

PRESCRIBING A FOURTH ORDER CONFORMAL INVARIANT ON THE STANDARD SPHERE — PART I: A PERTURBATION RESULT

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Dedicated to the memory of Stephen M. Paneitz

In this paper we study some fourth order elliptic equation involving the critical Sobolev exponent, related to the prescription of a fourth order conformal invariant on the standard sphere. We use a topological method to prove the existence of at least a solution when the function to be prescribed is close to a constant and a finite dimensional map associated to it has non-zero degree.

Keywords: Paneitz operator; positive solution; critical exponent; topological degree.

1. Introduction

1.1. Statement of the problem

In 1983, Paneitz [38] discovered a particularly interesting fourth order operator on 4-manifolds, given by

$$P_g^4 \varphi = \Delta_g^2 \varphi - \operatorname{div}_g \left(\frac{2}{3} \operatorname{Scal}_g g - 2 \operatorname{Ric}_g \right) d\varphi,$$

where Scal_g denotes the scalar curvature of (M^4, g) and Ric_g denotes the Ricci curvature of (M^4, g) . This operator enjoys many interesting properties (in particular, it is conformally invariant), and can be seen as a natural extension of the well known

second order conformal operator Δ_g on 2-manifolds. This operator gives rise to a fourth order conformal invariant ($|\text{Ric}_g|$ denoting the norm of Ric_g with respect to the metric)

$$Q_g = \frac{1}{12}(\Delta_g \text{Scal}_g + \text{Scal}_g^2 - 3|\text{Ric}_g|^2);$$

namely, under the conformal change $g' = e^{2w}g$, $Q_{g'}$ and Q_g are related by

$$P_g^4 w + 2Q_{g'} e^{4w} = 2Q_g.$$

There is also a natural fourth order Paneitz operator on manifolds of dimension greater than 4, introduced by Branson [10] and defined as follows. Given a smooth compact Riemannian n -manifold (M, g) , $n \geq 5$, let P_g^n be defined by

$$P_g^n u = \Delta_g^2 u - \text{div}_g(a_n \text{Scal}_g g + b_n \text{Ric}_g) du + \frac{n-4}{2} Q_g^n u,$$

where

$$a_n = \frac{(n-2)^2 + 4}{2(n-1)(n-2)}, \quad b_n = -\frac{4}{n-2},$$

$$Q_g^n = \frac{1}{2(n-1)} \Delta_g \text{Scal}_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} \text{Scal}_g^2 - \frac{2}{(n-2)^2} |\text{Ric}_g|^2.$$

If $\tilde{g} = \varphi^{\frac{4}{n-4}} g$ is a conformal metric to g , then for all $u \in C^\infty(M)$ we have

$$P_g^n(u\varphi) = \varphi^{\frac{n+4}{n-4}} P_{\tilde{g}}^n(u).$$

In particular

$$P_g^n(\varphi) = \frac{n-4}{2} Q_{\tilde{g}}^n \varphi^{\frac{n+4}{n-4}}. \tag{1.1}$$

Many interesting results on the Paneitz operator and related topics have been recently obtained by several authors. Among them, let us mention Branson [9], Branson–Chang–Yang [11], Chang–Yang [20], Chang–Gursky–Yang [15], Chang–Qing–Yang [16] and [17], Gursky [27]. See also the surveys Chang [13] and Chang–Yang [18] for more results on basic properties of the Paneitz operator.

In view of relation (1.1), a natural problem to propose is to prescribe the conformal invariant Q . Equation (1.1), where φ has to be considered as unknown, has a variational structure, and hence solutions occur as critical points of the associated Euler functional. A natural space to look in for solutions is the Sobolev space $H_2^2(M)$ (see Sec. 2). Due to the non-compactness of the injection of $H_2^2(M)$ into $L^{\frac{2n}{n-4}}(M)$, the Euler functional does not satisfy the Palais–Smale condition, which leads to the failure of the standard critical point theory.

In this paper we consider the problem of prescribing Q on the standard sphere (\mathbb{S}^n, h) , $n \geq 5$, where the expression of P_h^n is given by

$$P_h^n u = \Delta_h^2 u + c_n \Delta_h u + d_n u, \tag{1.2}$$

with

$$c_n = \frac{1}{2}(n^2 - 2n - 4), \quad d_n = \frac{n - 4}{16}n(n^2 - 4). \tag{1.3}$$

More precisely, given a smooth positive function f on \mathbb{S}^n , can one find a metric g conformal to h for which the Q_g -curvature is given by f ? Setting $g = u^{\frac{-4}{n-4}}h$, this amounts to solving the following equation

$$(E) \begin{cases} \Delta_h^2 u + c_n \Delta_h u + d_n u = \frac{n - 4}{2} f u^{\frac{n+4}{n-4}}, \\ u > 0 \text{ on } \mathbb{S}^n. \end{cases}$$

We recall that, by the regularity results of Djadli–Hebey–Ledoux [21] (see also Van der Vorst [40]), a weak solution of (E) is indeed a smooth solution.

