

Prescribing the Scalar Curvature under Minimal Boundary Conditions on the Half Sphere*

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Abstract

This paper is devoted to the problem of prescribing the scalar curvature under zero boundary conditions. Using dynamical and topological methods involving the study of critical points at infinity of the associated variational problem, we prove some existence results on the standard half sphere

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1 Introduction and the Main Results

In this paper we study some nonlinear problem arising from conformal geometry. Precisely, consider a Riemannian manifold with boundary (M^n, g) of dimension $n \geq 3$ and take $\tilde{g} = u^{4/(n-2)}g$, be a conformal metric to g , where u is a smooth positive function, then the following equations relate the scalar curvatures $R_g, R_{\tilde{g}}$ and the mean curvatures of the boundary $h_g, h_{\tilde{g}}$, with respect to g and \tilde{g} respectively.

$$(P_1) \quad \begin{cases} -c_n \Delta_g u + R_g u &= R_{\tilde{g}} u^{\frac{n+2}{n-2}} & \text{in } M \\ \frac{2}{n-2} \frac{\partial u}{\partial \nu} + h_g u &= h_{\tilde{g}} u^{\frac{n}{n-2}} & \text{on } \partial M \end{cases}$$

where $c_n = 4(n-1)/(n-2)$ and ν denotes the outward normal vector with respect to the metric g .

In view of the above equations, a natural question is whether it is possible to prescribe both the scalar curvature and the boundary mean curvature, that is : given two functions $K : M \rightarrow \mathbb{R}$ and $H : \partial M \rightarrow \mathbb{R}$, does exists a metric \tilde{g} conformally equivalent to g such that $R_{\tilde{g}} = K$ and $h_{\tilde{g}} = H$?

According to equations (P_1) , the problem is equivalent to finding a smooth positive solution u of the following equation

$$(P_2) \quad \begin{cases} -c_n \Delta_g u + R_g u &= K u^{\frac{n+2}{n-2}} & \text{in } M \\ \frac{2}{n-2} \frac{\partial u}{\partial \nu} + h_g u &= H u^{\frac{n}{n-2}} & \text{on } \partial M. \end{cases}$$

Such a problem was studied in [1] [14],[15] [16], [17], [18], [19] [21] . Yanyan Li [21], and Djadli-Malchiodi-Ould Ahmedou [15] studied this problem when the manifold is the three dimensional standard half sphere. Their approach involves a fine blow up analysis of some subcritical approximations and the use of the topological degree tools.

Regarding the above problem it is well known that the most interesting case is the so called positive one, that is when the quadratic part of the associated Euler functional is positive definite. Another interesting case is when a noncompact group of conformal transformations acts on the equation leading to topological obstructions. The half sphere represents the simplest case where such a noncompactness occurs, and in this paper we consider the case of the standard half sphere under minimal boundary conditions:

More precisely, let

$$S_+^n = \{x \in \mathbb{R}^{n+1} / |x| = 1, x_{n+1} > 0\}$$

$n \geq 3$. Given a C^2 function K on $\overline{S_+^n}$, we look for conditions on K to ensure the

existence of a positive solution of the problem

$$(1) \quad \begin{cases} -\Delta_g u + \frac{n(n-2)}{4}u = Ku^{\frac{n+2}{n-2}} & \text{in } S_+^n \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial S_+^n \end{cases}$$

where g is the standard metric of S_+^n .

Problem (1) is in some sense related to the well known scalar curvature problem on S^n

$$(2) \quad -\Delta_g u + \frac{n(n-2)}{4}u = Ku^{\frac{n+2}{n-2}} \quad \text{in } S^n$$

to which much work has been devoted. For details please see [3],[5] , [8],[9], [13], [12], [20], [22], [23],[27] and the references therein.

As for (2), there are topological obstructions of Kazdan-Warner type to solve (1) (see [10]) and so a naturel question arises: under which conditions on K , (1) has a positive solution. We propose to handle such a question, using some topological and dynamical tools of the theory of critical points at infinity, see Bahri [3] , [4].

Our approach follows closely the ideas developped in Aubin-Bahri [2], Bahri [3] and Ben Ayed-Chtioui-Hammami [9] where the problem of prescribing the scalar curvature on closed manifolds was studied using some algebraic topological tools. The main idea is to use the difference of topology between the level sets of the function K to produce a critical point of the Euler functional J associated to (1) and the main issue is under which conditions on K , a topological accident between the level sets of K induces a topological accident between the level sets of J . Such an accident is sufficient to prove the existence of a critical point when some compactness conditions are satisfied. However our problem presents a lack of compactness due to the presence of critical points at infinity, that is noncompact orbits for the gradient of J along which J is bounded and its gradient goes to zero. Therefore a careful study of such noncompact orbits is necessary, in order to take into account their contribution to the difference of topology between the level sets of J .

In order to state our main results, we need to introduce the assumptions that we are using in our results.

(A₁) We assume that $K_1 = K|_{\partial S_+^n}$ has only nondegenerate critical points y_0, \dots, y_s , where y_0 is the absolute maximum, such that

$$K(y_0) \geq K(y_1) \geq \dots \geq K(y_l) > K(y_{l+1}) \geq \dots \geq K(y_s)$$

with

$$\frac{\partial K}{\partial \nu}(y_i) > 0, \text{ for } 0 \leq i \leq l, \quad \frac{\partial K}{\partial \nu}(y_i) \leq 0, \text{ for } l+1 \leq i \leq s.$$

(A₂) Assume that there exists c a positive constant such that $c < K(y_l)$, and every y critical point of K , $K(y) < c$.

(A₃) Let Z be a pseudogradient of K_1 , of Morse-Smale type (that is the intersections of the stable and the unstable manifolds of the critical points of K_1 are transverse.)

Set

$$X = \overline{\cup_{0 \leq i \leq l} W_s(y_i)}$$

where $W_s(y_i)$ is the stable manifold of y_i for Z .

We assume that X is not contractible and denote by m the dimension of the first nontrivial reduced homological group.

(A₄) Assume that X is contractible in $K^c = \{x \in S_+^n / K(x) \geq c\}$, where c is defined in the assumption (A₂).

Now, we are able to state our first main results.

Theorem 1.1 *Assume that $n \geq 4$. Then, under the assumptions (A₁), (A₂), (A₃), (A₄), there exists a constant c_0 independent of K such that if $K(y_0)/c \leq 1 + c_0$, then (1) has a solution.*

Corollary 1.1 *The solution obtained in Theorem 1.1 has an augmented Morse index $\geq m$.*

Next, we state another kind of existence results for problem (1) based on a “topological invariant” for some Yamabe type problems introduced by Bahri see [3]. To state these results, we need to introduce the assumptions that we will be using and some notations.

(H₁) We assume that K_1 has only nondegenerate critical points and we assume that there exists $y_0 \in \partial S_+^n$ such that y_0 is the absolute maximum of K_1 and $(\partial K/\partial \nu)(y_0) > 0$.

(H₂) $W_s(y_i) \cap W_u(y_j) = \emptyset$ for any i such that $(\partial K/\partial \nu)(y_i) > 0$ and for any j such that $(\partial K/\partial \nu)(y_j) < 0$.

For $k \in \{1, 2, \dots, n-1\}$, we define X as

$$X = \overline{W_s(y_{i_0})}$$

where y_{i_0} satisfies

$$K_1(y_{i_0}) = \max\{K_1(y_i), / \text{ind}(y_i) = n-1-k, (\partial K/\partial \nu)(y_i) > 0\}$$

(Here $\text{ind}(y_i)$ denotes the Morse index of y_i for the function K_1).

(H₃) We assume that X is without boundary.

We denote by $C_{y_0}(X)$ the following set

$$C_{y_0}(X) = \{\alpha \delta_{y_0} + (1-\alpha) \delta_x / \alpha \in [0, 1], x \in X\}.$$

For λ large enough, we introduce a map $f_\lambda : C_{y_0}(X) \rightarrow \Sigma^+$, defined by

$$C_{y_0}(X) \ni (\alpha \delta_{y_0} + (1-\alpha) \delta_x) \longrightarrow \frac{\alpha \delta_{(y_0, \lambda)} + (1-\alpha) \delta_{(x, \lambda)}}{|\alpha \delta_{(y_0, \lambda)} + (1-\alpha) \delta_{(x, \lambda)}|} \in \Sigma^+.$$

Then $C_{y_0}(X)$ and $f_\lambda(C_{y_0}(X))$ are manifolds in dimension $k+1$, that is, their singularities arise in dimension $k-1$ and lower, see [3]. Observe that $C_{y_0}(X)$ and $f_\lambda(C_{y_0}(X))$ are contractible while X is not contractible.

For λ large enough, we also define the intersection number(modulo 2) of $f_\lambda(C_{y_0}(X))$ with $W_s(y_0, y_{i_0})_\infty$

$$\mu(y_0) = f_\lambda(C_{y_0}(X)) \cdot W_s(y_0, y_{i_0})_\infty$$

where $W_s(y_0, y_{i_0})_\infty$ is the stable manifold of $(y_0, y_{i_0})_\infty$ for a decreasing pseudogradient V for J which is transverse to $f_\lambda(C_{y_0}(X))$. Thus this number is well defined [25].

We then have the following result:

Theorem 1.2 *Assume that $n \geq 4$. Under assumptions (H_1) , (H_2) and (H_3) , if $\mu(y_0) = 0$ then (1) has a solution of index k or $k + 1$.*

Now, we state a statement more general than Theorem 1.2. For this we define X to be

$$X = \overline{\cup_{y_i \in B_k} W_s(y_i)}, \quad \text{with } B_k = \{y_i / \text{ind}(y_i) = n - 1 - k \text{ and } (\partial K / \partial \nu)(y_i) > 0\}.$$

For $y_i \in B_k$, we define, for λ large enough, the intersection number(modulo 2)

$$\mu_i(y_0) = f_\lambda(C_{y_0}(X)) \cdot W_s(y_0, y_i)_\infty.$$

By the above arguments, this number is well defined [25].

