq 0$,

$$\Delta_{j+1} = \Delta_j + f\left(\left(\varkappa\left(\Delta_j + z, \Delta_j + y + z\right)\right)_{y\in\mathbb{Z}^d}, U_j(x+z,i)\right).$$

It follows by induction over j that Δ_j is a function of j, x, i and $\varphi_z(U, \bar{\varkappa})$. This implies (27). Thus, (ERG) follows from the ergodicity of $(U, \bar{\varkappa})$.

For the proof of (ID) it suffices due to stationarity to consider the case $v = 0$. We construct the measure P_0 on $(\Omega, \mathcal{F})$ as follows. Set $\widetilde{P} := P[\,\cdot\,|\,\#\mathcal{C}(0) = \infty]$. By the Radon-Nikodym theorem there is a strictly positive density $h := d\bar{P}/d\widetilde{P}_{\bar{\varkappa}} : \bar{\Pi} \to (0,\infty)$, where $\widetilde{P}_{\bar{\varkappa}}$ denotes the distribution of $\bar{\varkappa}$ under $\widetilde{P}$. Define $P_0[A] := \widetilde{E}\left[h(\bar{\varkappa}); A\right]$ for all $A \in \mathcal{F}$ and note that

$$\text{the distribution of } \bar{\varkappa} \text{ under } P_0 \text{ is } \bar{P}. \tag{28}$$

As required, P_0 and $\widetilde{P}$ are equivalent.

Observe that the environment viewed from $z \in \mathbb{Z}^d$ as defined in (6) can be written as $Z(z) = \left(\left(\mathbf{1}_{R_{x+z}}\right)_{x\in\mathbb{Z}^d}, \varphi_z(\varkappa), 0\right)$. Therefore, there is due to Lemma 14 and (27) a deterministic function g_2 such that $Z(z) = g_2\left(\varphi_z(U, \bar{\varkappa})\right)$ for all $z \in \mathbb{Z}^d$. Abbreviate $S_j := S_j(0,0)$. We need to show that the distribution of $Z(S_j) = g_2(\varphi_{S_j}(U, \bar{\varkappa})) = g_2(\varphi_{S_j}(U), \varphi_{S_j}(\bar{\varkappa}))$ under P_0 does not depend on j. Note that $U, U(0,0)$, and $\bar{\varkappa}$ are

[8] The proof is similar to the one of (4) $\Rightarrow$ (5) in [Pet83, Theorem 6.1, p. 65].

independent under P and hence also under P_0. Moreover, U is stationary under P and so it is under P_0. Since S_j is measurable w.r.t. $\sigma(U(0,0),\bar\varkappa)$, this implies that $Z(S_j)$ has the same distribution under P_0 as $g_2(U,\varphi_{S_j}(\bar\varkappa))$. Therefore and again by independence it suffices to show that $(\varphi_{S_j}(\bar\varkappa))_{j\ge 0}$ is stationary under P_0. This is the case due to (25), (28) and the invariance of $\bar P$ w.r.t. $\bar K$. Consequently, (ID) is fulfilled and Theorem 13 yields the claim. $\square$

Proof of Theorem 2. The theorem trivially holds if $P[\#\mathcal{C}(0)=\infty]=0$. Therefore, we assume that $P[\#\mathcal{C}(0)=\infty]>0$. Since $(\eta(x))_{x\in\mathbb{Z}^d}$ is i.i.d. and independent of the conductances and of the frog trajectories, $\bar\varkappa$ is stationary and ergodic[9]. We only need to produce an invariant measure $\bar P$ which is equivalent to $\widetilde P_{\bar\varkappa}$, the distribution of $\bar\varkappa$ under $\widetilde P := P[\,\cdot\,|\,\#\mathcal{C}(0)=\infty]$. First, define

$$Q[B] := \frac{E[q(0);\{\varkappa\in B\}\cap\{\#\mathcal{C}(0)=\infty\}]}{E[q(0);\#\mathcal{C}(0)=\infty]},\quad B\in\mathcal{B}(\Pi).$$

Then Q is equivalent to $\widetilde P_\varkappa$ since $q(0)>0$ on the event $\{\#\mathcal{C}(0)=\infty\}$. Moreover, Q is invariant for K defined in (24) due to the following variation of a standard argument, cf. [Bis11, Lemma 2.1]. For every bounded measurable function $f:\Pi\to\mathbb{R}$ we have by stationarity of the conductances

$$\begin{aligned}
E[q(0);\#\mathcal{C}(0)=\infty]\int_\Pi (Kf)(\pi)\,dQ(\pi) &= \sum_{\|e\|_1=1} E\left[c(\{0,e\})f(\varphi_e(\varkappa));\#\mathcal{C}(0)=\infty\right]\\
&= \sum_{\|e\|_1=1} E\left[c(\{-e,0\})f(\varkappa);\#\mathcal{C}(-e)=\infty\right] = \sum_{\|e\|_1=1} E\left[c(\{e,0\})f(\varkappa);\#\mathcal{C}(e)=\infty\right]\\
&= \sum_{\|e\|_1=1} E\left[c(\{0,e\})f(\varkappa);\{\#\mathcal{C}(e)=\infty\}\cap\{c(\{0,e\})>0\}\right]\\
&= \sum_{\|e\|_1=1} E\left[c(\{0,e\})f(\varkappa);\#\mathcal{C}(0)=\infty\right] = E[q(0);\#\mathcal{C}(0)=\infty]\int_\Pi f(\pi)\,dQ(\pi).
\end{aligned}$$

Denote by P_η the distribution of η under P. Then $\bar P := P_\eta\otimes Q$ is invariant w.r.t. $\bar K$ and equivalent to $P_{\bar\varkappa}$. The claim now follows from Theorem 22. $\square$

