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The prescribed boundary mean curvature problem on $\mathbb{B}^4$

Zindine Djadli,[a,b,*] Andrea Malchiodi,[b] and Mohameden Ould Ahmedou[c]

[a] *Département de Mathématiques, Université de Cergy-Pontoise, Site de Saint-Martin, 2 Avenue Adolphe Chauvin, F 95302 Cergy, Pontoise Cedex, France*
[b] *School of Mathematics, Institute for Advanced Study, 1 Einstein Drive, Princeton, NJ 08540, USA*
[c] *Rheinische Friedrich-Wilhelms-Universitat Bonn, Mathematisches Institut, Beringstrasse 4, D-53115 Bonn, Germany*

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Abstract

In this paper, we perform a fine blow-up analysis for a boundary value elliptic equation involving the critical trace Sobolev exponent related to the conformal deformation of the metrics on the standard ball, namely the problem of prescribing the boundary mean curvature. From this analysis some a priori estimates in low dimension are obtained. With these estimates, we prove the existence of at least one solution when an index-counting formula associated to the prescribed mean curvature is different from zero.

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[*] Corresponding author. Fax: +33-1-34-25-66-45.

E-mail addresses: zindine.djadli@math.u-cergy.fr (Z. Djadli), malchiod@ias.edu (A. Malchiodi), ahmedou@math.uni-bonn.de (M.O. Ahmedou).

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1. Introduction

This paper is devoted to the study of an nonlinear elliptic equation involving the Sobolev trace critical exponent. Such an equation is associated to the conformal deformation of Riemannian metrics on manifolds with boundary. Given a Riemannian manifold with boundary (M, g) of dimension $n \geqslant 3$, with scalar curvature R_g and boundary mean curvature H_g, let $g' = v^{\frac{4}{n-2}} g$, where v is a smooth positive function, be a conformal metric; then new curvatures $R_{g'}$ and $H_{g'}$ are given by

$$\begin{cases} -4\dfrac{n-1}{n-2}\Delta_g v + R_g v = R_{g'} v^{\frac{n+2}{n-2}} & \text{in } M, \\ \dfrac{2}{n-2}\dfrac{\partial v}{\partial \nu} + H_g v = H_{g'} v^{\frac{n}{n-2}} & \text{on } \partial M, \end{cases} \tag{1}$$

see e.g. [3]. In the above equation, ν denotes the outward unit vector perpendicular to ∂M, with respect to the metric g.

A problem arises naturally when looking at Eq. (1): given a function $H : \partial M \to \mathbb{R}$, does there exists a metric g' conformal to g such that $R_{g'} \equiv 0$ and $H_{g'} \equiv H$? From Eq. (1), the problem is equivalent to finding a smooth positive solution v of the equation

$$\begin{cases} -4\dfrac{n-1}{n-2}\Delta_g v + R_g v = 0 & \text{in } M, \\ \dfrac{2}{n-2}\dfrac{\partial v}{\partial \nu} + H_g v = H v^{\frac{n}{n-2}} & \text{on } \partial M. \end{cases} \tag{2}$$

The requirement about the positivity of v is necessary for the metric g' to be Riemannian. For the two-dimensional case, there are analogous equations involving exponential nonlinearities.

In this paper we are interested in the case where a noncompact group of conformal transformations acts on the equation so that Kazdan–Warner type conditions give rise to obstructions, as in the Nirenberg problem (see [19]). The simplest situation is the following: let B^n be the unit ball in $\mathbb{R}^n$ with Euclidean metric g_0. Its boundary will be denoted by S^{n-1} and will be endowed with the standard metric still denoted by g_0. Let H be a smooth function on S^{n-1}.

In this case our problem becomes

$$\begin{cases} \Delta_{g_0} u = 0, \ u > 0 & \text{in } B^n, \\ \dfrac{\partial u}{\partial \nu} + \dfrac{n-2}{2} u = \dfrac{n-2}{2} H u^{\frac{n}{n-2}} & \text{on } S^{n-1}. \end{cases} \tag{3}$$

Our aim is to give sufficient conditions on H such that problem (3) admits a positive solution (of class C^2).

It is easy to see that a necessary condition for solving the problem is that H has to be positive somewhere.

As already mentioned, there is at least another obstruction to solving the problem, the so-called Kazdan–Warner conditions. In [14] Escobar studied this problem on manifolds which are not conformally equivalent to the standard ball, and in [11], Chang et al. gave a perturbative result for Eq. (3) (that is, for H close to a constant function). In [15] Escobar and Garcia studied problem (3) for $n = 3$ in the nonperturbative case, proving that blow ups of solutions (of subcritical approximations) occur at one point and also obtained compactness and existence results. Related problems were studied in [16,13].

In contrast with the case $n = 3$, where there is only a single blow-up point, for $n \geqslant 4$ multiple blow-up points may occur, as for scalar curvature in dimension $n \geqslant 4$, see [6,8,20] (see also [7,9,10]).

In this paper we study the four-dimensional case, where a phenomenon of balance between the "self-interaction" (of the bubbles among themselves) and the interactions of the bubbles with H occurs, allowing us to provide a Hopf formula criterion for the existence of solutions. More precisely we establish some compactness and existence result, that we state after introducing some notation.

Consider the following problem in B^4:

$$\begin{cases} \Delta v = 0 & \text{in } B^4, \\ \dfrac{\partial v}{\partial \nu} + v = f & \text{on } \partial S^3. \end{cases} \tag{4}$$

It is standard, see e.g. [1], that if $f \in C^{1,\alpha}(S^3)$ for some $\alpha \in (0,1)$, then there exists a solution $v \in C^{2,\alpha}$ of (4). We denote by Ξ the operator which associates to f the solution v of (4), and we extend the definition of Ξ also to the case of weak solutions of (4).

For $H \in C^2(S^3)$, H positive, let

$$\begin{aligned} \mathcal{H} &= \{p \in S^3 : \nabla_{g_0} H(p) = 0\}, \\ \mathcal{H}^+ &= \{p \in S^3 : \nabla_{g_0} H(p) = 0, -\Delta_{g_0} H(p) > 0\}, \\ \mathcal{H}^- &= \{p \in S^3 : \nabla_{g_0} H(p) = 0, -\Delta_{g_0} H(p) < 0\}, \\ \mathcal{M}_H &= \{v \in C^2(\overline{B^4}) : v \text{ satisfies } (3)\}. \end{aligned}$$

To each $H \in C^2(S^3)$ and to each $\{p^1, \dots, p^k\} \subseteq \mathcal{H} \setminus \mathcal{H}^-$, $k \geqslant 1$, we associate a $k \times k$ symmetric matrix $M = M(p^1, \dots, p^k)$ defined by

$$\begin{cases} M_{ii} = \dfrac{\Delta H(p^i)}{(H(p^i))^2}, \\ M_{ij} = -\dfrac{24}{5} \dfrac{G_{p^i}(p^j)}{(H(p^i)H(p^j))^{\frac{1}{4}}} & \text{if } i \neq j. \end{cases} \tag{5}$$

Here $G_q(\cdot)$ denotes the Green function for the operator Ξ with pole q defined as follows. For $q \in S^3$, let $\pi_q : B^4 \to \mathbb{R}^4_+$ denote the stereographic projection with pole $-q$. In π_q-stereographic coordinates, we define the function $G_q : B^4 \to \mathbb{R}$ by

$$G_q(x) = \left(\frac{1+|x|^2}{2}\right)\frac{1}{|x|^2}, \quad x \in \mathbb{R}^4_+. \tag{6}$$

Let $\rho = \rho(p^1, \dots, p^k)$ denote the least eigenvalue of M.

Define the set $\mathcal{A}$ to be

$$\mathcal{A} = \{H \in C^2(S^3) : H \text{ is a positive Morse function on } S^3 \text{ such that}$$

$$\Delta_{g_0} H \neq 0 \text{ on } \mathcal{H}, \text{and } \rho = \rho(p^1, \dots, p^k) \neq 0, \ \forall p^1, \dots, p^k \in \mathcal{H}^+\}.$$

Let us observe that $\mathcal{A}$ is open in $C^2(S^3)$ and dense in the space of positive C^2 functions on S^3 (with respect to the C^2 norm).

We introduce an integer-valued continuous function denoted by Index $: \mathcal{A} \to \mathbb{N}$ by the following formula:

$$\mathrm{Index}(H) = -1 + \sum_{k=1}^{m} \sum_{\substack{\rho(p^{i_1}, \dots p^{i_k}) > 0, \\ 1 \leqslant i_1 < i_2 < \dots < i_k \leqslant m}} (-1)^{k-1+\sum_{j=1}^{k} i(p^{i_j})},$$

where $i(p^{i_j})$ denotes the Morse index of H at p^{i_j}.

Our main result concerning compactness and existence is the following.