Then we have the following theorem

Theorem 1.3 *Assume that $n \geq 4$. Under assumptions (H_1) , (H_2) and (H_3) , if $\mu_i = 0$ for each $y_i \in B_k$, then (1) has a solution of index k or $k + 1$.*

The remainder of the present paper is organized as follows. In section 2, we set up the variational structure and recall some preliminaries. In section 3, we perform an expansion of the Euler functional associated to (1) and its gradient near the potential critical points at infinity, then we prove a Morse Lemma at infinity in section 4. In section 5, we provide the proof of Theorem 1.1 and Corollary 1.1, while section 6 is devoted to the proof of Theorems 1.2 and 1.3.

2 Variational Structure and Preliminaries

In this section we recall the functional setting and the variational problem and its main features. Problem (1) has a variational structure. The functional is

$$J(u) = \frac{\int_{S_+^n} |\nabla u|^2 + \frac{n(n-2)}{4} \int_{S_+^n} u^2}{\left(\int_{S_+^n} K u^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}}$$

defined on $H^1(\overline{S_+^n}, \mathbb{R})$ equipped with the norm

$$\|u\|^2 = \int_{S_+^n} |\nabla u|^2 + \frac{n(n-2)}{4} \int_{S_+^n} u^2.$$

We denote by Σ the unit sphere of $H^1(\overline{S_+^n}, \mathbb{R})$ and we set $\Sigma^+ = \{u \in \Sigma / u \geq 0\}$. The Palais-Smale condition fails to be satisfied for J on Σ^+ . Its failure has been studied by various authors (see Brezis-Coron [11], Lions [24], Struwe [28]).

In order to characterize the sequences failing the Palais-Smale condition, we need to introduce some notations.

For $a \in \overline{S_+^n}$ and $\lambda > 0$, let

$$\tilde{\delta}_{a,\lambda}(x) = c_0 \frac{\lambda^{\frac{n-2}{2}}}{(\lambda^2 + 1 + (\lambda^2 - 1) \cos d(a, x))^{\frac{n-2}{2}}}$$

where d is the geodesic distance on (S_+^n, g) and c_0 is chosen so that

$$-\Delta \tilde{\delta}_{a,\lambda} + \frac{n(n-2)}{4} \tilde{\delta}_{a,\lambda} = \tilde{\delta}_{a,\lambda}^{\frac{n+2}{n-2}}, \quad \text{in } S_+^n.$$

For $\varepsilon > 0$ and $p \in \mathbb{N}^*$, let us define

$$\begin{aligned} V(p, \varepsilon) = & \{u \in \Sigma^+ / \exists a_1, \dots, a_p \in \overline{S_+^n}, \exists \lambda_1, \dots, \lambda_p > 0, \exists \alpha_1, \dots, \alpha_p > 0 \\ & \text{s.t. } \|u - \sum_{i=1}^p \alpha_i \tilde{\delta}_i\| < \varepsilon \text{ and } \left| \frac{\alpha_i^{\frac{4}{n-2}} K(a_i)}{\alpha_j^{\frac{4}{n-2}} K(a_j)} - 1 \right| < \varepsilon, \\ & \lambda_i > \varepsilon^{-1}, \varepsilon_{ij} < \varepsilon \text{ and } \lambda_i d_i < \varepsilon \text{ or } \lambda_i d_i > \varepsilon^{-1}\} \end{aligned}$$

where $\tilde{\delta}_i = \tilde{\delta}_{a_i, \lambda_i}$, $d_i = d(a_i, \partial S_+^n)$ and $\varepsilon_{ij} = (\lambda_i/\lambda_j + \lambda_j/\lambda_i + \lambda_i \lambda_j d^2(a_i, a_j))^{\frac{2-n}{2}}$. The failure of Palais-Smale condition can be described as follows:

Proposition 2.1 (see [6], [24] and [28]) *Assume that J has no critical point in Σ^+ and let $(u_k) \in \Sigma^+$ be a sequence such that $J(u_k)$ is bounded and $\nabla J(u_k) \rightarrow 0$. Then, there exist an integer $p \in \mathbb{N}^*$, a sequence $\varepsilon_k > 0$ ($\varepsilon_k \rightarrow 0$) and an extracted subsequence of u_k , again denoted (u_k) , such that $u_k \in V(p, \varepsilon_k)$.*

Now, we consider the following subset of $V(p, \varepsilon)$

$$V_b(p, \varepsilon) = \{u \in V(p, \varepsilon) / \lambda_i d_i < \varepsilon\}.$$

The following lemma defines a parametrization of the set $V_b(p, \varepsilon)$.

Lemma 2.1 (see [4], [6], [26]) *There is $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ and $u \in V_b(p, \varepsilon)$, then the problem*

$$\min \left\{ \|u - \sum_{i=1}^p \alpha_i \tilde{\delta}_i\|, \alpha_i > 0, \lambda_i > 0, a_i \in \partial S_+^n \right\}$$

has a unique solution (up to permutation). In particular, we can write $u \in V_b(p, \varepsilon)$ as follows

$$u = \sum_{i=1}^p \bar{\alpha}_i \tilde{\delta}_{\bar{a}_i, \bar{\lambda}_i} + v,$$

where $(\bar{\alpha}_1, \dots, \bar{\alpha}_p, \bar{a}_1, \dots, \bar{a}_p, \bar{\lambda}_1, \dots, \bar{\lambda}_p)$ is the solution of the minimization problem and where $v \in H^1(S_+^n)$ such that for each $i = 1, \dots, p$

$$(v, \tilde{\delta}_i) = (v, \partial \tilde{\delta}_i / \partial \lambda_i) = (v, \partial \tilde{\delta}_i / \partial a_i) = 0.$$

Here (\cdot, \cdot) denoted the scalar inner defined on $H^1(\overline{S_+^n})$ by

$$(u, v) = \int_{S_+^n} \nabla u \nabla v + \frac{n(n-2)}{4} \int_{S_+^n} uv.$$

Before ending this section, we mention that it will be convenient to perform some stereographic projection in order to reduce our problem to \mathbb{R}_+^n . Let $D^{1,2}(\mathbb{R}_+^n)$ denote the completion of $C_c^\infty(\mathbb{R}_+^n)$ with respect to Dirichlet norm. The stereographic projection π_a through a point $a \in \partial S_+^n$ induces an isometry $i : H^1(S_+^n) \rightarrow D^{1,2}(\mathbb{R}_+^n)$ according to the following formula

$$(iv)(x) = \left(\frac{2}{1+|x|^2} \right)^{(n-2)/2} v(\pi_a^{-1}(x)), \quad v \in H^1(S_+^n), x \in \mathbb{R}^n.$$

In particular, one can check that the following holds true, for every $v \in H^1(S_+^n)$

$$\int_{S_+^n} (|\nabla v|^2 + \frac{n(n-2)}{4} v^2) = \int_{\mathbb{R}_+^n} |\nabla(iv)|^2 \quad \text{and} \quad \int_{S_+^n} |v|^{\frac{2n}{n-2}} = \int_{\mathbb{R}_+^n} |iv|^{\frac{2n}{n-2}}.$$

In the sequel, we will identify the function K and its composition with the stereographic projection π_a . We will also identify a point b of S_+^n and its image by π_a . These facts will be assumed as understood in the sequel.

3 Expansion of J and its gradient at infinity

This section is devoted to an useful expansion of J and its gradient near a potential boundary critical point at infinity consisting of two masses.

Proposition 3.1 *For $\varepsilon > 0$ small enough and $u = \sum_{i=1}^2 \alpha_i \tilde{\delta}_{a_i, \lambda_i} + v \in V_b(2, \varepsilon)$, we have the following expansion*

$$\begin{aligned} J(u) = & \frac{(\alpha_1^2 + \alpha_2^2)(S_n/2)^{(2/n)}}{(\alpha_1^{\frac{2n}{n-2}} K(a_1) + \alpha_2^{\frac{2n}{n-2}} K(a_2))^{\frac{n-2}{n}}} \left[1 + \frac{4(n-2)}{n\beta} c_1 \sum_{i=1}^2 \frac{\alpha_i^{\frac{2n}{n-2}}}{\lambda_i} \frac{\partial K}{\partial \nu}(a_i) \right. \\ & - 2c_2 \alpha_1 \alpha_2 \varepsilon_{12} \left(-\frac{1}{\gamma} + \frac{1}{\beta} \left(\sum_{i=1}^2 \alpha_i^{\frac{4}{n-2}} K(a_i) \right) \right) + f(v) + Q(v, v) \\ & \left. + O \left(\varepsilon_{12}^{\frac{n}{n-2}} \log(\varepsilon_{12}^{-1}) + \sum \left(\frac{1}{\lambda_i^2} + \frac{\varepsilon_{12}}{\lambda_i} (\log(\varepsilon_{12}^{-1}))^{\frac{n-2}{n}} \right) + \|v\|^{\inf(3, \frac{2n}{n-2})} \right) \right], \end{aligned}$$

where

$$\begin{aligned} Q(v, v) &= \frac{1}{\gamma} \|v\|^2 - \frac{n+2}{n-2} \frac{1}{\beta} \int_{S_+^n} K(\alpha_1 \tilde{\delta}_1 + \alpha_2 \tilde{\delta}_2)^{\frac{4}{n-2}} v^2, \\ f(v) &= -\frac{2}{\beta} \int_{S_+^n} K(\alpha_1 \tilde{\delta}_1 + \alpha_2 \tilde{\delta}_2)^{\frac{n+2}{n-2}} v, \quad S_n = c_0^{\frac{2n}{n-2}} \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^n}, \\ \beta &= \frac{S_n}{2} (\alpha_1^{\frac{2n}{n-2}} K(a_1) + \alpha_2^{\frac{2n}{n-2}} K(a_2)), \quad \gamma = \frac{S_n}{2} (\alpha_1^2 + \alpha_2^2), \\ c_1 &= c_0^{\frac{2n}{n-2}} \int_{\mathbb{R}_+^n} \frac{x_n dx}{(1+|x|^2)^n}, \quad c_2 = c_0^{\frac{2n}{n-2}} \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{\frac{n+2}{2}}}. \end{aligned}$$