Example 23 (RW in i.i.d. environment). There are several examples of RWRE known in which the random environment $(\varkappa(x,x+\cdot))_{x\in\mathbb{Z}^d}$ is i.i.d. and for which there is a measure Q on Π that is invariant w.r.t. K defined in (24) and absolutely continuous w.r.t. $P_\varkappa$, see e.g. [BS02a] and the references mentioned therein after (0.12) and in the introduction of [Sab13]. For more recent results see, e.g. [Sab13, Theorem 1(i)] and [BCR14]. In these cases the environment is elliptic, i.e. $\varkappa(x,y)>0$ a.s. for all nearest neighbors $x,y\in\mathbb{Z}^d$ such that $\mathcal{C}(0)=\mathbb{Z}^d$. It has been noticed, e.g. in [Sab13, Lemma 4], see also [BS02b, Lecture 1, Theorem 1.2], that such Q is automatically equivalent to $P_\varkappa$. Also in the case of balanced and possibly

[9]See the previous footnote.

non-elliptic environments considered in [BD14] there exists such an equivalent Q provided that there is at least one direction $e_i \in \mathbb{Z}^d$ such that $\varkappa(x, x+e_i) > 0$ a.s. for all x [Ber16].

In all these cases, if we choose $(\eta(x))_{x\in\mathbb{Z}^d}$ i.i.d. and independently of $\varkappa$, then, as in the proof of Theorem 2, the measure Q can be augmented to a probability measure which is invariant w.r.t. $\bar{K}$ so that Theorem 22 is applicable. □

Counterexample 24 (RWRE dynamics, (ID) is not satisfied). This example shows that in Theorem 22 the requirement of the existence of an invariant probability measure $\bar{P}$ cannot be dropped. In [BZZ06], a Π-valued environment $\varkappa$ on $\mathbb{Z}^d, d \geq 3$, is constructed which is stationary and ergodic w.r.t. the shifts on $\mathbb{Z}^d$ and also uniformly elliptic[10], but for which the corresponding RWRE $(X_j)_{j\geq 0}$ starting at 0 disobeys the so-called 0–1 law for directional transience. (For $d = 2$ such an example is constructed in [Hei13]. For a simpler, but not uniformly elliptic example see [ZM01, Section 3].) In particular, there are $\{0, 1, 2\}$-valued random variables $N(x)$, $x \in \mathbb{Z}^d$, such that $(\varkappa, N)$ is stationary and ergodic w.r.t. the shifts and

$$P[N(X_j) = i \text{ for all } j \geq 0] > 0 \tag{29}$$

for $i = 1, 2$ (see the proof of [BZZ06, Theorem 3] on page 847). Now let $(\widetilde{\eta}(x))_{x\in\mathbb{Z}^d}$ be i.i.d. and independent of $(\varkappa, N)$ such that $E[\log_+(\widetilde{\eta}(0))] = \infty$. Set $\eta(x) := \widetilde{\eta}(x)\mathbf{1}_{N(x)=1}$. The 0–1 law for recurrence and transience fails in this case.

Indeed, on the one hand if $\eta(0) \geq 1$ and if the first frog woken up at 0 stays forever in the set $\{x \,|\, N(x) = 1\}$ then it will wake up for each $m \geq 1$ at least an independent $\widetilde{\eta}(0)$-distributed number of frogs at ℓ_1-distance m from 0. Each such frog has a chance of at least ε^m to reach the origin. Lemma 29 ($d = 1$) and the Borel-Cantelli lemma then imply $P[R_0] \geq P[\eta(0) \geq 1, \forall j \geq 0 : N(S_j(0, 1)) = 1]$, which is strictly positive due to (29) for $i = 1$.

On the other hand, $P[R_0] < P[\eta(0) \geq 1]$. Indeed, there is a.s. a finite nearest-neighbor path from 0 to a site v with $N(v) = 2$. If $N(0) < 2$ and if we wake up the frogs at 0 then with positive probability they and all the frogs woken up by them will follow this path to v and then stay forever within $\{x : N(x) = 2\}$ without ever returning to 0 due to (29) for $i = 2$. □

5. Recurrence and transience of some inhomogeneous frog models on $\mathbb{Z}^d$

Proof of Theorem 5. We generate the trajectories $(S_j(x, i))_{j\geq 0}$, $x \in \mathbb{Z}^d$, $i \geq 0$, in the following way. Enumerate the $2d$ unit vectors of $\mathbb{Z}^d$ as $e_1, \ldots, e_{2d}$. Let $U_j(x, i)$, $j \geq 0, x \in \mathbb{Z}^d$, $i \geq 0$, be i.i.d. random variables, each one uniformly distributed on $[0, 1]$. Since for each $x \in \mathbb{Z}^d$ and $i \geq 1$, $((S_k(x, i))_{0\leq k\leq j})_{j\geq 0}$ is a Markov chain (with state space $\bigcup_{j\geq 0}(\mathbb{Z}^d)^{j+1}$) we may assume without loss of generality that there are functions $f_{j,x,i} : \mathbb{Z}^{d(j+1)} \times [0, 1] \to \mathbb{Z}^d, j \geq 0, x \in \mathbb{Z}^d$, $i \geq 1$, such that

$$S_{j+1}(x, i) = f_{j,x,i}\left((S_k(x, i))_{0\leq k\leq j}, U_j(x, i)\right)$$

[10] i.e. there is an $\varepsilon > 0$ such that $\varkappa(x, y) > \varepsilon$ P-.a.s. for all nearest neighbors $x, y \in \mathbb{Z}^d$

and $f_{j,x,i}((x_k)_{0\le k\le j},u)=x_j+e_m$ if $u\in[(m-1)\varepsilon,m\varepsilon)$ for some $m\in\{1,\dots,2d\}$. We define the extra frogs by setting for $x\in\mathbb{Z}^d,\ j\ge 0$,

$$S_{j+1}(x,0):=f_{j,x,1}\left((S_k(x,0))_{0\le k\le j},U_j(x,0)\right).$$

Let $T(x,i):=\inf\{j\ge 0: U_j(x,i)\ge 2d\varepsilon\},\ x\in\mathbb{Z}^d,\ i\ge 1$. Then the random variables $T(x,i)$, $x\in\mathbb{Z}^d,\ i\ge 1$ are independent and geometrically distributed with parameter $1-2d\varepsilon$. Finally, define for all $x,y,v\in\mathbb{Z}^d$, $o(x):=0\in\mathbb{Z}^d$, $X_v:=v$, and set $\varkappa(x,y):=\varepsilon$ if $\|x-y\|_1=1$ and $\varkappa(x,y):=0$ otherwise.