Theorem 1.1. *Let* $n = 4$ *and suppose* $H \in \mathcal{A}$. *Then for all* $0 < \alpha < 1$, *there exists some constant* C *depending only on* $\min_{S^3} H$, $||H||_{C^2(S^3)}$, $\min_{\mathcal{H}} |\Delta_{g_0} H|$ *and* $\min\{|\rho(p^1, \dots, p^k)| : p^1, \dots, p^k \in \mathcal{H}^+, k \geqslant 2\}$ *such that*

$$\frac{1}{C} \leqslant v \leqslant C, \quad ||v||_{C^{2,\alpha}(\overline{B^4})} \leqslant C, \tag{7}$$

for all solutions v *of Eq.* (3). *Furthermore, for all* $R \geqslant C$

$$deg(v - \Xi^{-1}(Hv^2), \mathcal{O}_R, 0) = Index(H), \tag{8}$$

where

$$\mathcal{O}_R = \left\{ v \in C^{2,\alpha}(\bar{B}^4) : \frac{1}{R} \leqslant v \leqslant R, \ ||v||_{C^{2,\alpha}(\bar{B}^4)} \leqslant R \right\}$$

and where deg denotes the Leray–Schauder degree in $C^{2,\alpha}(\bar{B}^4)$. *As a consequence* $\mathcal{M}_H \neq \emptyset$, *provided* $Index(H) \neq 0$.

Since the situation here is similar to the scalar curvature problem on S^4 for a Morse function K, our Theorem 1.1 can be considered as a counterpart of the results in [8,20] for manifolds with boundary. Notice that only the least eigenvalue of $M(q^1, \dots, q^N)$ plays a role in counting the total degree of solutions of (3) and in the compactness result. In act, the points $p^{i_1}, \dots, p^{i_k}$ for which $\rho(p^{i_1}, \dots, p^{i_k})$ is positive characterize the so-called *asymptotics* in the theory of itical point at infinity developed by A. Bahri. For instance, considering a continuous family of functions (K_t, H_t), the total degree changes when the least eigenvalue of $M_t(q^1, \dots, q^N)$ crosses zero, while it remains unchanged when other eigenvalues cross zero.

We expect that in higher dimension it is possible to obtain from the above blow-up analysis some compactness and existence results under suitable flatness conditions on the function H near its critical points, just as Yanyan Li did in [20] for the scalar curvature problem.

Our paper is organized as follows. In Sections 2 and 3 we provide the main local blow-up analysis, in Section 4 we provide, along the direction initiated by R. Schoen and YY. Li, a-priori estimates in $H^2(\overline{B^n})$ norm and $L^\infty(\overline{B^n})$ norm for solutions of Eq. (3) in dimension 4. In Section 5, we give the proof of the existence and compactness results stated above. In the Appendix, we provide some useful technical results. For references on some of the analytical tools used in this paper, the reader can see [4,18,22–24].

2. Blow-up analysis

For a smooth bounded domain $\Omega \subseteq \mathbb{R}^n$ set $\Omega^+ = \Omega \cap \{x_n > 0\}$, $\partial_1\Omega = \Omega \cap \partial\mathbb{R}^n_+$ and $\partial_2\Omega = \partial\Omega \cap \mathbb{R}^n_+$, hence $\partial\Omega^+ = \overline{\partial_1\Omega} \cup \partial_2\Omega$. We also assume that $\partial\Omega$ and $\partial\mathbb{R}^n_+$ intersect transversally so that $\partial\Omega \cap \partial\mathbb{R}^n_+$ is a smooth manifold of dimension $n-2$. Let ν denote the unit exterior normal to Ω, and let ν' denote the exterior unit normal of $\partial_1\Omega$ in $\partial\mathbb{R}^n_+$. Given $w : \partial\mathbb{R}^n_+ \to \mathbb{R}$, the expression $\nabla' w$ stands for the gradient in $\mathbb{R}^{n-1}$. If w is defined on $\overline{\Omega^+}$, the same symbol will be used for the gradient of the restriction of w to $\partial_1\Omega$. In the following $B_\sigma(x)$ denotes the open ball in $\mathbb{R}^n$ of radius σ centered at x; we just write B_σ if $x = 0$.

We will consider Eq. (3) when the exponent of u in the right-hand side is replaced by some q_i converging to $\frac{n}{n-2}$ from below. We also allow the function H to vary; more precisely, we consider positive solutions u_i of the sequence of problems (for $q_i \to \frac{n}{n-2}$)

$$\begin{cases} \Delta u_i = 0 & \text{in } \Omega^+, \\ -\dfrac{\partial u_i}{\partial x_n} = \dfrac{n-2}{2} H_i u_i^{q_i} & \text{on } \partial_1\Omega. \end{cases} \tag{9}$$

We are mainly concerned with what happens to $\{u_i\}_i$ when i tends to infinity. It follows from standard elliptic theory that if $\{u_i\}_i$ remains bounded in $L^\infty_{\mathrm{loc}}(\overline{\Omega^+})$, then $\{u_i\}_i$ tends to some u in $C^2_{\mathrm{loc}}(\overline{\Omega^+})$ along a subsequence. Otherwise, we say that $\{u_i\}_i$

blows up. In the following, we give the definition of *isolated and isolated simple blow up point*, which were first introduced by Schoen [25], and adapted to the framework of boundary value problems by Li [19].

Let $\{H_i\}_i \subset L^\infty(\partial_1\Omega)$ satisfy for all $i \in \mathbb{N}$ and for some positive constant A_1 independent of i

$$\frac{1}{A_1} \leqslant H_i(x) \leqslant A_1 \quad \text{for all } x \in \partial_1\Omega. \tag{10}$$

Definition 2.1. The point $\bar{x} \in \Omega^+ \cup \partial_1\Omega$ is called a *blow up point* for $\{u_i\}_i$ if there exists a sequence of points $x_i \in \Omega^+ \cup \partial_1\Omega$ tending to $\bar{x}$ such that $u_i(x_i) \to +\infty$.

Definition 2.2. Let $\bar{x} \in \Omega^+ \cup \partial_1\Omega$, and let $\{x_i\}$ be a sequence of local maxima of u_i such that $x_i \to \bar{x}$ and $u_i(x_i) \to +\infty$. The point $\bar{x}$ is called an *isolated blow up point* if there exist $0 < \bar{r} < dist(\bar{x}, \partial_2\Omega)$ and $\bar{C} > 0$ such that

$$u_i(x) \leqslant \bar{C}\,|x - x_i|^{-\frac{2}{q_i - 1}} \quad \text{for all } |x - x_i| \leqslant \bar{r},\ \ x \in \Omega^+.$$

If $x_i \to \bar{x}$ is a simple blow up point for $\{u_i\}_i$ and if $\bar{r}$ is given by Definition 2.2, we define

$$\bar{u}_i(r) = \frac{1}{|\partial B_r(x_i) \cap \Omega^+|} \int_{\partial B_r(x_i) \cap \Omega^+} u_i \quad r \in (0, \bar{r}) \tag{11}$$

and

$$\tilde{u}_i(r) = r^{\frac{2}{q_i - 1}} \bar{u}_i(r), \quad r \in (0, \bar{r}).$$

Definition 2.3. The isolated blow up point $x_i \to \bar{x}$ is called *isolated simple* if there exists $\varrho \in (0, \bar{r})$ such that for large i there holds

$$\tilde{u}_i \text{ has precisely one critical point in } (0, \varrho). \tag{12}$$

If $\bar{x}$ is a blow up point, we will call it *interior blow up point* if $\bar{x} \in \Omega^+$, or *boundary blow up point* if $\bar{x} \in \partial_1\Omega$.

An isolated blow-up point has the following interesting properties.

Lemma 2.4 ("Harnack inequality"). *Let* $\{H_i\}_i \subset L^\infty(\partial_1\Omega)$ *satisfy* (10), $\{u_i\}_i$ *satisfy* (9), *and let* $y_i \to \bar{y} \in \Omega^+ \cup \partial_1\Omega$ *be an isolated blow-up point. Then for any* $r \in (0, \frac{1}{3}\bar{r})$, *we have the following Harnack inequality*:

$$\max_{y \in B^+_{2r} \setminus B^+_{\frac{r}{2}}} u_i(y) \leqslant C \min_{y \in B^+_{2r} \setminus B^+_{\frac{r}{2}}} u_i(y). \tag{13}$$

where C is some positive constant depending only on n, $\bar{C}$, *and* A_1.

Proof. Set $v_i(y) = r^{\frac{2}{q_i-1}} u_i(y_i + ry)$, $y \in \frac{1}{r}(\Omega^+ \cup \partial_1\Omega) - y_i$. It is easy to see that v_i satisfies

$$\begin{cases} \Delta v_i = 0, \ v_i > 0 & \text{in } \frac{1}{r}(\Omega^+) - y_i, \\ -\dfrac{\partial v_i}{\partial x_n} = \frac{n-2}{2} H_i(y_i + ry) v_i^{q_i} & \text{on } y \in \frac{1}{r}(\partial_1\Omega) - y_i, \\ 0 < v_i(y) \leqslant \bar{C}|y|^{-\frac{2}{q_i-1}}. \end{cases} \tag{14}$$

So the Lemma follows from Lemma A.2 of [17]. □

Proposition 2.5. *Let* $\{H_i\}_i \subset C^2(\overline{\partial_1\Omega})$ *satisfy for all* $i \in \mathbb{N}$

$$\|\nabla' H_i\|_{L^\infty(\overline{\partial_1\Omega})} + \|\nabla'^2 H_i\|_{L^\infty(\overline{\partial_1\Omega})} \leqslant A_2 \tag{15}$$

for some positive constant A_2. *Let* $\{u_i\}_i$ *satisfy* (9), *and let* $y_i \to \bar{y} \in \Omega^+ \cup \partial_1\Omega$ *be an isolated blow-up point of* $\{u_i\}_i$. *Then, for any* $R_i \to +\infty$ *and* $\varepsilon_i \to 0^+$, *we have, after passing to a subsequence* (*still denoted* $\{u_i\}_i$, y_i, *etc.*), *that*

$$\left\| u_i(y_i)^{-1} u_i(u_i(y_i)^{1-q_i} y + y_i) - ((1 + h_i y^n)^2 + h_i^2 |y'|^2)^{\frac{2-n}{2}} \right\|_{C^2(\overline{B^+_{2R_i}(0)})} \leqslant \varepsilon_i,$$

$$R_i\, u_i(y_i)^{1-q_i} \to 0 \quad \text{as } i \to +\infty,$$

where $h_i = \frac{1}{2} H_i(y_i')$, $y = (y', y^n) \in \mathbb{R}^{n-1} \times \mathbb{R}^+$, *and* $y_i = (y_i', y_i^n)$ (*here* y' *and* y_i' *denote, respectively, the projection onto* $\partial_1\Omega$ *of* y *and* y_i).