Let us observe that, in the case of the sphere (in view of the uniqueness result of Lin [35]), a solution of (E) cannot be achieved (unless f is a constant) as a minimum for the extremal problem

$$\inf \left\{ \frac{f_{\mathbb{S}^n} (\Delta_h u)^2 + c_n f_{\mathbb{S}^n} |\nabla u|^2 + d_n f_{\mathbb{R}^n} u^2}{(f_{\mathbb{S}^n} f |u|^{2^\#})^{\frac{2}{2^\#}}} \mid u \in H_2^2(\mathbb{S}^n) u \not\equiv 0 \right\}.$$

Moreover, due to the invariance of equation (E) under the group of conformal transformations of (\mathbb{S}^n, h) , Kazdan–Warner type obstructions occur. Precisely, the following theorem holds.

Theorem 1.1 ([21]). *Let (\mathbb{S}^n, h) be the standard n -dimensional unit sphere, $n \geq 5$, and f a smooth function on \mathbb{S}^n . If u is a smooth solution of the equation*

$$P_h^n u = \frac{n - 4}{2} f |u|^{\frac{-8}{n-4}} u,$$

then for any eigenfunction ξ of Δ_h associated to the first non zero eigenvalue $\lambda_1 = n$, it is

$$\int_{\mathbb{S}^n} \langle \nabla f, \nabla \xi \rangle |u|^{2^\#} dv_h = 0,$$

where $2^\# = \frac{2n}{n-4}$. In particular, for any eigenfunction ξ of Δ_h associated to the first non zero eigenvalue and for any $\varepsilon \in \mathbb{R}$, equation (E) with $f = 1 + \varepsilon \xi$ admits no positive solution.

In fact, this theorem is proved in Djadli–Hebey–Ledoux [21] for positive solution, but, with the same arguments, one can prove that it remains true for changing sign solutions.

The problem of finding conditions on f such that (E) possesses a solution can be seen as the generalization to the Paneitz operator of the so-called “Nirenberg problem”; namely: which functions $K \in C^\infty(\mathbb{S}^n)$ are the scalar curvature of a metric conformal to the standard one? Theorem 1.1 is the analogue of the classical obstruction for the Nirenberg problem (see Kazdan–Warner [31]). The Nirenberg problem has been studied by several authors; among others, we mention Ambrosetti–Garcia

Azorezo–Peral [2], Ambrosetti–Malchiodi [3], Aubin [4], Bahri–Coron [6], Chang–Yang [19], Chang–Gursky–Yang [14], Hebey [28], Li [33], Lin [34], Schoen–Zhang [39].

1.2. Statement of the results

It is well known (see for instance Berger–Gauduchon–Mazet [8]), that denoting

$$\mathbb{H} = \{ \xi \in C^\infty(\mathbb{S}^n) \mid \xi \neq 0, \Delta_h \xi = n\xi \},$$

it is $\dim \mathbb{H} = n + 1$, and $\xi \in \mathbb{H}$ if and only if ξ is the restriction to \mathbb{S}^n of a linear form on \mathbb{R}^{n+1} . It follows that, denoting by $(x_i)_{i \in \{1, \dots, n+1\}}$ the canonical coordinates of \mathbb{R}^{n+1} and by ξ_i the restriction to \mathbb{S}^n of the i th canonical projection associated to these coordinates, $(\xi_i)_{i \in \{1, \dots, n+1\}}$ is a basis of \mathbb{H} . In the sequel of the paper we set

$$\xi = (\xi_1, \dots, \xi_{n+1}) \in \mathbb{H}^{n+1}.$$

Let us now introduce some notation (following those introduced by Chang–Yang [19]). First we will denote

$$f = \frac{1}{\text{Vol}_h(\mathbb{S}^n)} \int.$$

For $P \in \mathbb{S}^n$ and $t \in [1; +\infty)$, we denote by $p = \frac{t-1}{t}P \in B$, B being the unit ball of \mathbb{R}^{n+1} . Using stereographic coordinates with projection through the point P , we set

$$\varphi_{P,t}(y) = ty.$$

This set of conformal transformations is diffeomorphic to the unit ball $B \subset \mathbb{R}^{n+1}$ with the identity transformation identified with the origin and $\varphi_{P,t} \leftrightarrow \frac{t-1}{t}P = p \in B$. In the sequel of the paper, we will consider the map $G : B \rightarrow \mathbb{R}^{n+1}$ given by

$$G(P, t) = \int_{\mathbb{S}^n} (f \circ \varphi_{P,t}) \xi dv_h,$$

where ξ is defined above (note that, here and in the sequel of the paper, we identify $(P, t) \in \mathbb{S}^n \times [1; +\infty)$ and $p = \frac{t-1}{t}P \in B$).