We claim that this collection of random variables satisfies the assumptions of Theorem 13. It is obvious that the assumptions (EL), (UC), (EX) and (Xv) hold. As in the proof of Theorem 16, Assumption (ERG) is satisfied since the walks $(\bar{S}_j(x,i))_{j\ge 0},\ x\in\mathbb{Z}^d,\ i\ge 1$, are independent, simple symmetric random walks on $\mathbb{Z}^d$ which are stopped after the i.i.d. times $T(x,i)$. For the same reason $(Z(x))_{x\in\mathbb{Z}^d}$ is stationary. Moreover, $(Z(x))_{x\in\mathbb{Z}^d}$ is independent of the extra frogs. Hence (ID) holds (even though the environment viewed from the extra frog is not stationary in general). Assumption (T) is satisfied since $P[\bar{R}_v]=P[\bar{R}_0]$ for all $v\in V$ and

$$P[\bar{R}_0]>0 \tag{30}$$

as we shall show below.

Therefore, Theorem 13 applies. Since $P[R_v]\ge P[\bar{R}_v]>0$ due to (30), we obtain $P[R_v]=P[\eta(v)\ge 1]$, i.e. the claim of Theorem 5.

It remains to show (30). To this end we shall first show that if the frogs at 0 are woken up and all the woken up frogs are stopped after their respective times $T(x,i)$, it happens with positive probability that every site is eventually visited, i.e.

$$P\left[W^0(\eta,\bar{S})=\mathbb{Z}^d\right]>0. \tag{31}$$

For $x\in\mathbb{Z}^d$ and $r\ge 0$ let $B(x,r):=\left\{y\in\mathbb{Z}^d:\|y-x\|_1\le r\right\}$ and $\partial B(x,r):=\left\{y\in\mathbb{Z}^d:\|y-x\|_1=r\right\}$ be the ball and sphere, respectively, with center x and ℓ_1-radius r. Then

$$\begin{aligned} P\left[W^0(\eta,\bar{S})=\mathbb{Z}^d\right] &= P\left[\forall r\ge 1:\ B(0,r)\subseteq W^0(\eta,\bar{S})\right] = \prod_{r\ge 1}(1-a_r), \quad \text{where} \\ a_r &:= P\left[\partial B(0,r)\not\subseteq W^0(\eta,\bar{S})\,|\,B(0,r-1)\subseteq W^0(\eta,\bar{S})\right]. \end{aligned}$$

Note that $a_r < 1$. Therefore, the above product does not vanish iff the sequence $(a_r)_{r\geq 1}$ is summable. We estimate

$$\begin{aligned}
a_r &\leq \sum_{y\in\partial B(0,r)} P\left[y \notin W^0(\eta,\bar S)\,|\,B(0,r-1)\subseteq W^0(\eta,\bar S)\right]\\
&\leq \sum_{y\in\partial B(0,r)} P\left[y \notin \left\{\bar S_j(z,i) : j\geq 0, z\in B(y,r)\cap B(0,r-1), 1\leq i\leq \eta(z)\right\}\right]\\
&= \sum_{y\in\partial B(0,r)} \prod_{z\in B(y,r)\cap B(0,r-1)} E\left[P\left[y\notin \left\{\bar S_j(z,i) : j\geq 0, 1\leq i\leq \eta(z)\right\}\,|\,\eta(z)\right]\right]\\
&\leq \sum_{y\in\partial B(0,r)} \prod_{k=1}^{r} \prod_{z\in\partial B(y,k)\cap B(0,r-1)} E\left[(1-\varepsilon^k)^{\eta(z)}\right]\\
&\leq c_3 r^{d-1}\prod_{k=1}^{r}\left(E\left[(1-\varepsilon^k)^{\eta(0)}\right]\right)^{c_4 k^{d-1}} \;\leq\; c_3 r^{d-1}\prod_{k=1}^{r}\left(\varphi(\varepsilon^k)\right)^{c_4 k^{d-1}},
\end{aligned}$$

where $c_3(d) < \infty$ and $c_4(d) > 0$ are constants such that $\#\partial B(0,r) \leq c_3 r^{d-1}$ and $\#\partial B(y,k)\cap B(0,r-1) \geq c_4 k^{d-1}$ for all $1\leq k\leq r$ and $y\in\partial B(0,r)$ and where $\varphi(t) := E[e^{-t\eta(0)}]$ denotes the Laplace transform of $\eta(0)$ at $t\geq 0$. It is well-known that $P[\eta(0)\geq t^{-1}] \leq 2(1-\varphi(t))$ for all $t>0$, see, e.g. [Kal02, Lemma 5.1 (3)]. Therefore, it suffices to show that

$$\sum_{r\geq 1} b_r < \infty \qquad \text{where} \qquad b_r := c_3 r^{d-1}\prod_{k=1}^{r}\left(1 - P[\eta(0) > \varepsilon^{-k}]/2\right)^{c_4 k^{d-1}}.$$

We are going to apply Raabe's criterion. Due to (1) we have for large r,

$$\begin{aligned}
r\left(\frac{b_{r+1}}{b_r}-1\right) &\leq r\left(\left(1+\frac{1}{r}\right)^{d-1}\left(1-\frac{c_1}{2(r+1)^d(\log\varepsilon^{-1})^d}\right)^{c_4(r+1)^{d-1}} - 1\right)\\
&= r\left(\left(1+\frac{d-1}{r}+o\left(r^{-1}\right)\right)\left(1-\frac{c_1c_5(d,\varepsilon)}{r+1}+o\left(r^{-1}\right)\right)-1\right)\\
&= r\left(\frac{d-1}{r}-\frac{c_1c_5}{r+1}+o\left(r^{-1}\right)\right) \underset{r\to\infty}{\longrightarrow} d-1-c_1c_5,
\end{aligned}$$

which is less than -1 for c_1 sufficiently large. Consequently, $\sum_r b_r$ and $\sum_r a_r$ are finite and (31) follows.