Proof. Writing $y_i = (y_i', y_i^n) \in \mathbb{R}^{n-1} \times \mathbb{R}^+$, and setting $T_i = u_i(y_i)^{1-q_i} y_i^n$ consider

$$\xi_i(y) = u_i(y_i)^{-1} u_i(u_i(y_i)^{1-q_i} y + y_i) \quad \text{in } \{y^n \geqslant -T_i\}.$$

Notice that $\xi_i(0) = 1$, that $y = 0$ is a local maximum of ξ_i in $\{y^n \geqslant -T_i\}$, that $0 < \xi_i(y) < \bar{C}\,|y|^{\frac{-1}{q_i-1}}$, and that ξ_i satisfies

$$\begin{cases} -\Delta \xi_i = 0, \ \xi_i > 0 & \text{in } \{y^n > -T_i\}, \\ -\dfrac{\partial \xi_i}{\partial x_n} = \frac{n-2}{2} H_i(y_i + ry) \xi_i^{q_i} & \text{on } \{y^n = -T_i\}. \end{cases} \tag{16}$$

It follows from Lemma 2.4 that for $0 < r < 1$, we have

$$\max_{\partial_2 B_r^+} \xi_i \leqslant C \min_{\partial_2 B_r^+} \xi_i, \tag{17}$$

where C is some positive constant depending only on n, $\bar{C}$, and A_1.

Noticing that $-\Delta\xi_i \geqslant 0$, $\xi_i(0) = 1$, $\frac{\partial \xi_i}{\partial y_n} \geqslant 0$, and using (17), we obtain (by the maximum principle and standard elliptic theory)

$$\max_{B_1^+} \xi_i \leqslant C \tag{18}$$

for some constant C, depending only on n, $\bar{C}$ and $\sup_i ||H_i||_{L^\infty(\overline{\partial_1 \Omega})}$. It follows from (14), (18) and standard elliptic theory that there exists some function $\xi \in C^2(\mathbb{R}^n)$ such that, after passing to a subsequence, $\xi_i \to \xi$ in $C^2_{\text{loc}}(\{y^n \geqslant T\})$, where $T = \lim_i T_i$ belongs to $[0, \infty]$ and where ξ satisfies

$$\begin{cases} -\Delta\xi = 0 & \text{in } \{y_n > -T\}, \\ -\dfrac{\partial \xi}{\partial x_n} = \dfrac{n-2}{2} \lim\limits_{i \to \infty} H_i(y_i + ry)\xi^{\frac{n}{n-2}} & \text{on } \{y^n = -T\} \text{ if } T < +\infty. \end{cases}$$

By the Liouville Theorem the case $T = +\infty$ cannot occur. Moreover, by the uniqueness result of Li and Zhu [21], we have $T = 0$ and

$$\xi(y', y_n) = \left(\left(1 + \lim_{i \to \infty} h_i y_n \right)^2 + \left(\lim_{i \to \infty} h_i \right)^2 |y'|^2 \right)^{\frac{2-n}{2}}.$$

The proof of Proposition 2.5 is thereby complete. □

We now state some technical results useful in performing blow up analysis. The proof of Proposition 2.6 and Lemma 2.7 below is omitted since it is similar to that of [13].

Proposition 2.6. *Let* $\{H_i\}_i \subset C^2(\partial_1 \Omega)$ *satisfy* (10) *with* $\Omega = B_2$ *and* (15) *for some positive constant* A_2. *Suppose that* $\{u_i\}_i$ *satisfy* (9), *and let* $y_i \to 0$ *be an isolated simple blow-up point of* $\{u_i\}_i$ *such that for some positive constant* A_3 *we have*

$$|y - y_i|^{\frac{1}{q_i - 1}} u_i(y_i) \leqslant A_3 \quad \textit{for all } y \in B_2^+. \tag{19}$$

Then there exists some positive constant $C = C(n, A_1, A_2, A_3, \rho)$ *such that*

$$u_i(y) \leqslant C u_i(y_i)^{-1} |y - y_i|^{2-n} \quad \textit{for } |y - y_i| \leqslant 1,$$

where ρ *is the constant in Definition* 2.3. *Furthermore, for some regular harmonic function* b *in* B_1^+, *satisfying* $\frac{\partial b}{\partial \nu} = 0$ *on* $\partial_1 B_1^+$, *we have, after passing to a subsequence*

$$u_i(y_i) u_i(y) \to h(y) = a|y|^{2-n} + b(y) \quad \textit{in } C^2_{\text{loc}}(\overline{B_1^+} \setminus \{0\}),$$

where $a = \frac{2^{n-1}}{\sqrt{2}} \Gamma(\frac{n}{2}) \frac{1}{(H(0))^{\frac{n-3}{2}}}$.

Lemma 2.7. *Under the assumptions of Proposition* 2.6 *there holds*

$$\tau_i := \frac{n}{n-2} - q_i = O\left(u_i(y_i)^{-\frac{2}{n-2}+o(1)}\right)$$

and therefore

$$u_i(y_i)^{\tau_i} = 1 + o(1).$$

The proof of the next lemma is an easy consequence of Propositions 2.5, 2.6 and Lemma 2.7.

Lemma 2.8. *Under the hypotheses of Proposition* 2.6, we have

$$\int_{\partial\mathbb{R}^n_+ \,\cap\, B_{r_i}(y_i)} |y-y_i|^s u_i(y)^{q_i+1} = \begin{cases} u_i(y_i)^{-\frac{2s}{n-2}}\left(\int_{\mathbb{R}^{n-1}} |z|^s (1+h_i^2\,|z|^2)^{-(n-1)}\,dz + o(1)\right), & -(n-1)<s<n-1,\\ O\left(u_i(y_i)^{-\frac{2(n-1)}{n-2}}\log(u_i(y_i))\right), & s=n-1,\\ o\left(u_i(y_i)^{-\frac{2(n-1)}{n-2}}\right), & s>n, \end{cases}$$

$$\int_{\partial\mathbb{R}^n_+ \cap (B_1\setminus B_{r_i})} |y-y_i|^s u_i(y)^{q_i+1} = \begin{cases} o\left(u_i(y_i)^{-\frac{2s}{n-2}}\right), & -(n-1)<s<n-1,\\ O\left(u_i(y_i)^{-\frac{2(n-1)}{n-2}}\log(u_i(y_i))\right), & s=n-1,\\ O\left(u_i(y_i)^{-\frac{2(n-1)}{n-2}}\right), & s>n, \end{cases}$$

where $h_i = \frac{1}{2}H_i(y_i')$.

Lemma 2.9. *Suppose that* $\{H_i\}_i$ *is bounded in* $C^2(\overline{B_{\bar r}^+(y_i)})$, *and suppose again that* $\{u_i\}_i$ *satisfy* (9). *Let* $y_i \to 0$ *be an isolated simple blow up of* $\{u_i\}_i$. *Then*

$$|\nabla' H_i(y_i')| = O\left(u_i(y_i)^{-\frac{2}{n-2}}\right).$$

Proof. Without loss of generality we can assume that $\bar r > 1$. Consider a cutoff function $\eta \in C_c^\infty(\overline{B_1^+})$ satisfying

$$\begin{cases} \eta(x) = 1, & x \in \overline{B^+_{\frac{1}{4}}},\\ \eta(x) = 0, & x \in \overline{\mathbb{R}^n_+}\setminus B^+_{\frac{1}{2}}. \end{cases}$$

Multiplying Eq. (9) by $\eta\frac{\partial u_i}{\partial x_1}$ and integrating by parts, taking into account the support of η and $\nabla\eta$, it follows (denoting $\Gamma_1 = \partial_1 B_1$)

$$\frac{1}{q_i+1}\int_{\Gamma_1}\frac{\partial H_i}{\partial x_1}u_i^{q_i+1}\,\eta = \int_{(B_{\frac{1}{2}}\backslash B_{\frac{1}{2}})^+}\nabla u_i\frac{\partial u_i}{\partial x_1}\nabla\eta - \frac{1}{2}\int_{(B^+_{\frac{1}{2}}\backslash B^+_{\frac{1}{4}})}\frac{\partial\eta}{\partial x_1}|\nabla u_i|^2$$
$$- \int_{B^+_{\frac{1}{2}}\backslash B^+_{\frac{1}{4}}}\frac{\partial\eta}{\partial x_1}H_i\,u_i^{q_i+1}.$$