Proof. We need to estimate

$$N(u) = \|u\|^2 \text{ and } D^{\frac{n}{n-2}} = \int_{S_+^n} K(x) u^{\frac{2n}{n-2}}.$$

In order to simplify the notations, in the remainder, we write $\tilde{\delta}_i$ instead of $\tilde{\delta}_{a_i, \lambda_i}$. We now have

$$N(u) = \alpha_1^2 \|\tilde{\delta}_1\|^2 + \alpha_2^2 \|\tilde{\delta}_2\|^2 + \|v\|^2 + 2\alpha_1 \alpha_2 \left(\int_{S_+^n} \nabla \tilde{\delta}_1 \nabla \tilde{\delta}_2 + \frac{n(n-2)}{4} \int_{S_+^n} \tilde{\delta}_1 \tilde{\delta}_2 \right)$$

Observe that

$$\|\tilde{\delta}\|^2 = \int_{\mathbb{R}_+^n} |\nabla \delta|^2 = \frac{S_n}{2} \quad (3.1)$$

and

$$\int_{S_+^n} \nabla \tilde{\delta}_1 \nabla \tilde{\delta}_2 + \frac{n(n-2)}{4} \int_{S_+^n} \tilde{\delta}_1 \tilde{\delta}_2 = \int_{\mathbb{R}_+^n} \nabla \delta_1 \nabla \delta_2 = \int_{\mathbb{R}_+^n} \delta_1^{\frac{n+2}{n-2}} \delta_2$$

where δ_i denotes δ_{a_i, λ_i} and, for $a \in \mathbb{R}^n$ and $\lambda > 0$, $\delta_{a, \lambda}$ denotes the family of solutions of Yamabe problem on \mathbb{R}^n defined by

$$\delta_{a, \lambda}(x) = c_0 \frac{\lambda^{\frac{n-2}{2}}}{(1 + \lambda^2 |x - a|^2)^{\frac{n-2}{2}}}.$$

A computation similar to the one performed in [4] shows that

$$\int_{\mathbb{R}_+^n} \delta_1^{\frac{n+2}{n-2}} \delta_2 = \frac{1}{2} c_2 \varepsilon_{12} + O(\varepsilon_{12}^{\frac{n}{n-2}} \log(\varepsilon_{12}^{-1})). \quad (3.2)$$

Thus

$$N = \gamma + \alpha_1 \alpha_2 c_2 \varepsilon_{12} + \|v\|^2 + O(\varepsilon_{12}^{\frac{n}{n-2}} \log(\varepsilon_{12}^{-1}))$$

For the denominator, we write

$$\begin{aligned} D^{\frac{n}{n-2}} &= \int_{S_+^n} K(\alpha_1 \tilde{\delta}_1 + \alpha_2 \tilde{\delta}_2)^{\frac{2n}{n-2}} + \frac{2n}{n-2} \int_{S_+^n} K(\alpha_1 \tilde{\delta}_1 + \alpha_2 \tilde{\delta}_2)^{\frac{n+2}{n-2}} v \\ &+ \frac{n(n+2)}{(n-2)^2} \int_{S_+^n} K(\alpha_1 \tilde{\delta}_1 + \alpha_2 \tilde{\delta}_2)^{\frac{4}{n-2}} v^2 \\ &+ O\left(\int_{S_+^n} (\alpha_1 \tilde{\delta}_1 + \alpha_2 \tilde{\delta}_2)^{\frac{4}{n-2}-1} \inf((\alpha_1 \tilde{\delta}_1 + \alpha_2 \tilde{\delta}_2), |v|)^3 + \int |v|^{\frac{2n}{n-2}} \right). \end{aligned}$$

We also write

$$\begin{aligned} \int_{S_+^n} K(\alpha_1 \tilde{\delta}_1 + \alpha_2 \tilde{\delta}_2)^{\frac{2n}{n-2}} &= \int_{S_+^n} K(\alpha_1 \tilde{\delta}_1)^{\frac{2n}{n-2}} + \int_{S_+^n} K(\alpha_2 \tilde{\delta}_2)^{\frac{2n}{n-2}} + \frac{2n}{n-2} \int_{S_+^n} K(\alpha_1 \tilde{\delta}_1)^{\frac{n+2}{n-2}} \alpha_2 \tilde{\delta}_2 \\ &+ \frac{2n}{n-2} \int_{S_+^n} K(\alpha_2 \tilde{\delta}_2)^{\frac{n+2}{n-2}} \alpha_1 \tilde{\delta}_1 + O\left(\int_{S_+^n} \sup(\tilde{\delta}_1, \tilde{\delta}_2)^{\frac{4}{n-2}} \inf(\tilde{\delta}_1, \tilde{\delta}_2)^2 \right). \end{aligned}$$

Expansions of K around a_1 and a_2 give

$$\int_{S_+^n} K(\tilde{\delta}_i)^{\frac{2n}{n-2}} = K(a_i) \frac{S_n}{2} - \frac{2c_1}{\lambda_i} \frac{\partial K}{\partial \nu}(a_i) + O\left(\frac{1}{\lambda_i^2}\right) \quad (3.3)$$

$$\int_{S_+^n} K(\tilde{\delta}_i)^{\frac{n+2}{n-2}} \tilde{\delta}_j = K(a_i) \frac{c_2}{2} \varepsilon_{12} + O\left(\varepsilon_{12}^{\frac{n}{n-2}} \log(\varepsilon_{12}^{-1}) + \frac{\varepsilon_{12}}{\lambda_i} (\log(\varepsilon_{12}^{-1}))^{\frac{n-2}{n}} \right) \quad (3.4)$$

It easy to check

$$\int_{S_+^n} \frac{4}{\sup(\tilde{\delta}_1, \tilde{\delta}_2)} \inf(\tilde{\delta}_1, \tilde{\delta}_2)^2 = O(\varepsilon_{12}^{\frac{n}{n-2}} \log(\varepsilon_{12}^{-1})) \quad \text{if } n \geq 4 \quad (3.5)$$

and

$$\int_{S_+^n} (\alpha_1 \tilde{\delta}_1 + \alpha_2 \tilde{\delta}_2)^{\frac{4}{n-2}-1} \inf((\alpha_1 \tilde{\delta}_1 + \alpha_2 \tilde{\delta}_2), |v|)^3 + \int |v|^{\frac{2n}{n-2}} = O\left(\|v\|^{\inf(3, \frac{2n}{n-2})} \right) \quad (3.6)$$

Combining (3.1), (3.2), (3.3), (3.4), (3.5) and (3.6), we easily derive our proposition.

□

A natural improvement of Proposition 3.1 is obtained by taking care of the v -part, in order to show that it can be neglected with respect to the concentration phenomenon.

Set

$$E_\varepsilon = \{v \in H^1(S_+^n) / \|v\| \leq \varepsilon \text{ and } v \text{ satisfies } (V_0)\}$$

where (V_0) is the following condition

$$(V_0) \quad (v, \tilde{\delta}_i) = (v, \partial \tilde{\delta}_i / \partial \lambda_i) = (v, \partial \tilde{\delta}_i / \partial a_i) = 0, \quad \text{for } i = 1, 2.$$

Notice that, one can prove arguing as in [4] (see also [26]), that for ε small enough, there exists $\rho > 0$ such that for all $v \in E_\varepsilon$

$$Q(v, v) \geq \rho \|v\|^2.$$

It follows the following lemma whose proof is similar, up to minor modifications to corresponding statements in [4] (see also [26]).

Lemma 3.1 *There exists a C^1 -map which, to each (α, a, λ) such that $\alpha_1 \tilde{\delta}_1 + \alpha_2 \tilde{\delta}_2 \in V_b(2, \varepsilon)$ with small ε , associates $\bar{v} = \bar{v}_{(\alpha, a, \lambda)}$ satisfying*

$$J(\alpha_1 \tilde{\delta}_1 + \alpha_2 \tilde{\delta}_2 + \bar{v}) = \min\{J(\alpha_1 \tilde{\delta}_1 + \alpha_2 \tilde{\delta}_2 + v), v \text{ satisfies } (V_0)\}.$$

Moreover, there exists $c > 0$ such that the following holds

$$\|\bar{v}\| \leq c \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \varepsilon_{12}^{\frac{n+2}{2(n-2)}} \log(\varepsilon_{12}^{-1}) + (\text{if } n \leq 5) \varepsilon_{12} (\log(\varepsilon_{12}^{-1}))^{\frac{n-2}{n}} \right).$$

Proposition 3.2 *Let $n \geq 4$, for $u = \alpha_1 \tilde{\delta}_1 + \alpha_2 \tilde{\delta}_2 \in V_b(2, \varepsilon)$, we have the following expansion*

$$\begin{aligned} (\nabla J(u), \lambda_1 \partial \tilde{\delta}_1 / \partial \lambda_1) &= 2J(u) \left[\frac{c_2}{2} \alpha_2 \lambda_1 \frac{\partial \varepsilon_{12}}{\partial \lambda_1} (1 - J(u)^{\frac{n}{n-2}} (\alpha_1^{\frac{4}{n-2}} K(a_1) + \alpha_2^{\frac{4}{n-2}} K(a_2))) \right. \\ &\quad \left. - 2J(u)^{\frac{n}{n-2}} \alpha_1^{\frac{n+2}{n-2}} \frac{c_3}{\lambda_1} \frac{\partial K}{\partial \nu}(a_1) \right] + O\left(\frac{1}{\lambda_1^2}\right) \\ &\quad + O\left(\varepsilon_{12}^{\frac{n}{n-2}} \log(\varepsilon_{12}^{-1}) + \varepsilon_{12} (\log(\varepsilon_{12}^{-1}))^{\frac{n-2}{n}} \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)\right). \end{aligned}$$

where $c_3 = \frac{n-2}{2} c_0^{2n/(n-2)} \int_{\mathbb{R}_+^n} \frac{x_n(|x|^2-1)}{(1+|x|^2)^{n+1}} dx$