Having proven (31), we obtain (30) if we show that waking up all initial frogs on $\mathbb{Z}^d$ causes a.s. infinitely many of them to visit 0 before they are stopped. Define the independent events $A(x,i) := \left\{\exists j\geq 0 : \bar S_j(x,i) = 0\right\}$, $x\in\mathbb{Z}^d$, $i\geq 1$, and observe that $P[A(x,i)\,|\,\eta] = P[A(x,i)] \geq \varepsilon^{\|x\|_1}$ a.s. for all x,i. Moreover, (1) implies (2). Therefore, a.s.

$$\sum_{x\in\mathbb{Z}^d, 1\leq i\leq\eta(x)} P[A(x,i)\,|\,\eta] \geq \sum_{r\geq 0}\varepsilon^r\sum_{\|x\|_1=r}\eta(x) = \infty$$

due to Lemma 29. By the second part of the Borel-Cantelli lemma $P[\cdot\,|\,\eta]$-a.s. infinitely many of the events $A(x,i), x\in\mathbb{Z}^d, 1\le i\le \eta(x)$, occur. Consequently, P-a.s. infinitely many of the events $A(x):=\cup_{i\le\eta(x)}A(x,i)$ occur. This proves (30) and concludes the proof of the theorem for $d\ge 2$.

For the final claim regarding $d=1$ note that in this case every frog trajectory starting at 0 is a.s. infinite and therefore covers a.s. $\mathbb{N}$ or $-\mathbb{N}$. Consequently, waking up a frog at 0 results a.s. in waking up all frogs on $\mathbb{N}$ or $-\mathbb{N}$. As above, this implies due to $E[\log_+\eta(0)]=\infty$ that infinitely many frogs visit 0. □

Problem 25. Does Theorem 5 still hold true if we replace the tail condition (1) by the weaker moment condition (2)?

If the answer to Problem 25 were positive then condition (2) in Theorem 5 would be sharp as the following result shows.

Proposition 26. *Let $d\ge 1$, $(\eta(x))_{x\in\mathbb{Z}^d}$ be identically distributed, and*

$$E\left[(\log_+\eta(0))^d\right]<\infty. \tag{32}$$

Suppose that K is a stochastic matrix on $\mathbb{Z}^d$ for which there exist $M\in\mathbb{N}$ and $\delta\in(0,M]$ such that $K(x,y)=0$ if $\|x-y\|_1\ge M$ and

$$\Big\|\sum_{y\in\mathbb{Z}^d}K(x,y)y\Big\|_1\ge\|x\|_1+\delta\qquad\text{for all } x\in\mathbb{Z}^d. \tag{33}$$

Assume that the frogs $S_\cdot(x,i)$, $x\in\mathbb{Z}^d$, $i\ge 1$, evolve as Markov chains with transition matrix K starting at x. Moreover, let the quantities $\eta(x)$, $S_\cdot(x,i)$, $x\in\mathbb{Z}^d$, $i\ge 1$, be independent. Then waking up all the initial frogs on $\mathbb{Z}^d$ results a.s. only in finitely many visits to 0. In particular, 0 is a.s. transient.

Proof. Let $A(x,i):=\{\exists j\ge 0: S_j(x,i)=0\}$ for $x\in\mathbb{Z}^d$, $i\ge 1$. Then for every $a>1$

$$P[A(x,i)]\le\sum_{j\ge0}P[S_j(x,i)=0]=\sum_{j\ge0}P[a^{-\|S_j(x,i)\|_1}\ge 1]\le\sum_{j\ge0}E[a^{-\|S_j(x,i)\|_1}]. \tag{34}$$

For $a>1$ let $\ell_a:\mathbb{R}\to\mathbb{R}$ be the linear function whose graph passes through $(-M,a^M)$ and (M,a^{-M}). By convexity, $a^{-s}\le\ell_a(s)$ for all $s\in[-M,M]$. Note that $\ell_a(s)<1$ for all $s>x_a:=M(a^M-1)(a^M+1)^{-1}$ and that $x_a\searrow 0$ as $a\searrow 1$. Therefore, given δ as in (33), there is an $a>1$ such that $\ell_a(\delta)<1$. Choose such an a. Then for any Markov chain $(S_j)_{j\ge0}$ with transition matrix K and $S_0=x$,

$$\begin{aligned}E[a^{-\|S_j\|_1}] &= E\left[E[a^{-(\|S_j\|_1-\|S_{j-1}\|_1)}\,|\,S_{j-1}]a^{-\|S_{j-1}\|_1}\right]\\ &\le E\left[E[\ell_a(\|S_j\|_1-\|S_{j-1}\|_1)\,|\,S_{j-1}]a^{-\|S_{j-1}\|_1}\right]\\ &\overset{(33)}{\le} \ell_a(\delta)E[a^{-\|S_{j-1}\|_1}]\ \le\ (\ell_a(\delta))^j a^{-\|x\|_1}\end{aligned}$$

by induction over j. Consequently, there is due to (34) a constant c_6 such that a.s. $P[A(x,i)] \le c_6 a^{-\|x\|_1}$ for all x, i. Therefore, by independence, a.s.

$$\sum_{x\in\mathbb{Z}^d, 1\le i\le \eta(x)} P[A(x,i)\,|\,\eta] = \sum_{x\in\mathbb{Z}^d, 1\le i\le \eta(x)} P[A(x,i)] \le c_6 \sum_{r\ge 0} a^{-r} \sum_{\|x\|_1=r} \eta(x) < \infty$$

due to Lemma 29 and (32). By the Borel-Cantelli lemma $P[\,\cdot\,|\,\eta]$-a.s. only finitely many of the events $A(x,i), x\in\mathbb{Z}^d, 1\le i\le \eta(x)$, occur. This implies the claim. $\square$

Note that Theorem 5 and Proposition 26 resemble Theorems 1.12 and 1.10 in [AMP02] respectively. In [AMP02] frogs have finite, geometric life times.