Using Proposition 2.6 we derive

$$\int_{\Gamma_1}\frac{\partial H_i}{\partial x_1}u_i^{q_i+1}\eta = O(u_i(y_i)^{-2}). \tag{20}$$

Now, taking into account of the boundedness of $\{H_i\}_i$ in $C^2(\partial_1 B_1)$ we have

$$\frac{\partial H_i}{\partial x_1}(y_i)\int_{\Gamma_1}u_i^{q_i+1}\eta = \int_{\Gamma_1}\left(\frac{\partial H_i}{\partial x_1}(y_i) - \frac{\partial H_i}{\partial x_1}\right)u_i^{q_i+1}\,\eta + \int_{\Gamma_1}\frac{\partial H_i}{\partial x_1}u_i^{q_i+1}\eta$$
$$= O\left(\int_{\Gamma_1}|y-y_i|u_i^{q_i+1}\right) + \int_{\Gamma_1}\frac{\partial H_i}{\partial x_1}u_i^{q_i+1}\eta.$$

Using Lemma 2.8 and (20), we finally get

$$\frac{\partial H_i}{\partial x_1}(y_i) = O(u_i(y_i)^{-\frac{2}{n-2}}).$$

Clearly we can estimate $(\frac{\partial H_i}{\partial x_k})(y_i')$, $2\leqslant k\leqslant n-1$, in a similar way; so Lemma 2.9 follows immediately. □

3. Isolated blow-ups are isolated simple blow-ups

In this section we prove that, under suitable assumptions on $\{H_i\}_i$, an isolated blow-up point has to be an isolated simple blow-up point.

Proposition 3.1. *Suppose that $\{u_i\}_i$ satisfies Eq.* (9) *for $n=4$ and $\Omega = B_2$ and suppose that $\{H_i\}_i$ satisfies* (10) *and* (15). *Let $y_i\to 0$ with $|y-y_i|^{\frac{1}{q_i-1}}u_i(y)\leqslant A_3$, be an isolated blow-up point for $\{u_i\}_i$. Then* 0 *is an isolated simple blow-up point.*

Proof. It follows from Proposition 2.5 that $r^{\frac{1}{q_i-1}}\bar{u}_i(r)$ has precisely one critical point in the interval $(0, r_i)$, where $r_i = R_i\,u_i(y_i)^{1-q_i}$, as before. Suppose, arguing by contradiction, that 0 is not an isolated simple blow-up, and let μ_i be the second critical point of $r^{\frac{1}{q_i-1}}\bar{u}_i(r)$. We know, by Proposition 2.5, that $\mu_i \geqslant r_i$ and, by the contradiction argument that $\mu_i \to 0$. Without loss of generality, we assume that $y_i = 0$. Set

$$\xi_i(y) = \mu_i^{\frac{1}{q_i-1}} u_i(\mu_i y), \quad |y| \leqslant \frac{1}{\mu_i} \text{ and } y^n \geqslant 0.$$

It follows from (9) and from the properties of μ_i that ξ_i satisfies

$$\begin{cases} \Delta\xi_i(y) = 0, & |y| < \frac{1}{\mu_i},\ y^n > 0, \\ -\frac{\partial \xi_i}{\partial y^n} = H_i(\mu_i y')\xi_i(y)^{q_i}, & |y| < \frac{1}{\mu_i},\ y^n = 0, \\ |y|^{\frac{1}{q_i-1}}\xi_i(y) \leqslant A_3, & |y| < \frac{1}{\mu_i},\ y^n \geqslant 0, \\ \lim_i \ \xi_i(0) = +\infty. \end{cases} \tag{21}$$

Moreover, by our choice of μ_i there holds

$$r^{\frac{1}{q_i-1}}\bar{\xi}_i(r) \text{ has precisely one critical point in } 0 < r < 1,$$

$$\frac{d}{dr}(r^{\frac{1}{q_i-1}}\bar{\xi}_i(r))|_{r=1} = 0,$$

where $\bar{\xi}_i(r) = \frac{1}{|\partial B_r^+|}\int_{\partial B_r^+} \xi$.

It follows that 0 is an isolated simple blow-up for $\{\xi_i\}_i$. Therefore, applying Proposition 2.6, there exist some positive constant $a > 0$ and some regular harmonic function b in $\mathbb{R}^n_+$, satisfying $\frac{\partial b}{\partial v} = 0$ on ∂R^n_+ such that

$$\xi_i(0)\xi_i(y) \to h(y) = a|y|^{2-n} + b(y) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^n_+ \backslash \{0\}). \tag{22}$$

It follows from the maximum principle and the Liouville Theorem that b is a nonnegative constant.

The value of b can be derived as follows. Since, by our choice of μ_i, 1 is a critical point of $r^{\frac{4}{q_i-1}}\bar{\xi}_i(r)$, we have

$$0 = \lim_i \frac{d}{dr}(r^{\frac{1}{q_i-1}}\bar{\xi}_i(r))|_{r=1} = \frac{d}{dr}(r^{\frac{n-2}{2}}h(r))|_{r=1},.$$

it follows immediately that

$$b = a > 0.$$

Applying Proposition 6.1, and using Proposition 2.6 for all $0<\sigma<1$, we have

$$\int_{\Gamma_2} F(\sigma,x,\xi_i,\nabla\xi_i)\,d\sigma \geqslant \frac{c(n)}{q_i+1}\sum_{i=1}^{i=n-1}\int_{\Gamma_1} x_i\frac{\partial H_i}{\partial x_i}(\mu_i y)\xi_i^{q_i+1}\,dx + O(\xi_i(0)^{-q_i-1}),$$

where F is given in Proposition 6.1 and where $\Gamma_2=\partial B_\sigma^+\cap\mathbb{R}_+^n$.

Multiplying the above by $\xi_i(0)^2$ and sending i to infinity, we have

$$\begin{aligned}\int_{\Gamma_2} F(\sigma,x,h,\nabla h) &= \lim_i\ \xi_i(0)^2\int_{\partial B_\sigma} F(\sigma,x,\xi_i,\nabla\xi_i)\\ &\geqslant \lim_i\ \xi_i(0)^2\frac{n-2}{2(q_i+1)}\sum_j\int_{\Gamma_1} x_j\frac{\partial H_i(\mu_i\cdot)}{\partial x_j}\xi_i^{q_i+1},\end{aligned}$$

where h is the function given in (22).

Now we want to estimate the last expression. We recall that we are assuming that $\{H_i\}_i$ is uniformly bounded in $C^2(\partial_1 B_2)$, and we proceed as follows. We have, using the Taylor expansion of H_i at 0

$$\left|\sum_j\int_{\Gamma_1} x_j\frac{\partial H_i(\mu_i\cdot)}{\partial x_j}\xi_i^{q_i+1}\right| \leqslant \left|\int_{\Gamma_1} y\cdot\nabla H_i(0)\xi_i^{q_i+1}\right| + \mu_i^2\max_{B_{\sigma\mu_i}}|\nabla'^2 H_i(0)|\int_{B_\sigma}|y|^2\xi_i^{q_i+1}.$$

Applying Lemma 2.9 we have $|\nabla' H_i(0)| = O(\xi_i(0)^{-\frac{q_i-1}{2}})$. Therefore, using Lemma 2.8 and the fact that $\mu_i\to 0$, we deduce

$$\int_{\Gamma_2} F(\sigma,x,h,\nabla h) \geqslant \lim_i\ \xi_i(0)^2\frac{n-2}{2(q_i+1)}\sum_j\int_{\Gamma_1} x_j\frac{\partial H_i(\mu_i\cdot)}{\partial x_j}\xi_i^{q_i+1} = 0. \qquad (23)$$

Now, by Proposition 6.2, we know that for $\sigma>0$ sufficiently small

$$\int_{\Gamma_2} F(\sigma,x,h,\nabla h)<0,$$

and this contradicts (23). This concludes the proof of the Proposition. □

4. A priori estimates

Consider, for $n \geqslant 3$, the following equation

$$\begin{cases} \Delta v = 0, \ v > 0 & \text{on } B^n, \\ \dfrac{\partial}{\partial v} v + \dfrac{n-2}{2} v = \dfrac{n-2}{2} H v^q & \text{on } S^{n-1}, \\ q = \dfrac{n}{n-2} - \tau, \ 0 \leqslant \tau \leqslant \dfrac{1}{n-2}. \end{cases} \tag{24}$$

Proposition 4.1. *Suppose $H \in C^2(S^{n-1})$ satisfies for some positive constants A_1, A_2*

$$H(p) \geqslant \frac{1}{A_1} \quad \textit{for all } p \in S^{n-1}, \quad \|\nabla' H\|_{L^\infty(S^{n-1})} \leqslant A_2.$$

Then, for any $0 < \varepsilon < 1$ and any $R > 1$, there exist some positive constants $C_0^* > 1$, $C_1^* > 1$, such that, if v is a solution of (24) *with*

$$\max_{S^n} v > C_0^*,$$

then there exists $1 \leqslant k = k(v) < +\infty$ and a set

$$\mathcal{S}(v) = \{p_1, \ldots, p_k\} \subseteq S^{n-1}$$

$(p_i = p_i(v))$ such that

(1) $0 \leqslant \tau < \varepsilon$,

(2) *$p_1, \ldots, p_k$ are local maxima of v, and for each $1 \leqslant j \leqslant k$ (y being some geodesic normal coordinates centered at p_j), we have*

$$\|v(0)^{-1} v(v(0)^{1-q} y) - \delta_j(y)\|_{C^2(B^+_{2R}(0))} < \varepsilon$$

and

$$\{B^+_{Rv(q_j)^{-\frac{q-1}{4}}}(p_j)\}_{1 \leqslant j \leqslant k} \quad \textit{are disjoint balls.}$$

Here

$$\delta_j(y) = ((1 + h_i y_n)^2 + h_j^2 |y|^2)^{\frac{2-n}{2}}$$

is the unique solution of

$$\begin{cases} \Delta \delta_j = 0 \quad \textit{in } \mathbb{R}^n_+, \\ -\dfrac{\partial \delta_j}{\partial x_n} = \dfrac{n-2}{2} h(p_i)\, \delta_j^{\frac{n}{n-2}} \quad \textit{on } \partial \mathbb{R}^n_+, \\ \delta_j > 0, \ \delta_j(0) = 1, \ \nabla' \delta_j(0) = 0, \end{cases}$$

and

$$h_j = \tfrac{1}{2}H(p_j).$$

(3) $v(p) \leqslant C_1^*\{dist(p, \mathcal{S}(v))\}^{-\frac{1}{q_i-1}}$, *for all* $p \in S^n$, *and* $dist(p_i, p_j)^{1-q_i} v(p_j) \geqslant C_0^*$.