Proof. We have

$$(\nabla J(u), h) = 2J(u) \left[\int_{S_+^n} \nabla u \nabla h + \frac{n(n-2)}{4} \int_{S_+^n} u h - J(u)^{\frac{n}{n-2}} \int_{S_+^n} K u^{\frac{n+2}{n-2}} h \right] \quad (3.7)$$

Observe that(see [4])

$$\int_{\mathbb{R}_+^n} \nabla \delta_1 \nabla (\lambda_1 \frac{\partial \delta_1}{\partial \lambda_1}) = \int_{\mathbb{R}_+^n} \delta_1^{\frac{n+2}{n-2}} \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1} = 0 \quad (3.8)$$

$$\begin{aligned} \int_{\mathbb{R}_+^n} \nabla \delta_2 \nabla (\lambda_1 \frac{\partial \delta_1}{\partial \lambda_1}) &= \int_{\mathbb{R}_+^n} \delta_2^{\frac{n+2}{n-2}} \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1} \\ &= \frac{1}{2} c_2 \lambda_1 \frac{\partial \varepsilon_{12}}{\partial \lambda_1} + O\left(\varepsilon_{12}^{\frac{n}{n-2}} \log(\varepsilon_{12}^{-1})\right) \end{aligned} \quad (3.9)$$

$$\begin{aligned} \int_{\mathbb{R}_+^n} K \delta_1^{\frac{n+2}{n-2}} \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1} &= 2\nabla K(a_1) \int_{\mathbb{R}_+^n} \delta_1^{\frac{n+2}{n-2}} \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1} (x - a_1) + O\left(\frac{1}{\lambda_1^2}\right) \\ &= -\frac{2c_3}{\lambda_1} \nabla K(a) e_n + O\left(\frac{1}{\lambda_1^2}\right), \end{aligned} \quad (3.10)$$

$$\begin{aligned} \int_{\mathbb{R}_+^n} K \delta_2^{\frac{n+2}{n-2}} \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1} &= K(a_2) \frac{1}{2} c_2 \lambda_1 \frac{\partial \varepsilon_{12}}{\partial \lambda_1} + O\left(\frac{1}{\lambda_2} \varepsilon_{12} (\text{Log}(\varepsilon_{12}^{-1}))^{\frac{n-2}{n}}\right) \\ &\quad + O\left(\varepsilon_{12}^{\frac{n}{n-2}} \log(\varepsilon_{12}^{-1})\right), \end{aligned} \quad (3.11)$$

$$\begin{aligned} \frac{n+2}{n-2} \int_{\mathbb{R}_+^n} K \delta_1^{\frac{4}{n-2}} \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1} &= K(a_1) \frac{1}{2} c_2 \lambda_1 \frac{\partial \varepsilon_{12}}{\partial \lambda_1} + O\left(\frac{1}{\lambda_1} \varepsilon_{12} (\text{Log}(\varepsilon_{12}^{-1}))^{\frac{n-2}{n}}\right) \\ &\quad + O\left(\varepsilon_{12}^{\frac{n}{n-2}} \log(\varepsilon_{12}^{-1})\right). \end{aligned} \quad (3.12)$$

Combining (3.7), (3.8), (3.9), (3.10), (3.11) and (3.12), we easily derive our proposition. \square

Proposition 3.3 *Let $n \geq 4$. For $u = \sum \alpha_i \tilde{\delta}_i \in V_b(2, \varepsilon)$, we have the following expansion:*

$$\begin{aligned} \left(\nabla J(u), \frac{1}{\lambda_1} \frac{\partial \tilde{\delta}_1}{\partial a_1} \right) &= 2J(u) \alpha_1 e_n \left[c_4 \left(1 - J(u)^{\frac{n-2}{n-2}} \alpha_1^{\frac{4}{n-2}} K(a_1) \right) + J(u)^{\frac{n-2}{n-2}} \alpha_1^{\frac{4}{n-2}} \frac{c_5}{\lambda_1} \frac{\partial K}{\partial \nu}(a_1) \right] \\ &\quad - J(u) \alpha_2 c_2 \frac{1}{\lambda_1} \frac{\partial \varepsilon_{12}}{\partial a_1} \left(-1 + J(u)^{\frac{n-2}{n-2}} \sum \alpha_i^{\frac{4}{n-2}} K(a_i) \right) \\ &\quad - 4J(u)^{\frac{2(n-1)}{n-2}} \alpha_1^{\frac{n+2}{n-2}} \frac{2c_5}{\lambda_1} \nabla_T K(a_1) + O\left(\varepsilon_{12}^{\frac{n-2}{n-2}} \text{Log}(\varepsilon_{12}^{-1}) + \varepsilon_{12}^{\frac{n+1}{n-2}} \lambda_2 |a_1 - a_2|\right) \\ &\quad + O\left(\varepsilon_{12} (\text{Log}(\varepsilon_{12}^{-1}))^{\frac{n-2}{n}} \sum \frac{1}{\lambda_k}\right) + O\left(\frac{1}{\lambda_1^2}\right) \end{aligned}$$

where

$$c_4 = (n-2) c_0^{\frac{2n}{n-2}} \int_{\mathbb{R}_+^n} \frac{x_n}{(1+|x|^2)^{n+1}} dx \quad \text{and} \quad c_5 = \frac{n-2}{2n} c_0^{\frac{2n}{n-2}} \int_{\mathbb{R}^n} \frac{x_n^2}{(1+|x|^2)^{n+1}} dx$$

Proof. An easy computation shows

$$\int_{\mathbb{R}_+^n} \nabla \delta_1 \nabla \left(\frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial a_1} \right) = \int_{\mathbb{R}_+^n} \delta_1^{\frac{n+2}{n-2}} \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial a_1} = c_4 e_n, \quad (3.13)$$

$$\begin{aligned} \int_{\mathbb{R}_+^n} \nabla \delta_2 \nabla \left(\frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial a_1} \right) &= \int_{\mathbb{R}_+^n} \delta_2^{\frac{n+2}{n-2}} \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial a_1} \\ &= \frac{1}{2} \frac{c_2}{\lambda_1} \frac{\partial \varepsilon_{12}}{\partial a_1} + O\left(\varepsilon_{12}^{\frac{n}{n-2}} \text{Log}(\varepsilon_{12}^{-1}) + \varepsilon_{12}^{\frac{n+1}{n-2}} \lambda_2 |a_1 - a_2|\right) \end{aligned} \quad (3.14)$$

$$\int_{\mathbb{R}_+^n} K \delta_1^{\frac{n+2}{n-2}} \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial a_1} = K(a_1) c_4 e_n + 2 \frac{c_5}{\lambda_1} \nabla K(a_1) + O\left(\frac{1}{\lambda_1^2}\right), \quad (3.15)$$

$$\begin{aligned} \int_{\mathbb{R}_+^n} K \delta_2^{\frac{n+2}{n-2}} \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial a_1} &= K(a_2) \frac{1}{2} c_2 \frac{1}{\lambda_1} \frac{\partial \varepsilon_{12}}{\partial a_1} + O\left(\varepsilon_{12}^{\frac{n+1}{n-2}} \lambda_2 |a_1 - a_2|\right) \\ &+ O\left(\varepsilon_{12}^{\frac{n}{n-2}} \text{Log}(\varepsilon_{12}^{-1})\right) + O\left(\frac{1}{\lambda_2} \varepsilon_{12} (\text{Log}(\varepsilon_{12}^{-1}))^{\frac{n-2}{n}}\right), \end{aligned} \quad (3.16)$$

$$\begin{aligned} \frac{n+2}{n-2} \int_{\mathbb{R}_+^n} K \delta_1^{\frac{4}{n-2}} \delta_2 \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial a_1} &= K(a_1) \frac{1}{2} c_2 \frac{1}{\lambda_1} \frac{\partial \varepsilon_{12}}{\partial a_1} + O\left(\varepsilon_{12}^{\frac{n+1}{n-2}} \lambda_2 |a_1 - a_2|\right), \\ &+ O\left(\varepsilon_{12}^{\frac{n}{n-2}} \text{Log}(\varepsilon_{12}^{-1})\right) + O\left(\frac{1}{\lambda_1} \varepsilon_{12} (\text{Log}(\varepsilon_{12}^{-1}))^{\frac{n-2}{n}}\right). \end{aligned} \quad (3.17)$$

Using (3.7), (3.13), (3.14), (3.15), (3.16) and (3.17), our proposition follows. \square

Proposition 3.4 For $u = \sum \alpha_i \tilde{\delta}_i \in V_b(2, \varepsilon)$, we have the following expansion:

$$\begin{aligned} (\nabla J(u), \tilde{\delta}_1) &= 2J(u) \alpha_1 \frac{S_n}{2} \left(1 - J(u)^{\frac{n}{n-2}} \alpha_1^{\frac{4}{n-2}} K(a_1)\right) + 2J(u)^{\frac{2(n-1)}{n-2}} \alpha_1^{\frac{n+2}{n-2}} \frac{2c_6}{\lambda_1} \frac{\partial K}{\partial \nu}(a_1) \\ &- J(u) c_2 \varepsilon_{12} \alpha_2 \left(-1 + J(u)^{\frac{n}{n-2}} \left(\frac{n+2}{n-2} \alpha_1^{\frac{4}{n-2}} K(a_1) + \alpha_2^{\frac{4}{n-2}} K(a_2)\right)\right) \\ &+ O\left(\frac{1}{\lambda_1^2} + \varepsilon_{12}^{\frac{n}{n-2}} \text{Log}(\varepsilon_{12}^{-1}) + \frac{1}{\lambda_2} \varepsilon_{12} (\text{Log}(\varepsilon_{12}^{-1}))^{\frac{n-2}{n}}\right) \end{aligned}$$

where $c_6 = c_0 \int_{\mathbb{R}_+^n} \frac{x_n}{(1+|x|^2)^n} dx$ and where S_n is defined in Proposition 3.1.