Appendix A. Discussion of Conjecture 2 from [GS09]

Conjecture 2 in [GS09] states that a frog model on a vertex transitive graph $G=(V,E)$ such that the frog numbers $\eta(x)$, $x\in V\setminus\{o\}$, are i.i.d., there is exactly one active frog at o at time 0, and the frog dynamics are given by independent homogeneous random walks satisfies the following zero-one law for recurrence and transience: either with probability one o is visited infinitely often or with probability one o is visited only finitely many times.

If all $\eta(x)$, $x\in V$, were i.i.d. and the notions of recurrence and transience were defined as in Definition 7 then Theorem 1 would simply cover this conjecture, since a homogeneous walk on a vertex transitive graph is a transitive Markov chain. However, also the original conjecture follows from Theorem 1 as we shall show now.

Assume without loss of generality that $P[\eta(x)\ge 1]>0$ $(x\ne o)$ and that the Markov chain for the individual frog dynamics in the [GS09] model is transient. Use the same ingredients in Theorem 1. Denote by R_v the event that v is recurrent according to Definition 7. Suppose that the conclusion of Theorem 1 is that $P[R_v]=0$ for all $v\in V$. Then $P[R_o\,|\,\eta(o)\ge 1]=0$. By monotonicity in the initial number of frogs, the fact that the number of frogs at each site is finite, and the assumed transience of the individual frog dynamics, the probability that o is visited infinitely many times (by any frog) starting with a single frog at o is equal to 0 as well.

Suppose now that the conclusion of Theorem 1 is that $P[R_v]=P[\eta(v)\ge 1]$ for all $v\in V$. It follows that for every $m\in\mathbb{N}$ such that $P[\eta(o)=m]>0$ we have $P[R_0\,|\,\eta(o)=m]=1$. Let $M:=\min\{m\in\mathbb{N}:\ P[\eta(o)=m]>0\}$. If $M=1$ then the proof of the conjecture is complete. Suppose $M>1$. We claim that if we follow, say, the first one of these M frogs and let the rest $M-1$ frogs remain inactive forever then o will still be recurrent with probability one. The argument follows closely the procedure of removing the extra frog in the proof of Theorem 13 after (16). Set $P_M[\,\cdot\,]:=P[\,\cdot\,|\,\eta(o)=M]$ and denote by $(n,s)^{(i)}$, $i\in\{1,\dots,M\}$, the frog configuration which is obtained from (n,s) by making all but the i-th frog at site o inactive. Note that as in (17) if o is recurrent for (n,s) then o is recurrent for at least one of $(n,s)^{(i)}$, $i\in\{1,\dots,M\}$. Denote by $\mathcal{G}$ the σ-algebra generated by

$(\eta(x), S(x,i)),\ x\in V\backslash\{o\},\ i\ge 1.$ Then by the above observation and independence

$$\begin{aligned}0=P_M[R_o^c] &\ge P_M\left[\cap_{i=1}^M\{o \text{ is transient for } (\eta,S)^{(i)}\}\right]\\ &= E_M\left[P_M\left[\cap_{i=1}^M\{o \text{ is transient for } (\eta,S)^{(i)}\}\,|\,\mathcal{G}\right]\right]\\ &= E_M\left[\left(P_M\left[o \text{ is transient for } (\eta,S)^{(1)}\,|\,\mathcal{G}\right]\right)^M\right].\end{aligned}$$

This gives that $P_M\left[o \text{ is transient for } (\eta,S)^{(1)}\,|\,\mathcal{G}\right]=0$ P_M-a.s., and after integration we obtain that $P_M[o \text{ is transient for } (\eta,S)^{(1)}]=0$. The proof of the conjecture is now complete.

Appendix B. Technical results

The following result generalizes [LP, Proposition 7.3].

Proposition 27 (i.i.d. $\Rightarrow$ ergodic). *Let V be a countably infinite set and $\Phi\subseteq \mathrm{Sym}(V)$ be closed under composition and such that all orbits $\Phi_x:=\{\varphi(x)\,|\,\varphi\in\Phi\}$, $x\in V$, are infinite. Let $H(x)$, $x\in V$, be independent random variables with values in some measurable space $(\mathbb{S},\mathcal{S})$ and $\mathcal{F}=\sigma(H(x),\ x\in V)$. For each $\varphi\in\Phi$ let $g_\varphi:\mathbb{S}\to\mathbb{S}$ be a measurable map and define $H_\varphi(x):=g_\varphi(H(\varphi(x)))$, $x\in V$. Finally suppose that*

$$\forall\varphi_1,\varphi_2\in\Phi\quad (H_{\varphi_1}(x))_{x\in V}\overset{d}{=}(H_{\varphi_2}(x))_{x\in V}. \tag{35}$$

Then for all $A\in\mathcal{I}:=\left\{A\in\mathcal{F}\,|\,\exists B\in\mathcal{S}^{\otimes V}\ \forall\varphi\in\Phi:\ A\overset{P}{=}\left\{(H_\varphi(x))_{x\in V}\in B\right\}\right\}$ we have that $P[A]\in\{0,1\}$.

For the proof we shall need the following fact.