Proof. This can be proved by quite standard blow-up arguments. □

Proposition 4.2. *Let* $n = 4$, *and suppose that* $H \in C^2(S^{n-1})$ *satisfies for some positive constant* A_1

$$H \geqslant A_1.$$

Then, for every $\varepsilon > 0$ *and* $R > 1$, *there exist some positive constant* $\delta^* > 0$, *depending on* $n, \varepsilon, R, A_1, ||H||_{C^2}$, *such that for any solution* v *of* (24) *with* $\max_{\overline{B^n}} v > C_0^*$ *we have*

$$|p_j - p_l| \geqslant \delta^* \quad \text{for all } 1 \leqslant j \neq l \leqslant k,$$

where $p_j = p_j(v),\ p_l = p_l(v)$ *are as in Proposition* 4.1.

Proof. Suppose the contrary, that is for some constants ε, R, A_1 there exist $\{q_i\}_i$, $\{H_i\}_i$ satisfying the hypotheses and a sequence of corresponding solutions v_i such that

$$\lim_i \min_{j \neq l} |p_j - p_l| = 0.$$

Without loss of generality, we assume that

$$|p_1(v_i) - p_2(v_i)| = \min_{j \neq l} |p_j(v_i) - p_l(v_i)| \to 0. \tag{25}$$

Since $B^+_{Rv_i(p_1)^{1-q_i}}(p_1)$ and $B_{Rv_i(p_2)^{1-q_i}}(p_2)$ are disjoint, we have, according to Eq. (25), that $v_i(p_1) \to +\infty$ and $v_i(p_2) \to +\infty$. Therefore, we can pass to a subsequence with $R_i \to +\infty$, $\varepsilon_i \to 0$ as in Proposition 2.5 such that, for y being any geodesic normal coordinate system centered at p_j, $(j = 1, 2)$, we have

$$||v(0)^{-1} v(v(0)^{1-q_i} y) - ((1 + h_{ij} y_n)^2 + h_{ij}^2 |y'|^2)^{\frac{2-n}{2}}||_{C^2(B^+_{2R_i}(0))} < \varepsilon_i,$$

where $h_{ij} = \frac{1}{2} h_i(p_j)$, $j = 1, 2,\ i = 1, 2, \ldots$.

Let us work on $\mathbb{R}^n_+$, instead of B^n. Without loss of generality, we may assume that $p_1 = (0, 1) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$. Consider the conformal map $\phi : \mathbb{R}^n_+ \to B$ defined as

$$\phi(x', x_n) = \left(\frac{2x'}{|x'|^2 + (x_n + 1)^2}, \frac{|x|^2 - 1}{|x'|^2 + (x_n + 1)^2} \right)$$

and let y denotes the ϕ-coordinates of x, and set

$$u_i(y) = \left(\frac{2}{|x'|^2 + (x_n+1)^2}\right)^{\frac{n-2}{2}} v_i(y).$$

It is easy to see, using the conformal invariance of the boundary operator, that u_i satisfies

$$\begin{cases} \Delta u_i = 0, \ u_i > 0 \quad \text{on } \mathbb{R}^n_+, \\ -\dfrac{\partial}{\partial x_n} u_i = \frac{n-2}{2} H_i(x) H(x)^{\tau_i} u_i^{q_i} \quad \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{26}$$

with

$$H(y) = \left(\frac{2}{(1+y_n)^2 + |y|^2}\right)^{\frac{n-2}{2}}.$$

Let us still denote by $p_2 \in \mathbb{R}^n$ the ϕ-coordinates of $p_2 \in S^{n-1}$, and set $\sigma_i = |p_2| \to 0$. For simplicity we assume that p_2 is a local maximum of u_i.

It is easy to see that for some constant $C(n) > 1$ depending only on n we have

$$\sigma_i > \frac{1}{C(n)} \max\{R_i u_i(0)^{1-q_i}, R_i u_i(p_2)^{1-q_i}\}.$$

Set now

$$w_i(y) = \sigma_i^{\frac{1}{q_i-1}} u_i(\sigma_i y), \quad |y| < \frac{1}{\sigma_i}, \ y_n \geqslant 0.$$

It follows that w_i satisfies the following equation:

$$\begin{cases} \Delta w_i = 0, \ w_i > 0 \quad \text{in } |y| \leqslant \dfrac{1}{\sigma_i}, \ y_n > 0, \\ \dfrac{\partial}{\partial y_n} w_i = \frac{n-2}{2} H_i(\sigma_i y) H(\sigma_i y)^{\tau_i} w_i^{q_i} \quad \text{in } |y| \leqslant \dfrac{1}{\sigma_i}, \ y_n = 0. \end{cases} \tag{27}$$

Note that v_i satisfies (see Proposition 4.1)

$$\begin{cases} v_i(y) \leqslant C_1 |y|^{-\frac{1}{q_i-1}}, \quad |y| \leqslant \dfrac{1}{2}\sigma_i, \\ v_i(y) \leqslant C_1 |y-p_2|^{-\frac{1}{q_i-1}}, \quad |y-p_2| \leqslant \dfrac{1}{2}\sigma_i. \end{cases}$$

Clearly,

$$\begin{cases} \lim_i w_i(0) = +\infty, \quad \lim_i w_i(|p_2|^{-1}p_2) = +\infty, \\ |y|^{\frac{1}{q_i-1}} w_i(y) \leqslant C_1, \quad |y| \leqslant \frac{1}{2}, \\ |y - |p_2|^{-1}p_2|^{\frac{4}{q_i-1}} w_i(y) \leqslant C_1, \quad |y - |p_2|^{-1}p_2| \leqslant \frac{1}{2}. \end{cases}$$

It follows that 0 and $\bar{q} = \lim_i |p_2|^{-1}p_2$ are both isolated blow-up points for w_i. We notice that both 0 and $\bar{q}$ are actually isolated simple blow-up points of w_i. In fact, this can be proved by using the same arguments as in Proposition 3.1.

By property (3) in Proposition 4.1 and Eq. (25), there exist an at most countable set $\tilde{\mathcal{S}}_1 \subseteq \mathbb{R}^n$ such that

$$\min\{|x-y| \mid x, y \in \tilde{\mathcal{S}}_1\} \geqslant 1$$

and

$$\begin{cases} \lim_i w_i(0) w_i(y) = h^*(y) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^n \setminus \tilde{\mathcal{S}}_1), \\ h^*(y) > 0, \ y \in (\mathbb{R}^n_+ \setminus \tilde{\mathcal{S}}_1). \end{cases}$$

Let $\mathcal{S}_1 \subseteq \tilde{\mathcal{S}}_1$ contain those points near which h^* is singular. From Proposition 2.6 we know that $w_i(0)$ and $w_i(|p_2|^{-1}p_2)$ are of the same order, hence both 0 and $\bar{p} = \lim_i |p_2|^{-1}p_2$ belong to $\mathcal{S}_1$. It follows from Eq. (27), the Harnack inequality and the maximum principle that there exist some nonnegative function $b^*(y)$ satisfying

$$\begin{cases} b^*(y) \geqslant 0, & y \in \mathbb{R}^n_+ \setminus \{\mathcal{S}_1 \setminus \{0, \bar{q}\}\}, \\ \Delta b^*(y) = 0, & y \in \mathbb{R}^n_+ \setminus \{\mathcal{S}_1 \setminus \{0, \bar{q}\}\}, \\ \Delta b^*(y) = 0, & y \in \mathbb{R}^n_+ \setminus \{\mathcal{S}_1 \setminus \{0, \bar{q}\}\} \end{cases}$$

and some positive constants a_1, a_2 such that

$$h^*(y) = a_1|y|^{2-n} + a_2|y-\bar{q}|^{2-n} + b^*(y), \quad y \in \mathbb{R}^n_+ \setminus \{\mathcal{S}_1 \setminus \{0, \bar{q}\}\}. \tag{28}$$

For $0 < \sigma < 1$ we apply Proposition 6.1 to Eq. (27) and we obtain

$$\begin{aligned} \int_{\Gamma_2} F(\sigma, x, h^*, \nabla h^*) &= \lim_i w_i(0)^2 \int_{\Gamma_2} F(\sigma, x, w_i, \nabla w_i) \\ &\geqslant \lim_i w_i(0)^2 \frac{\frac{n-2}{2}}{q_i+1} \sum_j \int_{\Gamma_1} x_j \frac{\partial h_i(\sigma_i \cdot) H_i^{\tau_i}(\sigma_i \cdot)}{\partial x_j} w_i^{q_i+1} = 0. \end{aligned} \tag{29}$$

The last inequality can be deduced reasoning as in the proof of Proposition 3.1.