Definition 1.1. Let $f \in C^\infty(\mathbb{S}^n)$. We say that f is uniformly non degenerated of order α if there exists $t_1 > 1$ and $C > 0$ such that for all $t \geq t_1$ and for all $P \in \mathbb{S}^n$

$$|G(P, t)| \geq \frac{C}{t^\alpha} \quad \text{when } \alpha < n;$$

$$|G(P, t)| \geq \frac{C}{t^n} \log t \quad \text{when } \alpha = n.$$

Our main theorem is the following

Theorem 1.2. *There exists $\varepsilon(n) > 0$, depending only on the dimension of \mathbb{S}^n , such that (E) admits a solution for all functions $f \in C^\infty(\mathbb{S}^n)$ satisfying the following assumptions:*

- (H1) $\|f - \frac{n(n^2-4)}{8}\|_\infty \leq \varepsilon(n)$;
- (H2) f is uniformly non degenerated of order at most n if n is even, and of order at most $n - 1$ if n is odd;
- (H3) $\deg(G, \{p|t < t_0\}, 0) \neq 0$ for some t_0 large enough.

Using Theorem 1.2, one can prove the following corollary.

Corollary 1.1. *Assume $f \in C^\infty(\mathbb{S}^n)$, $n \geq 5$, is a Morse function satisfying the condition (H1) of Theorem 1.2 and the following assumptions*

$$|\nabla f(P)| = 0 \implies \Delta_h f(P) \neq 0,$$

and

$$\sum_{P \in \mathbb{S}^n | \nabla f(P)=0 \text{ and } \Delta_h f(P)>0} (-1)^{m(P,f)} \neq -1,$$

where $m(P, f)$ is the Morse index of f at P . Then (E) admits a solution.

As observed in Chang–Gursky–Yang [14], the condition on the index in Corollary 1.1 is stronger than the condition (H3) of Theorem 1.2. Recently, Felli [26] proved an existence result for (E) related to Corollary 1.1 using a method introduced by Ambrosetti and Badiale [1].

1.3. Comments

Theorem 1.2 is an extension to the Paneitz operator of the results of Chang–Yang [19]. Apart from the difficulties mentioned above, differently from the case of the scalar curvature, there are other specific problems. The first is that obtaining positive solutions is not immediate; in the case of the scalar curvature equation, one can obtain positivity just by using the fact that $|\nabla|u|| = |\nabla u|$ a.e. This is not anymore true for the fourth order equation because of the term $\int_{\mathbb{S}^n} (\Delta_h u)^2$ (and the fact that $|u|$ not necessarily belongs to $H^2_2(\mathbb{S}^n)$). We also encounter the same difficulty in obtaining *a priori* estimates (see Sec. 5) since we cannot apply the Moser iteration scheme. Our main tool to handle such difficulties is an *a priori* estimate for fourth order operator (see Proposition 3.1) which goes back to a device introduced by Van der Vorst [40] (see also Lee–Parker [32] and Djadli–Hebey–Ledoux [21]) to prove regularity of solutions. Starting from Proposition 3.1, we propose a way to handle positivity and *a priori* estimates for the above fourth order operator (see Remark 5.2). Another main ingredient in the proof of Theorem 1.2 is a fourth order Sobolev–Aubin type inequality, stated in Theorem 4.1, which could be interesting also in itself.

In the second part of the paper (Djadli–Malchiodi–Ould Ahmedou [24]), we give some existence results without the “close to a constant condition” in dimension 5 and 6. Part of these results are obtained via a continuity argument whose starting point is Theorem 1.2 (and generalize to the Paneitz operator previous results of

Chang–Gursky–Yang [14], Bahri–Coron [6], Li [33] and Benayed–Chen–Chtioui–Hammami [7] for the Nirenberg problem). Our work in the second part of this paper is based on a fine blow-up analysis for a family of equations approximating (E) . This analysis extend to this framework the notion of isolated and isolated simple blow-up points introduced by R. Schoen (and used extensively by Y. Y. Li [33]). This blow-up analysis requires some Harnack type inequalities for fourth order operators obtained from the interpretation of higher order equations as systems (from which we extract some useful informations on the sign of the laplacian of the solutions). See Djadli–Malchiodi–Ould Ahmedou [24] for details.