Lemma 28. *Let V and Φ satisfy the assumptions of Proposition 27. Then for each pair of finite subsets F and J of V there is a $\varphi\in\Phi$ such that $\varphi[F]\cap J=\emptyset$.*

Proof. We use induction over k, the number of elements in F. The case $k=0$ is trivial. Assume as induction hypothesis that the statement has been shown for some $k\ge 0$ and let $F\subset V$ have exactly $k+1$ elements. The proof is by contradiction. Suppose that $J\subset V$ satisfies

$$\varphi[F]\cap J\neq\emptyset\quad\text{ for all }\varphi\in\Phi. \tag{36}$$

Let $(v_n)_{n\ge1}$ enumerate V, set $J_n:=J\cup\{v_1,\dots,v_n\}$ for all $n\ge1$, and choose $v\in F$. Then $F':=F\backslash\{v\}$ has exactly k elements. By the induction hypothesis for each $n\ge1$ there is a $\varphi_n\in\Phi$ such that

$$\varphi_n[F']\cap J_n=\emptyset \tag{37}$$

and, therefore, also $\varphi_n[F']\cap J=\emptyset$. Then $\varphi_n(v)\in J$ for all $n\ge1$ due to (36). Since J is finite, by the pigeon hole principle there is a $u\in J$ such that $\varphi_n(v)=u$ for infinitely many $n\ge1$. Since Φ_u is infinite, there is a $\psi\in\Phi$ such that $\psi(u)\notin J$. Choose n large enough so that $\psi^{-1}[J]\subseteq J_n$ and $\varphi_n(v)=u$. Set $\varphi:=\psi\circ\varphi_n\in\Phi$.

Then $\varphi(v) = \psi(\varphi_n(v)) = \psi(u) \notin J$ and, by (37), $\varphi[F'] = \psi[\varphi_n[F']] \subseteq \psi[J_n^c] = \psi[J_n]^c \subseteq J^c$. Thus $\varphi[F] \subseteq J^c$, and we get a contradiction with (36). □

Proof of Proposition 27. Let $(v_n)_{n\geq 1}$ be an arbitrary enumeration of V. Fix $A \in \mathcal{I}$ and let B be a measurable set such that $A \stackrel{P}{=} \left\{(H_\varphi(x))_{x\in V} \in B\right\}$ for all $\varphi \in \Phi$. Set $\mathcal{F}_n(\varphi) := \sigma\left(H_\varphi(v_k); 1 \leq k \leq n\right)$ and $Z_n(\varphi) := P[A \,|\, \mathcal{F}_n(\varphi)]$ for all $\varphi \in \Phi$ and $n \geq 1$. Fix $\varepsilon > 0$ and $\bar{\varphi} \in \Phi$. By Levy's 0–1 law, $Z_n(\bar{\varphi})$ converges in $\mathcal{L}^1$ to $\mathbf{1}_A$ as $n \to \infty$. Hence there is an $n \geq 1$ such that

$$E\left[|Z_n(\bar{\varphi}) - \mathbf{1}_A|\right] \leq \varepsilon/2. \tag{38}$$

Set $V' := \{v_k : k \leq n\}$. By Lemma 28 there is a $\varphi' \in \Phi$ such that

$$V' \cap \varphi'[V'] = \emptyset. \tag{39}$$

We claim that the distribution of $(\mathbf{1}_A, Z_n(\varphi))$ is the same for all $\varphi \in \Phi$. Indeed, a.s.

$$\begin{aligned}(\mathbf{1}_A, Z_n(\varphi)) =& (\mathbf{1}_B((H_\varphi(x))_{x\in V}), E[\mathbf{1}_B((H_\varphi(x))_{x\in V}) \,|\, H_\varphi(v_k),\, 1 \leq k \leq n]) \\ =& f_n((H_\varphi(x))_{x\in V})\end{aligned}$$

for some measurable function $f_n : \mathbb{S}^V \to \{0,1\} \times [0,1]$ which does not depend on φ due to (35). The claim follows by another application of (35). Therefore, by (38)

$$E\left[|Z_n(\bar{\varphi} \circ \varphi') - \mathbf{1}_A|\right] \leq \varepsilon/2. \tag{40}$$

Consequently,

$$\begin{aligned}\left|P[A] - P[A]^2\right| &= \left|E\left[(\mathbf{1}_A - Z_n(\bar{\varphi}) + Z_n(\bar{\varphi}))\, \mathbf{1}_A\right] - P[A]^2\right| \\ &\leq E\left[|\mathbf{1}_A - Z_n(\bar{\varphi})|\, \mathbf{1}_A\right] + \left|E\left[Z_n(\bar{\varphi})\left((\mathbf{1}_A - Z_n(\bar{\varphi} \circ \varphi')) + Z_n(\bar{\varphi} \circ \varphi')\right)\right] - P[A]^2\right| \\ &\leq E\left[|\mathbf{1}_A - Z_n(\bar{\varphi})|\right] + E\left[|Z_n(\bar{\varphi})||\mathbf{1}_A - Z_n(\bar{\varphi} \circ \varphi')|\right] + \left|E\left[Z_n(\bar{\varphi}) Z_n(\bar{\varphi} \circ \varphi')\right] - P[A]^2\right|.\end{aligned}$$

Due to (38), the first summand in the line above is less than or equal to $\varepsilon/2$. The same holds for the second term due to $|Z_n(\bar{\varphi})| \leq 1$ a.s. and (40). And the last summand vanishes since $Z_n(\bar{\varphi})$ and $Z_n(\bar{\varphi} \circ \varphi')$ are independent because $\mathcal{F}_n(\bar{\varphi})$ and $\mathcal{F}_n(\bar{\varphi} \circ \varphi')$ are independent due to (39). Letting $\varepsilon \searrow 0$ proves the claim. □

Lemma 29. *Let $d \in \mathbb{N}, 0 < c < \infty$, and $0 < a < 1$. Assume that $(Y_{i,n})_{i,n\geq 0}$ is an i.i.d. family of non-negative random variables. Then a.s.,*

$$E\left[\left(\log_+ Y_{0,0}\right)^d\right] < \infty \qquad \textit{iff} \qquad \sum_{n\geq 0} a^n \sum_{i=0}^{\lfloor cn^{d-1}\rfloor} Y_{i,n} < \infty. \tag{41}$$

For $d = 1$, this lemma is well-known, see [Luk75, Theorem 5.4.1] or [Bil95, Exercise 22.10]. For $d \geq 1$ it follows from [Zer02, Theorem 2], see [Bau14, Lemma 2.2] for details and note that by Kolmogorov's 0–1 law the double sum in (41) converges with probability 0 or 1.