On the other hand, we use (28) and apply Corollary 6.2 to obtain that

$$\int_{\partial B_\sigma} F(\sigma, x, h^*, \nabla h^*) < 0$$

for $\sigma > 0$ sufficiently small, which contradicts (29). Thus the Proposition is established. □

Now consider the following equations for $i = 1, 2, 3\ldots$

$$\begin{cases} -\Delta_{g_0} u = 0, \ u > 0 & \text{in } B^n, \\ \dfrac{\partial u}{\partial \nu} + \dfrac{n-2}{2} u = \dfrac{n-2}{2} H_i u^{q_i} & \text{on } S^{n-1}, \\ q_i = \dfrac{n}{n-2} - \tau_i, \ \tau_i \geqslant 0, \ \tau_i \to 0. \end{cases} \tag{30}$$

Our main result about blow-up analysis is the following.

Theorem 4.3. *Let $n = 4$, and assume that for some positive constants A_1 and A_2*

$$H_i(p) \geqslant \frac{1}{A_1} \quad \textit{for all } p \in S^{n-1} \quad \textit{and} \quad ||H_i||_{C^2} \leqslant A_2.$$

Let $\{v_i\}_i$ be solutions of (30), we have

$$||v_i||_{H^{1,2}(\overline{B^n})} \leqslant C,$$

where C is a fixed constant. Furthermore, after passing to a subsequence, either $\{v_i\}_i$ stays bounded in $L^\infty(\bar{B}^n)$ (hence bounded in $C^{2,\alpha}(\bar{B}^n)$), or $\{v_i\}_i$ has only boundary isolated simple blow-up points and the distance between any two blow-up points is bounded below by some fixed positive constant independent of i.

Proof. It is a simple consequence of Propositions 2.5, 3.1, 4.2 and Lemma 2.8. □

5. Existence and compactness results on B^4

In this section we prove Theorem 1.1. We start by giving some further characterization of the blow up points for solutions of (9). We recall the definition of the matrix M_{lj} given in formula (5) and its least eigenvalue ρ.

Proposition 5.1. *Let $H \in C^2(S^3)$. Then there exists some number $\delta^* > 0$, depending only on $||H||_{C^2(S^3)}$, with the following properties:*

Let $\{q_i\}_i$ *be such that* $q_i \leqslant 2$, $q_i \to 2$, *let* $H_i \to H$ *in* $C^2(S^3)$, *and let* $v_i > 0$ *satisfy*

$$\begin{cases} \Delta v_i = 0 & \text{in } B^4, \\ \dfrac{\partial v_i}{\partial \nu} + v_i = H_i(x')v_i^{q_i} & \text{on } S^3 \end{cases} \tag{31}$$

with $\max_{\bar{B}^4} v_i \to +\infty$ *as* $i \to +\infty$. *Then, after passing to a subsequence, the following properties hold true*:

(i) $\{v_i\}_i$ *has only boundary isolated simple blow-up points* $(p^1, \dots, p^k) \in \mathcal{H} \setminus \mathcal{H}^- (k \geqslant 1)$, *with* $|p^j - p^i| \geqslant \delta^*$ *for all* $j \neq k$, *and* $\rho(p^1, \dots, p^k) \geqslant 0$. *Furthermore* $(p^1, \dots, p^k) \in \mathcal{H}^+$ *if* $k \geqslant 2$.

(ii)
$$\lambda_j := H(p^j)^{-1} \lim_i v_i(p_i^1)(v_i(p_i^j))^{-1} \in]0, +\infty[$$

and

$$\mu^j := \lim_i \tau_i v_i(p_i^j)^2 \in [0, +\infty[, \quad \forall j \in [1, k],$$

where $p_i^j \to p^j$ *is a local maximum of* v_i.

(iii) *When* $k = 1$

$$\mu^1 = -\frac{1}{12}\frac{\Delta H(p^1)}{(H(p^1))},$$

when $k \geqslant 2$

$$\sum_{l=1}^{k} M_{lj}\lambda_l = \frac{1}{12}\lambda_j \mu^j, \quad \forall j \in [1, k]. \tag{32}$$

(iv) $\mu^j \in]0, +\infty[\ \forall j \in [1, \dots, k]$ *if and only if* $\rho(p^1, \dots, p^k) > 0$.

Proof. Assertion (ii) follows from Proposition 2.6 and Lemmas 2.4 and 2.7. From another part, it follows from Propositions 3.1 and 4.2 that v_i has only isolated simple blow up points $q^1, \dots, q^N \in \mathcal{H}(N \geqslant 1)$ with $|q^j - q^l| \geqslant \delta^* (j \neq l)$ for a fixed $\delta^* > 0$ depending only on the above quantities.

Let $q_i^1 \to q^1$ be the local maximum of v_i for which $v_i(q_i^1) \to +\infty$; performing a stereographic projection through the point $-q^1$, Eq. (31) is transformed into

$$\begin{cases} \Delta u_i = 0 & \text{in } \mathbb{R}^4_+, \\ \dfrac{\partial v_i}{\partial \nu} = W(x')^{\tau_i} H_i(x') v_i^{q_i} & \text{on } \partial\mathbb{R}^4_+, \end{cases}$$

where $W(x') = \frac{2}{1+|x'|^2}$.

By our choice of the projection, it is clear that 0 is also an isolated simple blow up point for $\{u_i\}_i$. We can also suppose that none of the points $\{q^1,\dots,q^N\}$ is mapped to $+\infty$ by the stereographic projection, and we still denote their images by $q^1,\dots,q^N$ (in particular we have $q^1=0$). It follows from Proposition 2.6 that

$$u_i(q_i^1)u_i(x)\to h_1(x) := \frac{1}{2}\frac{1}{H(q^1)^2}|x|^{-2}+b_1(x) \quad \text{in } C^2_{\mathrm{loc}}(\overline{\mathbb{R}^3_+}\setminus\{q^1,\dots,q^N\}). \tag{33}$$

The function $b_1(x)$ is harmonic in $\overline{\mathbb{R}^4_+}\setminus\{q^2,\dots,q^N\}$, and we have still used the notation q_i^j for the local maxima of u_i converging to q^j.

Coming back to v_i on B^4 we have

$$\lim_i v_i(q_i^1)v_i(x)=\frac{1}{4H(q^1)^2}G_{q^1}(x)+\tilde{b}_1(x) \quad \text{in } C^2_{\mathrm{loc}}(\overline{B^4}\setminus\{q^1,\dots,q^N\}),$$

where $\tilde{b}_1$ is some regular function on $\overline{B^4}\setminus\{q^2,\dots,q^N\}$ satisfying $\Delta\tilde{b}_1=0$ with $\frac{\partial\tilde{b}_1}{\partial v}+\tilde{b}_1=0$ on $S^3\setminus\{q^2,\dots,q^N\}$.

If $N=1$, then $\tilde{b}_1=0$ by the maximum principle, while for $N\geqslant 2$, taking into account the contribution of all the poles, we deduce

$$\begin{aligned}\lim_i v_i(q_i^j)v_i(x) = {} & \frac{1}{4}\frac{1}{H(q^1)^2}G_{q^1}(x)+\frac{1}{4}\sum_{l\neq 1}\frac{1}{H(q^l)^2}G_{q^l}(x)\\ & \times\lim_i\frac{v_i(q_i^1)}{v_i(q_i^l)} \quad \text{in } C^4(\overline{B^4}\setminus\{q^1,\dots,q^N\}).\end{aligned}$$

In fact, subtracting all the poles from the limit function, we obtain a regular $r:\overline{B^4}\to\mathbb{R}$ such that $\Delta r=0$ and $\frac{\partial r}{\partial v}+r=0$ on S^3, so it must be $r\equiv 0$. In the above formulas, $G_q(x)$ is the function defined in the Introduction.