Proof. Using estimates (3.1), (3.2), (3.3) and (3.4), we easily derive our proposition. \square

Before ending this section, we state the above results in the case where we have only one mass instead of two masses.

Proposition 3.5 For $\varepsilon > 0$ small enough and $u = \alpha \tilde{\delta}_{a,\lambda} + v \in V_b(1, \varepsilon)$, we have the following expansion:

$$\begin{aligned} J(u) &= \frac{(S_n/2)^{(2/n)}}{(K(a))^{\frac{n-2}{n}}} \left[1 + \frac{8c_1(n-2)}{nK(a)S_n\lambda} \frac{\partial K}{\partial \nu}(a) + f(v)\right] \\ &+ Q(v, v) + O\left(\frac{1}{\lambda^2}\right) + O\left(\|v\|^{\inf(3, \frac{2n}{n-2})}\right) \end{aligned}$$

where

$$\begin{aligned} Q(v, v) &= \frac{2}{\alpha^2 S_n} \left[\|v\|^2 - \frac{n+2}{(n-2)K(a)} \int_{S_+^n} K(\tilde{\delta}_{a,\lambda})^{\frac{4}{n-2}} v^2 \right] \\ f(v) &= -\frac{4}{\alpha K(a) S_n} \int_{S_+^n} K(\tilde{\delta}_{a,\lambda})^{\frac{n+2}{n-2}} v. \end{aligned}$$

Proposition 3.6 For $u = \alpha\tilde{\delta}_{a,\lambda} \in V_b(1, \varepsilon)$, we have the following expansion:

$$\left(\nabla J(u), \lambda \frac{\partial \tilde{\delta}}{\partial \lambda} \right) = -4J(u) \frac{2(n-1)}{n-2} \alpha^{\frac{n+2}{n-2}} c_3 \frac{\partial K}{\partial \nu}(a) + O\left(\frac{1}{\lambda^2}\right)$$

Proposition 3.7 For $u = u = \alpha\tilde{\delta}_{a,\lambda} \in V_b(1, \varepsilon)$, we have the following expansion:

$$\begin{aligned} \left(\nabla J(u), \frac{1}{\lambda} \frac{\partial \tilde{\delta}}{\partial a} \right) &= 2J(u) \alpha e_n \left[c_4 \left(1 - J(u)^{\frac{n}{n-2}} \alpha^{\frac{4}{n-2}} K(a) \right) + 2c_5 J(u)^{\frac{n}{n-2}} \frac{\alpha^{\frac{4}{n-2}}}{\lambda} \frac{\partial K}{\partial \nu}(a) \right] \\ &\quad - 4J(u) \frac{2(n-1)}{n-2} \alpha^{\frac{n+2}{n-2}} c_5 \frac{\nabla_T K(a)}{\lambda} + O\left(\frac{1}{\lambda^2}\right). \end{aligned}$$

4 Morse Lemma at Infinity

In this section, we consider the case where we only have one mass and we perform a Morse lemma at infinity for J , which completely gets rid of the v -contribution and shows that the functional behaves, at infinity, as $J(\alpha\tilde{\delta}_{\tilde{a},\tilde{\lambda}}) + |V|^2$, where V is a variable completely independent of \tilde{a} , $\tilde{\lambda}$. Namely, we prove the following proposition.

Proposition 4.1 For $\varepsilon > 0$ small enough, there is a diffeomorphism

$$\alpha\tilde{\delta}_{a,\lambda} + v \mapsto \alpha\tilde{\delta}_{\tilde{a},\tilde{\lambda}} \in V_b(1, \varepsilon')$$

for some ε' such that

$$J(\alpha\tilde{\delta}_{a,\lambda} + v) = J(\alpha\tilde{\delta}_{\tilde{a},\tilde{\lambda}}) + |V|^2$$

where V belongs to a neighborhood of zero in a fixed Hilbert space.

The above Morse Lemma can be improved when the concentration point is near a critical point y of $K_1 = K|_{\partial S_+^n}$ with $\frac{\partial K}{\partial \nu}(y) > 0$, leading to the following normal form:

Proposition 4.2 For $u = \alpha\tilde{\delta}_{\tilde{a},\tilde{\lambda}} \in V_b(1, \varepsilon)$ such that $\tilde{a} \in \mathcal{V}(y, \rho)$, $\frac{\partial K}{\partial \nu}(y) > 0$, $\rho > 0$ and y is a critical point of K_1 , there is another change of variable $(\tilde{\tilde{a}}, \tilde{\tilde{\lambda}})$ such that

$$\begin{aligned} J(u) &= \psi(\tilde{\tilde{a}}, \tilde{\tilde{\lambda}}) \\ &:= \frac{(S_n/2)^{\frac{2}{n}}}{(K(\tilde{\tilde{a}}))^{\frac{n-2}{n}}} \left[1 + (c - \eta) \frac{1}{\lambda} \frac{\partial K}{\partial \nu}(\tilde{\tilde{a}}) \right] \end{aligned}$$

where η is a small positive real.

Here and in the sequel, $\mathcal{V}(y, \rho)$ denotes a neighborhood of y .

The proof of Propositions 4.1 and 4.2 can be easily deduced from the following lemma, arguing as in [3] and [8].

Lemma 4.1 *There exists a pseudogradient Z so that the following holds.*

There is a constant $c > 0$ independent of $u = \alpha\tilde{\delta}_{a,\lambda}$ in $V_b(1, \varepsilon)$ such that

i. $-(\nabla J(u), Z) \geq \frac{c}{\lambda}$

ii. $-\left(\nabla J(u + \bar{v}), Z + \frac{\partial \bar{v}}{\partial(\alpha, a, \lambda)}(Z)\right) \geq \frac{c}{\lambda}$

iii. Z is bounded

iv. *the only region where λ increases along Z is the region where $a \in \mathcal{V}(y, \rho)$, where y is a critical point of K_1 such that $\frac{\partial K}{\partial \nu}(y) > 0$.*

Before giving the proof of Lemma 4.1, we notice that combining Proposition 4.2 and Lemma 4.1, one can easily derive the following corollary.

Corollary 4.1 *Assume that J does not have any critical point. Then, the only critical points at infinity of J in $V_b(1, \varepsilon)$, for ε small enough, correspond to $\tilde{\delta}_{y, \infty}$, where y is a critical point of $K_1 = K|_{\partial S^2_+}$ such that $\frac{\partial K}{\partial \nu}(y) > 0$.*

Moreover such a critical point at infinity has a Morse index equal to $(n - 1 - \text{index}(K_1, y))$.

Proof of Lemma 4.1 Let $u = \alpha\tilde{\delta}_{a,\lambda} \in V_b(1, \varepsilon)$. We divide $V_b(1, \varepsilon)$ in three regions.

1st region. $a \notin \cup_{0 \leq i \leq s} \mathcal{V}(y_i, \rho)$, where $\rho < \frac{1}{2} \min_{i \neq j} d(y_i, y_j)$.

Set $Z_1 = \frac{1}{\lambda} \frac{\partial \tilde{\delta}}{\partial a} \nabla_T K(a)$, from Proposition 3.7, we have

$$\begin{aligned} -(\nabla J(u), Z_1) &= c \frac{|\nabla_T K(a)|^2}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \\ &\geq \frac{c}{\lambda} \end{aligned}$$

2nd region. $a \in \cup_{0 \leq i \leq l} \mathcal{V}(y_i, 2\rho)$

We set

$$Z_2 = \lambda \frac{\partial \tilde{\delta}}{\partial \lambda} + \frac{1}{\lambda} \frac{\partial \tilde{\delta}}{\partial a} \nabla_T K(a).$$

Using Propositions 3.6 and 3.7, we obtain

$$\begin{aligned} -(\nabla J(u), Z_2) &\geq c \frac{|\nabla_T K(a)|^2}{\lambda} + \frac{c}{\lambda} \frac{\partial K}{\partial \nu}(a) + O\left(\frac{1}{\lambda^2}\right) \\ &\geq \frac{c}{\lambda}. \end{aligned}$$

3rd region. $a \in \cup_{l+1 \leq i \leq s} \mathcal{V}(y_i, 2\rho)$.

We set

$$Z_3 = -\lambda \frac{\partial \tilde{\delta}}{\partial \lambda}.$$

Using Proposition 3.6, we deduce that

$$\begin{aligned} -(\nabla J(u), Z_3) &\geq -\frac{c}{\lambda} \frac{\partial K}{\partial \nu}(a) + O\left(\frac{1}{\lambda^2}\right) \\ &\geq \frac{c}{\lambda}. \end{aligned}$$

Hence our global vector field will be built using a convex combination of Z_1, Z_2 and Z_3 and will satisfy obviously i., iii. and iv. Regarding the estimate ii., it can be obtained once we have i. arguing as in [3] and [8]. \square

5 Proof of Theorem 1.1

Our proof follows the algebraic topological arguments introduced in [2]. Arguing by contradiction, we suppose that J has no critical points. It follows from Corollary 4.1, that under the assumptions of Theorem 1.1, the critical points at infinity of J under the level $c_1 = (S_n/2)^{\frac{2}{n}} (K(y_l))^{\frac{2-n}{n}} + \varepsilon$, for ε small enough, are in one to one correspondance with the critical points of $K_1 y_0, y_1, \dots, y_l$. The unstable manifold at infinity of such critical points at infinity, $W_u(y_0)_\infty, \dots, W_u(y_l)_\infty$ can be described, using Proposition 4.2, as the product of $W_s(y_0), \dots, W_s(y_l)$ (for a pseudogradient of K) by $[A, +\infty[$ domaine of the variable λ , for some positive number A large enough.

Since J has no critical points, it follows that $J_{c_1} = \{u \in \Sigma^+ / J(u) \leq c_1\}$ retracts by deformation on $X_\infty = \cup_{0 \leq j \leq l} W_u(y_j)_\infty$ (see Sections 7 and 8 of [7]) which can be parametrized as we said before by $X \times [A, +\infty[$.