Appendix C. Adding extra frogs

Lemma 30. *Let $(\Omega^0, \mathcal{F}^0, P^0)$ be a probability space, $(\mathbb{X}, \mathcal{X})$ a measurable space and $\mathbb{S}$ a Polish space with Borel σ-field $\mathcal{S}$. Suppose $X^0 : \Omega^0 \to \mathbb{X}$ and $Y^0 : \Omega^0 \to \mathbb{S}$ are measurable. Then there is a probability space $(\Omega, \mathcal{F}, P)$ and measurable maps X, Y, and Z on $(\Omega, \mathcal{F})$ such that*

(a) $(X^0, Y^0) \stackrel{d}{=} (X, Y)$ *and*
(b) Y *and* Z *are i.i.d. given* X.

Proof. There is a probability kernel $\mu : \mathbb{X} \times \mathcal{S} \to [0, 1]$ such that for all $B \in \mathcal{S}$, P^0-a.s. $P^0[Y^0 \in B \,|\, X^0] = \mu(X^0, B)$ (see, e.g. [Kal02, Theorem 6.3]). Let $\Omega := \mathbb{X} \times \mathbb{S} \times \mathbb{S}$, $\mathcal{F}$ be the product σ-field on Ω, denote by E^0 the expectation operator w.r.t. P^0, and define for $A \in \mathcal{X}$ and $B, C \in \mathcal{S}$,

$$P[A \times B \times C] := E^0 \left[\mu(X^0, B)\, \mu(X^0, C);\; X^0 \in A\right].$$

By [Bil95, Theorem 11.3], P can be extended to a probability measure on $\mathcal{F}$. We let (X, Y, Z) be the identity on Ω. Then (a) follows from our choice of μ. Moreover, it follows from $X^0 \stackrel{d}{=} X$ that P-a.s., $P[Y \in B, Z \in C|X] = \mu(X, B)\mu(X, C)$. This implies (b). □

Lemma 31 (Coupling). *Let V be countably infinite and suppose that $\mathbb{U}$ and $\mathbb{S}$ are Polish spaces with Borel σ-fields $\mathcal{U}$ and $\mathcal{S}$, respectively. Assume that for each $v \in V$ there is a probability space $(\Omega_v, \mathcal{F}_v, P_v)$, and random variables $U_v : \Omega_v \to \mathbb{U}$ and $Z_v : \Omega_v \to \mathbb{S}$ such that U_v, $v \in V$, are identically distributed.*

Then there is a probability space $(\Omega, \mathcal{F}, P)$ and random variables $U : \Omega \to \mathbb{U}$ and $E_v : \Omega \to \mathbb{S}$, $v \in V$, on it such that

$$(U, E_v) \stackrel{d}{=} (U_v, Z_v) \quad \textit{for all } v \in V. \tag{42}$$

Proof. For all $I \subseteq V$ define the product measurable space $(\Omega_I, \mathcal{F}_I) := (\mathbb{U}, \mathcal{U}) \otimes (\mathbb{S}, \mathcal{S})^{\otimes I}$. Choose $(\Omega, \mathcal{F}) := (\Omega_V, \mathcal{F}_V)$ and let $(U, (E_v)_{v \in V})$ be the identity on Ω. Denote by ν the common distribution of U_v, $v \in V$. For each $v \in V$ there is a probability kernel $\mu_v : \mathbb{U} \times \mathcal{S} \to [0, 1]$ such that for all $B \in \mathcal{S}$, P_v-a.s. $\mu_v(U_v, B) = P_v[Z_v \in B \,|\, U_v]$. For finite $I \subseteq V$ and $A \in \mathcal{U}, B_v \in \mathcal{S}$ we set

$$P_I \left[A \times \prod_{v \in I} B_v \right] := \int_A \prod_{v \in I} \mu_v(x, B_v)\, d\nu(x).$$

This defines a projective family of probability measures P_I on $(\Omega_I, \mathcal{F}_I)$. We choose P as its projective limit, see, e.g. [Kal02, Theorem 6.14]. Then (42) follows from the construction. □

Corollary 32. *Let V be countably infinite and let $\mathbb{W}$ and $\mathbb{S}$ be Polish spaces with Borel σ-fields $\mathcal{W}$ and $\mathcal{S}$, respectively. Suppose we are given a probability space $(\Omega^0, \mathcal{F}^0, P^0)$ and random variables $W^0 : \Omega^0 \to \mathbb{W}$ and $F_v^0 : \Omega^0 \to \mathbb{S}$ for each $v \in V$.*

Then there is a probability space $(\Omega, \mathcal{F}, P)$ and random variables $W : \Omega \to \mathbb{W}$ and $F_v, E_v : \Omega \to \mathbb{S}$, $v \in V$, such that

(a) $(W^0, (F_x^0)_{x\in V}) \overset{d}{=} (W, (F_x)_{x\in V})$ *and*
(b) *for all* $v \in V$*,* F_v *and* E_v *are i.i.d. given* $(W, \check{F}_v)$*, where* $\check{F}_v := (F_x)_{x\in V\setminus\{v\}}$*.*

Proof. Fix $v \in V$. Endow $\mathbb{X} := \mathbb{W} \times \mathbb{S}^{V\setminus\{v\}}$ with its product σ-field $\mathcal{X}$. Then $X^0 := (W^0, \check{F}_v^0) : \Omega^0 \to \mathbb{X}$ is measurable. Set $Y^0 := F_v^0$ and apply Lemma 30 to these quantities. Since we fixed v, the outcome may depend on v, which we indicate by a subscript v. Hence we get from Lemma 30 for every $v \in V$ a probability space $(\Omega_v, \mathcal{F}_v, P_v)$ and measurable maps X_v, Y_v, and Z_v on $(\Omega_v, \mathcal{F}_v)$ such that

(a′) $(W^0, (F_x^0)_{x\in V}) = (X^0, F_v^0) \overset{d}{=} (X_v, Y_v)$ and
(b′) Y_v and Z_v are i.i.d. given X_v.