Using the last expression, we can compute the value of $b_1(0)$ in (33), which is

$$b_1(0)=\frac{1}{4}\sum_{l\neq 1}\frac{\lambda_1}{\lambda_l}\frac{1}{H(q^l)H(q^l)}G_{q^l}(x). \tag{34}$$

Hence, using (34) and Proposition 6.2, we deduce

$$\lim_{\sigma\to 0}\int_{\partial_2 B_\sigma}F(\sigma,x,h_1,\nabla h_1)=-\frac{\pi^2}{4(H(q^1))^2}\sum_{l\neq 1}\frac{\lambda_l}{\lambda_1}\frac{1}{H(q^l)H(q^l)}G_{q^l}(q^1).$$

From another part, from Lemmas 6.1, 2.7, Proposition 2.5, and some computations as in Lemma 2.8, it follows that

$$\int_{\partial_2 B_\sigma}F(\sigma,x,h_1,\nabla h_1)=4\pi^3\frac{\Delta H(q^1)}{H(q^1)^3}+\frac{\pi^3}{3}\frac{1}{H(q^1)^2}\mu^1.$$

In the above formula we have used Proposition 2.5 and Lemma 2.7 to derive

$$\frac{1}{3}\lim_{i} u_i(q_i^1)^2 \int_{B_\sigma} x \cdot \nabla H_i u_i^{p_i+1} = 4\pi^3 \frac{\Delta H(q^1)}{H(q^1)^3}.$$

Taking into account (34) and Proposition 6.2, we obtain

$$4\pi^3 \frac{\Delta H(q^1)}{H(q^1)^3} + \frac{\pi^3}{3}\frac{1}{H(q^1)^2}\mu^1 = -\frac{\pi^2}{4}\frac{1}{H(q^1)^2}\frac{1}{4}\sum_{l\neq 1}\frac{\lambda_1}{\lambda_l}\frac{1}{H(q^l)H(q^l)}G_{q^l}(x).$$

Finally, using the expression of $\{\mu^l\}$ and λ_1 we get

$$-\frac{\Delta H(q^1)}{H(q^1)}\lambda_1 - \frac{\pi}{16}\sum_{l\neq 1}\frac{G_{q^l}(q^1)}{H(q^1)H(q^l)}\lambda_l = \lambda_1\mu^1.$$

Of course a similar formula holds for every q^j with $j\neq 1$. We have thus established (32) and completed the proof of (iii).

From the last formula it follows that $q^j \in \mathcal{H}\backslash\mathcal{H}^-, \forall j = 1, \dots, N$, and when $N \geqslant 2$, $q^j \in \mathcal{H}^+$. Furthermore, since $M_{jj} \geqslant 0$ for every j, and $M_{lj} < 0$ for $l \neq j$, it follows from linear algebra and the variational characterization of the least eigenvalue that there exists some $y = (y_1, \dots, y_N) \neq 0$, $y_l \geqslant 0$ $\forall l$, such that $\sum_{j=1}^N M_{lj} y_j = \rho y_l$.

Multiplying (32) by y_j and summing over j, we have

$$\rho \sum_l \mu^l y_l = \sum_{l,j} M_{lj} y_j \mu^l = \sum_j \lambda_j \mu^j y_j \geqslant 0.$$

It follows that $\rho \geqslant 0$, so we have verified part (i). Part (iv) follows from (i)–(iii). □

As a consequence of Proposition 5.1 we deduce the following corollary. It asserts that, in the *nondegenerate* situation $H \in \mathcal{A}$, one can obtain a priori estimates for positive solutions of the subcritical problem, provided q is sufficiently close to $\frac{n}{n-2}$.

Consider the following problem

$$\begin{cases} \Delta u = 0 & \text{in } B^4, \\ \dfrac{\partial u}{\partial v} + u = H u^{2-\tau} & \text{on } S^3. \end{cases} \tag{35}$$

Corollary 5.2. *Suppose $H \in \mathcal{A}$. Then there exist constants τ_* and C_*, depending only on $\|H\|_{C^2(S^3)}$ and $\min\{|\rho(q^1, \dots, q^N)| : q^1, \dots, q^N \in \mathcal{H}, N \geqslant 1\}$ such that*

$$\max_{\overline{B^4}} v \leqslant \frac{C_*}{\sqrt{\tau}}$$

for every positive solution v of (35) *with* $0 < \tau = \dfrac{n}{n-2} - q < \tau_*$.

For $\tau = \frac{n}{n-2} - p$, $\tau > 0$, let J_τ denote the Euler functional corresponding to problem (35), namely

$$J_\tau(v) = \frac{1}{2}\int_{B^4} |\nabla v|^2 + \frac{1}{2}\int_{S^3} v^2 - \frac{1}{3-\tau}\int_{S^3} H|v|^{3-\tau}, \quad v \in W^{1,2}(B^4).$$

For $q \in S^{n-1}$ and $t \in [1, +\infty)$ we denote by $\varphi_{q,t}$ the conformal map on B^n defined as follows: using stereographic coordinates with projection through the point q, we set

$$\varphi_{q,t}(y) = ty$$

and set $\delta_{a,\lambda}(x) = \varphi_{q,t}(1)$.

Let $q^1, \ldots, q^N \in \mathcal{H}^+$ be critical points of H with $\rho(q^1, \ldots, q^N) > 0$. For ε small, define the set $V_\varepsilon = V_\varepsilon(\tau, q^1, \ldots, q^N) \subseteq W^{1,2}(B^4)$ as

$$V_\varepsilon := \left\{ \sum_{i=1}^{N} \frac{1}{H(a_i)} \delta_{a_i, \gamma_i} : (\gamma, a) \in \mathbb{R}_+^N \times (S^3)^N,\ |a_i - q^i| < \varepsilon,\ \varepsilon < \tau\gamma_i < \frac{1}{\varepsilon} \right\}.$$

We also define $\mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon(\tau, q^1, \ldots, q^N)$ to be the ε-tubular neighborhood of V_ε, namely

$$\mathcal{U}_\varepsilon = \{ v + z : v \in V_\varepsilon, z \in (T_v V_\varepsilon)^\perp, ||z|| < \varepsilon \},$$

where $(T_v V_\varepsilon)^\perp$ denotes the subspace of $W^{1,2}(B^4)$ orthogonal to $T_v V_\varepsilon$.

For $R > 0$, set

$$\mathcal{O}_R = \left\{ v \in C^{2,\alpha}(\overline{B^4}) \,\middle|\, \frac{1}{R} \leqslant v \leqslant R,\ ||v||_{C^{2,\alpha}(\overline{B^4})} \leqslant R \right\}.$$

Using the last definitions and standard regularity results, Proposition 5.1 can be reformulated as follows.

Proposition 5.3. *Let $H \in \mathcal{A}$ and let $\alpha \in]0, 1[$. Then there exist a small positive constant ε, and a large positive constant R such that, when $\tau > 0$ is sufficiently small, there holds*

$$v \in \mathcal{O}_R \cup \{ \mathcal{U}_\varepsilon(\tau, q^1, \ldots, q^N) : q^1, \ldots, q^N \in \mathcal{H}^+, \rho(q^1, \ldots q^N) > 0, N \geqslant 1 \}$$

for all $v \in W^{1,2}(B^4)$ satisfying $v \geqslant 0$ a.e. and $J_\tau{}'(v) = 0$.

Using blow up analysis, we gave necessary conditions on blowing up solutions of (9) when q_i tends to $\frac{n}{n-2}$ from below. Now we are going to show that if $H \in \mathcal{A}$, one can construct solutions highly concentrating at arbitrary points $q^1, \ldots, q^N \in \mathcal{H}^+$ provided $\rho(q^1, \ldots, q^N) > 0$, see Proposition 5.4 below. The main tool is the Implicit Function

Theorem. Since the procedure is well-known, see [20,26], we just give a general idea of the proof omitting some details.

Following the original arguments in [5], one can prove that for τ sufficiently small

$$||J_\tau{}'(v)|| \leqslant O(\tau\,|\log\tau|) \quad \text{for } v \in V_\varepsilon.$$

Moreover, from Proposition 3.2 in [2] and standard computations, it follows that, for τ small, $I_\tau''(u)$ is invertible in $(T_v V_\varepsilon)^\perp$, uniformly with respect to τ and $v \in V_\varepsilon$. Hence by the local inversion theorem, see [5], there exists $\varepsilon > 0$ small (independent of τ) with the following property. For any $v \in V_\varepsilon$, there exists a unique $w(v,\tau)$ such that

$$w(v,\tau) \in (T_v V_\varepsilon)^\perp, \quad J_\tau{}'(v + w(v,\tau)) \in T_v V_\varepsilon. \tag{36}$$

Furthermore, the norm of $w(v,\tau)$ can be estimated as

$$||w(v,\tau)|| \leqslant C||J_\tau{}'(v)|| \leqslant C'\tau|\log\tau|, \tag{37}$$

where C and C' are fixed constants. By means of Eq. (36), the manifold

$$\tilde{V}_\varepsilon = \{v + w(v,\tau) : v \in V_\varepsilon\}$$

is a *natural constraint* for J_τ, namely a point u which is critical for $J_\tau|_{\tilde{V}_\varepsilon}$ is also critical for J_τ. In order to find critical points of $J_\tau|_{\tilde{V}_\varepsilon}$, we differentiate $J_\tau(v + w(v,\tau))$ with respect to the parameters a_i, γ_i. Using standard estimates we obtain

$$\frac{\partial}{\partial a_i} J_\tau(v + w(v,\tau)) = \frac{\partial H}{\partial a_i} + o(1), \quad v \in V_\varepsilon, \ \tau \to 0 \tag{38}$$

and

$$\begin{aligned}\frac{\partial}{\partial \lambda_i} J_\tau(v + w(v,\tau)) = {} & \tau\frac{\pi^2}{12}\frac{1}{H(a_i)^2}\frac{1}{\lambda_i} + \frac{\pi^2}{4}\frac{\Delta H(a_i)}{H(a_i)^4}\frac{1}{\lambda_i^3} \\ & + \pi^2 \sum_{i \neq j} \frac{G(a_i,a_i)}{H(a_i)H(a_i)}\frac{1}{\lambda_i^2\lambda_j} + o(\tau^3).\end{aligned} \tag{39}$$

Using (38) and the fact that H is a Morse function, one can prove that

$$deg_{W^{1,2}(B^4)}(J_\tau{}'|_{\tilde{V}_\varepsilon}, \tilde{V}_\varepsilon, 0) = (-1)^{\sum_{i=1}^N (3 - m(H,q^j))}. \tag{40}$$

Using the invertibility of J_τ'' in the normal direction to V_ε, and the fact that the functions δ_{a_i,γ_i} have Morse index 1, it follows from (40) that

$$deg_{W^{1,2}(B^4)}(J_\tau{}', \mathcal{U}_\varepsilon, 0) = (-1)^{N + \sum_{j=1}^n (3 - m(H,q^j))}. \tag{41}$$

Since the above degree is always different from zero, J_τ has at least one critical point in $\mathcal{U}_\varepsilon$; moreover it is standard to prove that critical points of J_τ in $\mathcal{U}_\varepsilon$ are nonnegative functions when τ is sufficiently small. Then, from [1,12] it follows that these solutions are also regular and strictly positive.