From another part, we have X_∞ is contractible in $J_{c_2+\varepsilon}$, where $c_2 = (S_n/2)^{\frac{2}{n}} c^{\frac{2-n}{n}}$. Indeed from (A_4) , it follows that there exists a contraction $h : [0, 1] \times X \rightarrow K^c$, h continuous such that for any $a \in X$ $h(0, a) = a$ and $h(1, a) = a_0$ a point of X . Such a contraction gives rise to the following contraction $\tilde{h} : X_\infty \rightarrow \Sigma^+$ defined by

$$[0, 1] \times X \times [0, +\infty[\ni (t, a_1, \lambda_1) \longmapsto \tilde{\delta}_{(h(t, a_1), \lambda)} + \bar{v} \in \Sigma^+, \quad a_1 \in X, \lambda_1 \geq A$$

For $t = 0$, $\tilde{\delta}_{(h(0, a_1), \lambda_1)} + \bar{v} = \tilde{\delta}_{a_1, \lambda_1} + \bar{v} \in X_\infty$. \tilde{h} is continuous and $\tilde{h}(1, a_1, \lambda_1) = \tilde{\delta}_{a_0, \lambda_1} + \bar{v}$, hence our claim follows.

Now, using Proposition 3.5, we deduce that

$$J(\tilde{\delta}_{h(t, a_1), \lambda_1} + \bar{v}) \sim \left(\frac{S_n}{2}\right)^{\frac{2}{n}} (K(h(t, a_1)))^{\frac{2-n}{n}} (1 + O(A^{-2}))$$

where $K(h(t, a_1)) \geq c$ by construction.

Therefore such a contraction is performed under $c_2 + \varepsilon$, for A large enough, so X_∞ is contractible in $J_{c_2+\varepsilon}$.

In addition, choosing c_0 small enough, $J_{c_2+\varepsilon}$ retracts by deformation on J_{c_1} , which retracts by deformation on X_∞ , therefore X_∞ is contractible leading to the contractibility of X , which is in contradiction with our assumption. Hence our theorem follows.

Before ending this section, we give the proof of Corollary 1.1.

Proof of Corollary 1.1 Arguing by contradiction, we may assume that the Morse index of the solution provided by Theorem 1.1 is $\leq m - 1$.

Perturbing, if necessary J , we may assume that all the critical points of J are

nondegenerate and have their Morse index $\leq m - 1$. Such critical points do not change the homological group in dimension m of level sets of J .

Since X_∞ defines a homological class in dimension m which is nontrivial in J_{c_1} , but trivial in $J_{c_2+\varepsilon}$, our result follows. \square

6 Proof of Theorems 1.2 and 1.3

First, we start by proving the following main results

Proposition 6.1 *Let y_0 be defined in (H_1) . Then $(y_0, y_0)_\infty$ is not a critical point at infinity for J , that is, there exists a decreasing pseudogradient W for J satisfying Palais-Smaile in the neighborhood of $(y_0, y_0)_\infty$.*

Proof. For $\varepsilon_0 > 0$ small enough, we set

$$C_{\varepsilon_0} = \{u = \alpha_1 \tilde{\delta}_{a_1, \lambda_1} + \alpha_2 \tilde{\delta}_{a_2, \lambda_2} \in V_b(2, \varepsilon_0) / a_1, a_2 \in \mathcal{V}(y_0) \cap \partial S_+^n\}.$$

Our goal is to build a pseudogradient vector field W for J satisfying the Palais-Smaile condition in C_{ε_0} such that for $u \in C_{\varepsilon_0}$, we have

- i. $-(\nabla J(u), W) \geq \gamma \left(\sum \frac{1}{\lambda_i^2} + \sum (1 - J(u)^{\frac{n}{n-2}} a_i^{\frac{4}{n-2}} K(a_i)) \right) + c\varepsilon_{12}^{\frac{n-\frac{1}{2}}{n-2}},$
- ii. $-(\nabla J(u + \bar{v}), W + \frac{\partial \bar{v}}{\partial(\alpha, a, \lambda)}(W)) \geq \frac{\gamma}{2} \left(\sum \frac{1}{\lambda_i^2} + \sum (1 - J(u)^{\frac{n}{n-2}} a_i^{\frac{4}{n-2}} K(a_i)) \right) + c\varepsilon_{12}^{\frac{n-\frac{1}{2}}{n-2}}.$
- iii. W is bounded
- iv. $\dot{\lambda}_{max} \leq 0.$

where γ is a positive constant large enough.

We can assume, without loss of generality, that $\lambda_1 \leq \lambda_2$. We divide C_{ε_0} in three principal regions.

1st region. $M\lambda_1 \leq \lambda_2$ and $\forall i |1 - J(u)^{\frac{n}{n-2}} a_i^{\frac{4}{n-2}} K(a_i)| \leq \frac{2C'}{\lambda_i}$, where M and C' are positive constants large enough.

We set

$$W_1 = \lambda_1 \frac{\partial \tilde{\delta}_1}{\partial \lambda_1} - \lambda_2 \frac{\partial \tilde{\delta}_2}{\partial \lambda_2} \sqrt{M}.$$

From Proposition 3.2, we derive

$$\begin{aligned} -(\nabla J(u), W_1) &\geq c \left(\frac{1}{\lambda_1} + O(\varepsilon_{12}) \right) + \sqrt{M} \left[-\frac{c}{\lambda_2} + c\varepsilon_{12} \right] + O \left(\sum \frac{1}{\lambda_k^2} \right) \\ &\quad + O \left(\varepsilon_{12}^{\frac{n}{n-2}} \text{Log}(\varepsilon_{12}^{-1}) + \varepsilon_{12} (\text{Log}(\varepsilon_{12}^{-1}))^{\frac{n-2}{n}} \sum \frac{1}{\lambda_k} \right) \\ &\geq \frac{c}{2} \left(\frac{1}{\lambda_1} + \frac{M}{\lambda_2} \right) + \frac{c\sqrt{M}}{2} \varepsilon_{12} \\ &\geq C \left(\sum \frac{1}{\lambda_i^2} + \sum (1 - J(u)^{\frac{n}{n-2}} a_i^{\frac{4}{n-2}} K(a_i)) + \varepsilon_{12} \right). \end{aligned}$$

2nd region. $2M\lambda_1 \geq \lambda_2$ and, $\forall i |1 - J(u)^{\frac{n}{n-2}} a_i^{\frac{4}{n-2}} K(a_i)| \leq \frac{2C'}{\lambda_i}$

In this region, two cases may occur.

Case 2.1 $\varepsilon_{12}^{\frac{n-1}{n-2}} \geq \frac{d_0}{\lambda_1}$, where $d_0 = \max(d(y_0, a_1), d(y_0, a_2), d(a_1, a_2))$.

We set

$$W_2 = \frac{1}{\lambda_1} \left[\alpha_1 \frac{\partial \tilde{\delta}_1}{\partial a_1} \left(\frac{a_2 - a_1}{d(a_2, a_1)} \right) - \alpha_2 \frac{\partial \tilde{\delta}_2}{\partial a_2} \left(\frac{a_2 - a_1}{d(a_2, a_1)} \right) \right].$$

From Proposition 3.3 and the fact that $\frac{\partial \varepsilon_{12}}{\partial a_2} = -\frac{\partial \varepsilon_{12}}{\partial a_1}$, we obtain

$$\begin{aligned} -(\nabla J(u), W_2) &= \frac{1}{\lambda_1} \left(\alpha_1 \alpha_2 c_2 2J(u) \frac{\partial \varepsilon_{12}}{\partial a_1} \right) \left(-1 + J(u)^{\frac{n}{n-2}} \sum \alpha_k^{\frac{4}{n-2}} K(a_k) \right) 2 \left(\frac{a_2 - a_1}{d(a_2, a_1)} \right) \\ &\quad + O \left(\sum \left(\frac{|\nabla_T K(a_k)|}{\lambda_k} + \frac{1}{\lambda_k^2} \right) \right) \\ &\quad + O \left(\varepsilon_{12}^{\frac{n}{n-2}} \text{Log}(\varepsilon_{12}^{-1}) + \sum \frac{1}{\lambda_k} \varepsilon_{12} (\text{Log}(\varepsilon_{12}^{-1}))^{\frac{n-2}{n}} + d(a_1, a_2) \varepsilon_{12}^{\frac{n+1}{n-2}} \sum \lambda_k \right). \end{aligned}$$

Observe that

$$\begin{aligned} \varepsilon_{12} &\sim (\lambda_1 \lambda_2 d(a_1, a_2)^2)^{\frac{2-n}{2}} \quad \text{Indeed } \lambda_1 \text{ and } \lambda_2 \text{ are the same order, and} \\ \frac{\partial \varepsilon_{12}}{\partial a_1} &= -(n-2) \lambda_1 \lambda_2 (a_1 - a_2) \varepsilon_{12}^{\frac{n}{n-2}}. \end{aligned}$$

Thus

$$\begin{aligned} -(\nabla J(u), W_2) &= \frac{4}{\lambda_1} \alpha_1 \alpha_2 c_2 (n-2) J(u) \lambda_1 \lambda_2 d(a_1, a_2) \varepsilon_{12}^{\frac{n}{n-2}} (1 + o(1)) + R \\ &\geq \frac{c \varepsilon_{12}}{\lambda_1 d(a_1, a_2)} + R \\ &\geq c \varepsilon_{12}^{\frac{n-1}{n-2}} + R \end{aligned} \tag{6.18}$$

where

$$\begin{aligned} R &= +O \left(\frac{d(a_1, y_0)}{\lambda_1} + \frac{d(a_2, y_0)}{\lambda_2} + \sum \frac{1}{\lambda_k^2} \right) \\ &\quad + O \left(\varepsilon_{12}^{\frac{n}{n-2}} \text{Log}(\varepsilon_{12}^{-1}) + \sum \frac{1}{\lambda_k} \varepsilon_{12} (\text{Log}(\varepsilon_{12}^{-1}))^{\frac{n-2}{n}} + d(a_1, a_2) \varepsilon_{12}^{\frac{n+1}{n-2}} \sum \lambda_k \right). \end{aligned}$$