Now we set $\mathbb{U} := \mathbb{W} \times \mathbb{S}^V$, denote by $\mathcal{U}$ the product σ-field on $\mathbb{U}$ and interpret $U_v := (X_v, Y_v)$ as random variable on Ω_v with values in $\mathbb{U}$. By (a′) the $U_v, v \in V$, are identically distributed. Therefore, Lemma 31 can be applied and yields random variables $U =: (W, (F_x)_{x\in V})$ and $E_v, v \in V$, on some probability space $(\Omega, \mathcal{F}, P)$ such that by (42), $\big((W, \check{F}_v), F_v, E_v\big) \overset{d}{=} (X_v, Y_v, Z_v)$. Therefore, (a) follows from (a′) and (b) from (b′). □

Proof of Remark 12. We apply Corollary 32 to $W^0 := (\varkappa, T, \eta, (S_\cdot(\cdot, i))_{i\geq 2})$ and the first frogs $F_v^0 := S_\cdot(v, 1), v \in V$, and obtain a new probability space and random variables $(W, (F_x)_{x\in V}, (E_x)_{x\in V})$ on it, which satisfy (a) and (b). The trajectory of the extra frog at $v \in V$ is defined by $S_\cdot(v, 0) := E_v$. Statement (b) then means (EX). □

Proof of Lemma 15. Fix $v \in V$, set $W_v := (\varkappa, T, \eta, (S_\cdot(x, i))_{x\in V, i\geq 1+\mathbf{1}_{x=v}})$, $F_v := S_\cdot(v, 1)$, $E_v := S_\cdot(v, 0)$, and let f be a measurable function on $V^{\mathbb{N}_0}$. First, we show that for all measurable sets $C \subseteq V^{\mathbb{N}_0}$,

$$P[E_v \in C \,|\, W_v, F_v, f(E_v)] = P[E_v \in C \,|\, W_v, f(E_v)] \quad P\text{-a.s.}. \tag{43}$$

For (43) it suffices to check that for sets G of the form $\{W_v \in A, F_v \in B, f(E_v) \in D\}$, where A, B, and D are measurable sets in appropriate spaces, we have

$$P\left[\{E_v \in C\} \cap G\right] = E\left[P[E_v \in C \,|\, W_v, f(E_v)]; G\right]. \tag{44}$$

The right-hand side of (44) is equal to

$$\begin{aligned}
&E\left[P[E_v \in C \cap f^{-1}(D) \,|\, W_v, f(E_v)]; F_v \in B, W_v \in A\right] \\
=&E\left[E\left[P[E_v \in C \cap f^{-1}(D) \,|\, W_v, f(E_v)]; F_v \in B, W_v \in A \,|\, W_v, f(E_v)\right]\right] \\
=&E\left[P[E_v \in C \cap f^{-1}(D) \,|\, W_v, f(E_v)]\, P\left[F_v \in B \,|\, W_v, f(E_v)\right]; W_v \in A\right].
\end{aligned}$$

It follows from independence in (EX) and [Kal02, Proposition 6.6] that a.s. $P[F_v \in B \,|\, W_v, f(E_v)] = P[F_v \in B \,|\, W_v]$. Using this fact and conditioning on W_v inside the above expectation we get that the last expression above is equal to

$$\begin{aligned}
&E\left[P\left[E_v\in C\cap f^{-1}(D)\,|\,W_v\right]P\left[F_v\in B\,|\,W_v\right];W_v\in A\right]\\
\overset{\text{(EX)}}{=}&E\left[P\left[F_v\in B, E_v\in C\cap f^{-1}(D)\,|\,W_v\right];W_v\in A\right]\\
=&P[W_v\in A, F_v\in B, E_v\in C\cap f^{-1}(D)],
\end{aligned}$$

which is the same as the left-hand side of (44). This proves (43).

Now let $j\ge 0$ and $y\in V$. Applying (43) to $f((s_k)_{k\ge 0}):=(s_k)_{0\le k\le j}$ and $\{E_v\in C\}=\{S_{j+1}(v,0)=y\}$ and subtracting $\varkappa(S_j(v,0),y)$ from both sides we obtain

$$\begin{aligned}
&P[S_{j+1}(v,0)=y\,|\,\mathcal{F}_j(v,0)]-\varkappa(S_j(v,0),y)\\
=\;&P\left[S_{j+1}(v,0)=y\,|\,\sigma(\mathcal{F}_0(v,1)\cup\sigma(S_k(v,0);0\le k\le j))\right]-\varkappa(S_j(v,0),y)\\
\overset{d}{=}\;&P\left[S_{j+1}(v,1)=y\,|\,\mathcal{F}_j(v,1)\right]-\varkappa(S_j(v,1),y)\overset{\text{(EL)}}{\ge}0\quad\text{a.s.}
\end{aligned}$$

since $(W_v,F_v,E_v)\overset{d}{=}(W_v,E_v,F_v)$ due to (EX). The claim of the lemma follows. □

Acknowledgment: E. Kosygina was partially supported by the Simons Foundation through Collaboration Grant for Mathematicians # 209493 and Simons Fellowship in Mathematics for 2014-2015. M. Zerner was partially supported by the European Research Council, Grant 208417-NCIRW. This research was supported through the program "Research in Pairs" by the Mathematisches Forschungsinstitut Oberwolfach in 2015.

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