We collect the above discussion in the following Proposition.

Proposition 5.4. *Let* $H \in \mathcal{A}$, *and let* $\varepsilon > 0$ *be small enough. Then, if* $q^1, \dots, q^N \in \mathcal{H}^+$ *with* $\rho(q^1, \dots, q^N) > 0$, *and if* $\tau > 0$ *is sufficiently small, the functional* J_τ *possesses a critical point in* $\mathcal{U}_\varepsilon(\tau, q^1, \dots, q^N)$. *Moreover, formula* (41) *holds true and all the critical points of* J_τ *are strictly positive functions on* $\bar{B}^4$.

When $H \in \mathcal{A}$ and the number τ is bounded from below, we have compactness result for positive solutions of (3) and we can compute their total degree. We recall the above definition of the set $\mathcal{O}_R$.

Proposition 5.5. *Suppose* $H \in \mathcal{A}$. *Then for any* $\tau_0 > 0$ *there exist constants* C_0 *and* δ_0, *depending only on* $\tau_0, \min_{S^3} H$ *and* $||H||_{C^2(S^3)}$ *with the following properties*:

(i) $\{v \in W^{1,2}(B^4) : v \geqslant 0 \ a.e., J_{\tau_0}{}'(v) = 0\} \subseteq \mathcal{O}_{C_0}$;

(ii) *for* $C, \delta > 0$ *set* $\mathcal{O}_{C,\delta} = \{u \in W^{1,2}(B^4) : \exists v \in \mathcal{O}_C$ *such that* $||u - v||_{W^{1,2}(B^4)} < \delta\}$; *then* $J_{\tau_0}{}' \neq 0$ *on* $\partial \mathcal{O}_{C_0,\delta_0}$, *and*

$$deg_{W^{1,2}(B^4)}(u - \Xi(H\,|u|^{3-\tau_0}u), \mathcal{O}_{C_0,\delta_0}, 0) = -1. \tag{42}$$

Proof. The proof follows from the same arguments used to prove Proposition 2.7 in [20], so we omit it. □

Proof of Theorem 1.1. From the Harnack inequality and standard elliptic estimates it is enough to prove upper bounds for v in (7). Arguing by contradiction, by Proposition 5.1 there exist a sequence of solutions $\{v_i\}$ of (3) blowing up at $q^1, \dots, q^N \in S^3$, and these blow ups are isolated simple. Taking into account that $H \in \mathcal{A}$ and that $\lambda_j = 0$ for all j (since $\tau_i = 0$ for all i), we get a contradiction from Proposition 5.1 (iv). Hence (7) is proved.

Using Proposition 5.3 and the homotopy invariance of the Leray–Schauder degree, we have

$$\deg_{C^{2,\alpha}(\overline{B^4})}(u - \Xi(H|u|^2u), \mathcal{O}_R, 0) = \deg_{C^{2,\alpha}(\overline{B^4})}(u - \Xi(H|u|^{3-\tau}u), \mathcal{O}_R, 0) \tag{43}$$

for τ sufficiently small. By Propositions 5.3 and 5.4, for a suitable value of ε and for τ small, we know that the nonnegative solutions of $J_\tau{}' = 0$ are either in $\mathcal{O}_R$ or in some $\mathcal{U}_\varepsilon(\tau, q^1, \dots, q^N)$; viceversa for all $q^1, \dots, q^N \in \mathcal{H}_+$ with $\rho(q^1, \dots, q^k) > 0$, there are (positive) solutions of $J_\tau{}' = 0$ in $\mathcal{U}_\varepsilon$, and degree of $J_\tau{}'$ on $\mathcal{U}_\varepsilon$ is given by (41).

Let $C_0 \gg R$, τ_0 and δ_0 be given by Proposition 5.5; take also $\delta_1 \ll \delta_0$. By Proposition 5.4, (42) and by the excision property of the degree, we have

$$\deg_{W^{1,2}(B^4)}(u - \Xi(H|u|^{3-\tau_0}u), \mathcal{O}_{R,\delta_1}, 0) = \text{Index}(H). \tag{44}$$

As in the proof of Proposition 5.5, one can check that there are no critical points of J_{τ_0} in $\overline{\mathcal{O}_{R,\delta_1}}\backslash\mathcal{O}_R$, hence Theorem B.2 of [20] applies and yields

$$\deg_{W^{1,2}(B^4)}(u-\Xi(K_0|u|^{3-\tau_0}u),\mathcal{O}_{R,\delta_1},0)=\deg_{C^{2,\alpha}(\overline{B^4})}(u-\Xi(K_0|u|^{3-\tau_0}u),\mathcal{O}_R,0). \tag{45}$$

Then the conclusion follows from (43)–(45). The proof of Theorem 1.1 is thereby completed. □

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Appendix

For σ and $\bar{x}\in\mathbb{R}^n$, we set $\mathbb{R}^n_+=\{(x_1,\dots,x_n)\in\mathbb{R}^n; x_n>0\}$, $B_\sigma(\bar{x})=\{x\in\mathbb{R}^n; |x|<\sigma\}$, and $B^+_\sigma(\bar{x})=\{x\in\mathbb{R}^n; |x|<\sigma\}\cap\mathbb{R}^n_+$, $\Gamma_1=\partial B^+_\sigma\cap\partial\mathbb{R}^n_+$ and $\Gamma_2=\partial B^+_\sigma\cap\mathbb{R}^n_+$.

Proposition 6.1 (Pohozaev identity). *Let* $n\geqslant 3$, $\sigma>0$, $q\geqslant 1$ *and let* $u\in C^2(B^+_\sigma)\cap C^1(\overline{B^+_\sigma})$ *be a positive solution of*

$$\begin{cases}\Delta u=0, & x\in B^+_\sigma,\\ -\dfrac{\partial u}{\partial x_n}=c(n)hu^q, & x\in\Gamma_1.\end{cases} \tag{A.1}$$

We have

$$\frac{c(n)}{q+1}\sum_{i=1}^{i=n-1}\int_{\Gamma_1}x_i\frac{\partial h}{\partial x_i}u^{q+1}\,dx+\left(\frac{n-1}{q+1}-\frac{n-2}{2}\right)c(n)\int_{\Gamma_1}hu^{q+1}\,dx$$

$$-\frac{\sigma c(n)}{q+1}\int_{\partial\Gamma_1}hu^{q+1}d\sigma=\int_{\Gamma_2}F(\sigma,x,u,\nabla u)\,d\sigma,$$

where

$$F(\sigma,x,u,\nabla u)=\frac{n-2}{2}u\frac{\partial u}{\partial \nu}-\frac{\sigma}{2}|\nabla u|^2+\sigma\left(\frac{\partial u}{\partial \nu}\right)^2.$$

We have also the following Proposition, which proof is elementary.

Proposition 6.2. *Suppose that the function* $h : \partial B_\sigma^+ \to \mathbb{R}$ *is of the form*

$$h(y) = a_1 |y|^{2-n} + b + O(|y|) \quad \textit{for } y \textit{ close to } 0,$$

with $a_1 > 0$ *and* $b \geqslant 0$. *Then, if* $b = 0$ *it is* $F \equiv 0$, *otherwise, for* $b > 0$ *there holds*

$$\lim_{\sigma \to 0} \int_{\partial B_\sigma \cap \mathbb{R}^n_+} F(\sigma, x, h, \nabla h) < 0.$$

Lemma 6.3 (Han and Li [17]). *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with piecewise smooth boundary* $\partial\Omega = \Gamma \cup \Sigma$, $V \in L^\infty(\Sigma)$. *Suppose that* $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, $u > 0$ *in* $\bar{\Omega}$, *satisfies*

$$\begin{cases} \Delta u + Vu \leqslant 0 & \textit{in } \Omega, \\ \dfrac{\partial u}{\partial v} \geqslant hu & \textit{on } \Sigma \end{cases}$$

and $v \in C^2(\Omega) \cup C^1(\bar{\Omega})$, *satisfies*

$$\begin{cases} \Delta v + Vv \leqslant 0 & \textit{in } \Omega, \\ \dfrac{\partial v}{\partial v} \geqslant hv & \textit{on } \Sigma, \\ v \geqslant 0 & \textit{on } \Gamma, \end{cases}$$

where v *denotes the unit outer normal of* Σ. *Then* $v \geqslant 0$ *in* $\bar{\Omega}$.

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