We also observe that

$$\frac{d_0}{\lambda_1} \leq \varepsilon_{12}^{\frac{n-1}{n-2}} = o \left(\varepsilon_{12}^{\frac{n-1}{n-2}} \right), \tag{6.19}$$

$$\frac{1}{\lambda_1 d(a_1, a_2)} \geq \left(\frac{1}{\lambda_1 \lambda_2 d(a_1, a_2)^2} \right)^{\frac{1}{2}} \sim \varepsilon_{12}^{\frac{1}{n-2}}, \tag{6.20}$$

$$\lambda_i d(a_1, a_2) \varepsilon_{12}^{(n+1)/(n-2)} = \varepsilon_{12}^{n/(n-2)} \left(\frac{\lambda_i}{\lambda_j} \right)^{1/2} = o(\varepsilon_{12}^{(n-1)/(n-2)}), \quad (6.21)$$

$$\begin{aligned} \frac{\varepsilon_{12}}{\lambda_1} (\text{Log}(\varepsilon_{12}^{-1}))^{(n-2)/n} &= \frac{\varepsilon_{12}}{\sqrt{\lambda_1 d_0}} \frac{\sqrt{d_0}}{\sqrt{\lambda_1}} (\text{Log}(\varepsilon_{12}^{-1}))^{(n-2)/n} \\ &= O\left(\frac{d_0}{\lambda_1} + \frac{\varepsilon_{12}^2 (\text{Log}(\varepsilon_{12}^{-1}))^{2(n-2)/n}}{\lambda_1 d_0} \right) \\ &= o\left(\varepsilon_{12}^{(n-1)/(n-2)} \right) + o\left(\frac{\varepsilon_{12}}{\lambda_1 d_0} \right). \end{aligned} \quad (6.22)$$

In the same way, we have

$$\frac{\varepsilon_{12}}{\lambda_2} (\text{Log}(\varepsilon_{12}^{-1}))^{(n-2)/n} = o\left(\varepsilon_{12}^{(n-1)/(n-2)} \right) + o\left(\frac{\varepsilon_{12}}{\lambda_2 d_0} \right) \quad (6.23)$$

We also have, since $\lambda_1 |a_1 - a_2| \rightarrow +\infty$,

$$\lambda_1^{-2} = o(d_0 \lambda_1^{-1}) = o\left(\varepsilon_{12}^{(n-1)/(n-2)} \right). \quad (6.24)$$

Similary, we have

$$\lambda_2^{-2} = o\left(\varepsilon_{12}^{(n-1)/(n-2)} \right). \quad (6.25)$$

Using (6.18), (6.19), (6.20), (6.21), (6.22), (6.23), (6.24) and (6.25), we find

$$\begin{aligned} -(\nabla J(u), W_2) &\geq C \left(\varepsilon_{12}^{(n-1)/(n-2)} + \frac{d_0}{\lambda_1} + \frac{d_0}{\lambda_2} \right) \\ &\quad \gamma \left(\varepsilon_{12}^{\frac{n-1}{n-2}} + \sum \frac{1}{\lambda_i^2} + \sum |1 - J(u)^{\frac{n-2}{n-2}} \alpha_i^{\frac{4}{n-2}} K(a_i)^2| \right) \end{aligned}$$

where γ is a large constant.

Case 2.2 $\varepsilon_{12}^{\frac{n-1}{n-2}} \leq 2 \frac{d_0}{\lambda_1}$

In this case, we set

$$W_3 = \frac{1}{\lambda_1} \left[\alpha_1 \frac{\partial \tilde{\delta}_1}{\partial a_1} \left(\frac{y_0 - a_1}{d_0} \right) + \alpha_2 \frac{\partial \tilde{\delta}_2}{\partial a_2} \left(\frac{y_0 - a_2}{d_0} \right) \right].$$

Using Proposition 3.3, we obtain

$$\begin{aligned} -(\nabla J(u), W_3) &= J(u) 2c_2 \frac{\alpha_1 \alpha_2}{\lambda_1} \frac{\partial \varepsilon_{12}}{\partial a_1} \left(\frac{a_2 - a_1}{d_0} \right) (1 + o(1)) \\ &\quad + J(u)^{\frac{2(n-1)}{n-2}} \frac{c_5}{\lambda_1} \sum \alpha_k^{\frac{2n}{n-2}} \nabla_T K(a_k) \frac{y_0 - a_k}{d_0} + R_1 \end{aligned}$$

where

$$R_1 = O\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \varepsilon_{12}^{\frac{n}{n-2}} \text{Log}(\varepsilon_{12}^{-1})\right) + O\left(\sum \frac{1}{\lambda_k} \varepsilon_{12} (\text{Log}(\varepsilon_{12}^{-1}))^{\frac{n-2}{n}} + d(a_1, a_2) \varepsilon_{12}^{\frac{n+1}{n-2}} \sum \lambda_k\right).$$

Thus, we find

$$\begin{aligned} -(\nabla J(u), W_3) &\geq \frac{c}{\lambda_1} \lambda_1 \lambda_2 \frac{d(a_1, a_2)^2}{d_0} \varepsilon_{12}^{\frac{n}{n-2}} + \frac{c}{\lambda_1 d_0} (d(a_1, y_0)^2 + d(a_2, y_0)^2) + R_1 \\ &\geq \frac{c}{\lambda_1 d_0} \varepsilon_{12} + \frac{c}{\lambda_1} d_0 + R_1. \end{aligned} \quad (6.26)$$

Observe that

$$\varepsilon_{12}^{\frac{n}{n-2}} \text{Log}(\varepsilon_{12}^{-1}) = o\left(\varepsilon_{12}^{\frac{n-\frac{1}{2}}{n-2}}\right) = o(d_0 \lambda_1^{-1}) \quad (6.27)$$

$$\lambda_k^{-2} = o(d_0 \lambda_k^{-1}) \quad \text{for } k = 1, 2, \quad (6.28)$$

$$\begin{aligned} \varepsilon_{12} (\text{Log}(\varepsilon_{12}^{-1}))^{\frac{n-2}{n}} \frac{1}{\lambda_1} &= \frac{\varepsilon_{12} (\text{Log}(\varepsilon_{12}^{-1}))^{\frac{n-2}{n}} \sqrt{d_0}}{\sqrt{\lambda_1 d_0} \sqrt{\lambda_1}} \\ &= o\left(\frac{d_0}{\lambda_1}\right) + O\left(\frac{\varepsilon_{12}^2 (\text{Log}(\varepsilon_{12}^{-1}))^{\frac{2(n-2)}{n}}}{\lambda_1 d_0}\right) \\ &= o\left(\frac{d_0}{\lambda_1} + \frac{\varepsilon_{12}}{\lambda_1 d_0}\right). \end{aligned} \quad (6.29)$$

Using (6.26), (6.27), (6.28) and (6.29), we obtain

$$\begin{aligned} -(\nabla J(u), W_3) &\geq C \left(\frac{d_0}{\lambda_1} + \frac{d_0}{\lambda_2} + \frac{\varepsilon_{12}}{\lambda_1 d_0}\right) \\ &\geq C \left(\frac{d_0}{\lambda_1} + \frac{d_0}{\lambda_2} + \varepsilon_{12}^{(n-\frac{1}{2})/(n-2)}\right) \\ &\quad \left(\sum \frac{1}{\lambda_i^2} + \sum (1 - J(u)^{\frac{n}{n-2}} a_i^{\frac{4}{n-2}} K(a_i))\right) + C \varepsilon_{12}^{\frac{n-\frac{1}{2}}{n-2}}. \end{aligned}$$

3rd region. $\exists i \in \{1, 2\}$ such that $|1 - J(u)^{\frac{n}{n-2}} a_i^{\frac{4}{n-2}} K(a_i)| \geq \frac{C'}{\lambda_i}$.
In this case, we set

$$Z = -\text{sign}\left(1 - J(u)^{\frac{n}{n-2}} a_i^{\frac{4}{n-2}} K(a_i)\right) \tilde{\delta}_i.$$

Using Proposition 3.4, we obtain

$$\begin{aligned} -(\nabla J(u), Z) &\geq C|1 - J(u)^{\frac{n}{n-2}} a_i^{\frac{4}{n-2}} K(a_i)| + O\left(\frac{1}{\lambda_i}\right) + O(\varepsilon_{12}) \\ &\geq \frac{C}{2}|1 - J(u)^{\frac{n}{n-2}} a_i^{\frac{4}{n-2}} K(a_i)| + \frac{C'}{4\lambda_i} + O(\varepsilon_{12}). \end{aligned}$$

We also set

$$Z_1 = \begin{cases} W_1 & \text{if } M\lambda_1 \leq \lambda_2 \\ -\lambda_2 \frac{\partial \tilde{\delta}_2}{\partial \lambda_2} & \text{if } 2M\lambda_1 \geq \lambda_2. \end{cases}$$

Setting $W_4 = Z + Z_1\sqrt{C'}$, we derive

$$\begin{aligned} -(\nabla J(u), W_4) &\geq C\left(|1 - J(u)^{\frac{n}{n-2}} a_i^{\frac{4}{n-2}} K(a_i)| + \sum \frac{1}{\lambda_k} + \varepsilon_{12}\right) \\ &\geq \gamma\left(\sum |1 - J(u)^{\frac{n}{n-2}} a_k^{\frac{4}{n-2}} K(a_k)|^2 + \sum \frac{1}{\lambda_k^2} + \varepsilon_{12}^{\frac{n-1}{n-2}}\right). \end{aligned}$$

Hence our global vector field will be built using a convex combination of W_1 , W_2 , W_3 and W_4 and will satisfy obviously i., iii. and iv. Next, we give the proof of ii. As in [3] and [8], it is easy to prove that

$$\begin{aligned} -\left(\nabla J(u + \bar{v}), \frac{\partial \bar{v}}{\partial(\alpha_i, a_i, \lambda_i)}(W)\right) &\leq c|\bar{v}||\nabla J(u + \bar{v})| \\ &= O(|\bar{v}|^2 + |\nabla J(u + \bar{v})|^2). \end{aligned}$$

By Propositions 3.2 and 3.3, we can prove that

$$|\nabla J(u + \bar{v})| = O\left(\sum |1 - J(u)^{\frac{n}{n-2}} a_k^{\frac{4}{n-2}} K(a_k)| + \sum \frac{1}{\lambda_k} + \varepsilon_{12} + |\bar{v}|\right).$$

Using now the estimate of \bar{v} , we easily derive ii. Thus our proposition follows. \square
Next, we state the following result whose proof is similar to corresponding statements in Lemma A1.1 [3]

Lemma 6.1 (Lemma A1.1 [3]) *Let $u = \alpha_{i_0} \delta_{(x_{i_0}, \lambda_{i_0})} + \alpha_{j_0} \delta_{(x_{j_0}, \lambda_{j_0})}$, where*

(i) $\varepsilon_{i_0 j_0} \geq \delta_1$, δ_1 a given constant

(ii) $B^2 \geq \lambda_{i_0}$, $\lambda_{j_0} \geq B$

If, given δ_1 , B is large enough, there is a pseudogradient vector field of J , built with the Yamabe gradient on u , which leads functions such as u in the neighborhood of functions of the type $\alpha \delta_{(y, \lambda)} + v$, where y is close to $\frac{1}{2}(x_{i_0} + x_{j_0})$ up to $O(\frac{1}{B})$, $\lambda \geq cB$ (c is a universal constant) and $\|v\| = o(1)$.

Now, we will use the above Lemma in the proof of the next main result.

Proposition 6.2 *Let $\varepsilon_0 > 0$ small enough. There exists a vector field Z_0 defined in*

$$W_{\varepsilon_0} = \{\alpha_1 \tilde{\delta}_{x, \lambda_1} + \alpha_2 \tilde{\delta}_{y_0, \lambda_2} / \alpha_i \geq 0, \alpha_1 + \alpha_2 = 1, x \in X, \lambda_i > \varepsilon_0^{-1}, \varepsilon_{12} < \varepsilon_0\}$$

which can be extended to

$$W(2, \varepsilon_0) = \{\alpha_1 \tilde{\delta}_{a_1, \lambda_1} + \alpha_2 \tilde{\delta}_{a_2, \lambda_2} + v \in \bar{V}_b(2, \varepsilon_0) / a_1, a_2 \in \overline{S_+^n}\}$$

so that the following holds:

$f_\lambda(C_{y_0}(X))$ retracts by deformation on $X \cup W_u(y_0, y_{i_0})_\infty \cup D$, where $D \subset \sigma$ is a stratified set (in the topological sense, that is, $D \in \Sigma_k(S_+^n)$, the group of chains of dimensions k) and where $\sigma = \cup_{y_i \in X \setminus \{y_{i_0}, y_0\}} W_u(y_0, y_i)_\infty$ is a manifold in dimension at most $k - 1$.

Here W_u denotes the unstable manifold for Z_0 .

Proof. First, we notice that assumption (H_2) implies that any critical point y of K_1 such that $y \in X$ satisfies $(\partial K / \partial \nu) > 0$. Now, we distinguish five cases

Case 1. There exists i such that α_i is far away from $J(u)^{\frac{-n}{4}} K(a_i)^{\frac{2-n}{4}}$ ($i = 1, 2$). We set

$$Z_1 = \tilde{\delta}_{a_i, \lambda_i} \tilde{\alpha}_i \quad \text{with } \tilde{\alpha}_i = \begin{cases} 1 & \text{if } \alpha_i > J(u)^{\frac{-n}{4}} K(a_i)^{\frac{2-n}{4}} + \eta \\ -1 & \text{if } \alpha_i < J(u)^{\frac{-n}{4}} K(a_i)^{\frac{2-n}{4}} - \eta \end{cases}$$

where η is a positive constant.

Using Proposition 3.4, we obtain

$$\begin{aligned} -(\nabla J(u), Z_1) &= c + O(\lambda_i^{-1}) + O(\varepsilon_{12}) \\ &\geq C > 0. \end{aligned}$$

Case 2. For each $i \in \{1, 2\}$, $\alpha_i = J(u)^{\frac{-n}{4}} K(a_i)^{\frac{2-n}{4}}$ and $x \notin \mathcal{V}(y_i, \rho)$, where $\rho < \frac{1}{2} \min_{i \neq j} d(y_i, y_j)$ and y_i is any critical point of $K_1 = K / \partial S_+^n$.

In this case, we have $d(x, y_0) \geq c$, thus $\varepsilon_{12} = o\left(\frac{1}{\lambda_i}\right)$ for $i = 1, 2$.

Two subcases may occur

If $\lambda_1 \leq C_1 \lambda_2$, where C_1 is a large enough positive constant, we set

$$Z_{21} = \frac{1}{\lambda_1} \frac{\partial \tilde{\delta}_1}{\partial a_1} \nabla_T K.$$

If $\lambda_1 \geq C_1 \lambda_2$, we set

$$Z_{22} = Z_{21} + \lambda_2 \frac{\partial \tilde{\delta}_2}{\partial \lambda_2}.$$

Using Propositions 3.2 and 3.3, we derive

$$-(\nabla J(u), Z_{2i}) \geq c \lambda_1^{-1} + c \lambda_2^{-1} + \varepsilon_{12} \quad \text{for } i = 1, 2.$$

Case 3. For each $i \in \{1, 2\}$, $\alpha_i = J(u)^{\frac{-n}{4}} K(a_i)^{\frac{2-n}{4}}$ and $x \in \mathcal{V}(y_i, 2\rho)$, where y_i is any critical point of K_1 such that $y_i \neq y_0$.

Since $x \in X$, $y_i \in X$ and therefore $(\partial K / \partial \nu)(y_i) > 0$. Now, we set

$$Z_3 = \lambda_1 \frac{\partial \tilde{\delta}_1}{\partial \lambda_1} + \lambda_2 \frac{\partial \tilde{\delta}_2}{\partial \lambda_2}.$$

Using Proposition 3.2, we obtain

$$-(\nabla J(u), Z_3) \geq c\lambda_1^{-1} + c\lambda_2^{-1} + \varepsilon_{12}.$$

Case 4. For each $i \in \{1, 2\}$, $\alpha_i = J(u)^{\frac{-n}{4}} K(a_i)^{\frac{2-n}{4}}$ and $x \in \mathcal{V}(y_0, 2\rho)$

In this case, we use the vector field defined in the proof of Proposition 6.1 which we combine with the vector field defined in Lemma 6.1.

Case 5. $\alpha_1 = 0$ or $\alpha_2 = 0$.

In this case, we only have one mass and we use the vector field defined in Lemma 4.1.

Our global vector field Z_0 will be built using a convex combination of vector fields defined in cases 1- 5.

Now, let $u = \alpha\delta_{x,\lambda} + (1 - \alpha)\delta_{y_0,\lambda} \in f_\lambda(C_{y_0}(X))$.

The action of the flow of the pseudogradient Z_0 is described as follow.

If $\alpha < 1/2$, the flow of Z_0 brings α to zero, and thus in this case u goes to $\overline{W_u((y_0)_\infty)} = \{y_0\}$.

If $\alpha > 1/2$, the flow of Z_0 brings α to 1, and thus u goes, in this case, to $\overline{W_u((y_{i_0})_\infty)} = X$.

If $\alpha = (1 - \alpha) = 1/2$, we have an action on $x \in X = \overline{W_s(y_{i_0})}$. In this case, u goes to $W_s(y_i)$, where y_i is a critical point of K_1 dominated by y_{i_0} and two cases may occur :

In the first case $y_i \neq y_0$, then x goes to $W_u(y_0, y_i)_\infty$.

In the second case $y_i = y_0$, u goes to $W_u(y_0)_\infty$ by the vector field defined in Lemma 6.1.

Then our result follows. \square

We now prove our theorems.

Proof of Theorem 1.2 We argue by contradiction. Assume that (1) has no solution. The strong retract defined in Proposition 6.2 does not intersect $W_u(y_0, y_{i_0})_\infty$ and thus it is contained in $X \cup D$ (see Proposition 6.2). Therefore $H_*(X \cup D) = 0$, for all $* \in \mathbb{N}^*$, since $f_\lambda(C_{y_0}(X))$ is a contractible set.

Using the exact homology sequence of $(X \cup D, X)$, we have

$$\dots \rightarrow H_{k+1}(X \cup D) \rightarrow^\pi H_{k+1}(X \cup D, D) \rightarrow^\partial H_k(X) \rightarrow^i H_k(X \cup D) \rightarrow \dots$$

Since $H_*(X \cup D) = 0$, for all $* \in \mathbb{N}^*$, then $H_k(X) = H_{k+1}(X \cup D, X)$.

In addition, $(X \cup D, X)$ is a stratified set of dimension at most k , then $H_{k+1}(X \cup D, X) = 0$, and therefore $H_k(X) = 0$. This yields a contradiction since X is a manifold in dimension k without boundary. Then our theorem follows. \square

Proof of Theorem 1.3 Assume that (1) has no solution. By the above arguments, $X \cup (\cup_{y_i \in B_k} W_u(y_0, y_i)) \cup D$ is a strong retract of $f_\lambda(C_{y_0}(X))$, where $D \subset \sigma$ is a stratified set and where $\sigma = \cup_{y_i \in X \setminus (B_k \cup \{y_{i_0}\})} W_u(y_0, y_i)_\infty$ is a manifold in dimension at most k .

Since $\mu_i(y_0) = 0$ for each $y_i \in B_k$, $f_\lambda(C_{y_0}(X))$ retracts by deformation on $X \cup D$, and therefore $H_*(X \cup D) = 0$, for all $* \in \mathbb{N}^*$. Using the exact homology sequence

of $(X \cup D, D)$, we obtain $H_{k+1}(X \cup D, X) = H_k(X) = 0$, a contradiction, and therefore our result follows. \square

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