The paper is organized as follows: in Sec. 2 we recall some useful facts, in Sec. 3 we derive some *a priori* estimates for some fourth order equations and in Sec. 4 we prove the Sobolev–Aubin type inequality (see Theorem 4.1); in Sec. 5 we perform a finite dimensional reduction of the problem, based on Theorem 4.1. In particular we give an uniform estimate on a family of solutions u_p of this reduced problem, depending on a $(n + 1)$ -dimensional parameter p . In Sec. 6 we show that the map $p \rightarrow u_p$ is well defined and continuous. In Sec. 7 we prove that the maps G and $p \rightarrow u_p$ are related by some degree formula; in the last section we prove Theorem 1.2 and Corollary 1.1.

2. Preliminaries on Paneitz Operator and Background Material

A natural space when studying the Paneitz operator P_h^n is the Sobolev space $H_2^2(\mathbb{S}^n)$ defined as the completion of $C^\infty(\mathbb{S}^n)$ with respect to the norm

$$\|u\|^2 = \|\nabla^2 u\|_2^2 + \|\nabla u\|_2^2 + \|u\|_2^2,$$

where $\|\cdot\|_p$ stands for the L^p -norm with respect to the Riemannian measure $dv(h)$. As is well known and easy to see, for all $u \in C^\infty(\mathbb{S}^n)$

$$(\Delta_h u)^2 \leq n|\nabla^2 u|^2.$$

Conversely, using the Bochner–Lichnerowicz–Weitzenbock formula, there holds

$$\int_{\mathbb{S}^n} |\nabla^2 u|^2 dv(h) \leq \int_{\mathbb{S}^n} |\Delta_h u|^2 dv(h) + k \int_{\mathbb{S}^n} |\nabla u|^2 dv(h),$$

where k is such that $\text{Ric}_g \geq -kg$. Hence $\|\cdot\|_{H_2^2}$ defined by

$$\|u\|_{H_2^2}^2 = \int_{\mathbb{S}^n} (P_h u)u dv(h)$$

is a norm on $H_2^2(\mathbb{S}^n)$ which is equivalent to the classical one $\|\cdot\|$.

By the Sobolev embedding theorem, for $n \geq 5$, one gets an embedding of $H_2^2(\mathbb{S}^n)$ in $L^{2^\#}(\mathbb{S}^n)$ where $2^\# = \frac{2n}{n-4}$. This embedding is critical and continuous; so there exists $A \in \mathbb{R}$ such that

$$\|u\|_{2^\#}^2 \leq A \int_{\mathbb{S}^n} (P_h u)u dv(h).$$

Setting

$$K_0 = \pi^2 n(n-4)(n^2-4) \Gamma\left(\frac{n}{2}\right)^{\frac{4}{n}} \Gamma(n)^{-\frac{4}{n}}, \tag{2.1}$$

we have the following theorem (see Djadli–Hebey–Ledoux [21]) regarding the best constant for the embedding $H_2^2(\mathbb{S}^n) \hookrightarrow L^{2^\#}(\mathbb{S}^n)$.

Theorem 2.1. *Let (\mathbb{S}^n, h) be the standard unit sphere of \mathbb{R}^{n+1} , $n \geq 5$, and P_h be the Paneitz operator $P_h = \Delta_h^2 + c_n \Delta_h + d_n$. Then*

$$\inf_{u \in C^\infty(\mathbb{S}^n) \setminus \{0\}} \frac{\int_{\mathbb{S}^n} (P_h u) u}{\left(\int_{\mathbb{S}^n} |u|^{2^\#}\right)^{\frac{2}{2^\#}}} = \frac{1}{K_0} \omega_n^{-\frac{4}{n}},$$

where ω_n denotes the volume of the standard unit n -sphere.

We also recall the following result of Djadli–Hebey–Ledoux [22].

Theorem 2.2. *Let (M, g) be a smooth compact n -dimensional Riemannian manifold, $n \geq 2$. Then for any positive real number α , the spectrum of the operator $\Delta_g^2 + \alpha \Delta_g$ is exactly $\{\lambda^2 + \alpha \lambda, \lambda \in Sp(\Delta_g)\}$, with the additional property that u is an eigenfunction of $\Delta_g^2 + \alpha \Delta_g$ associated to $\mu = \lambda^2 + \alpha \lambda$ if and only if u is an eigenfunction of Δ_g associated to λ .*

3. An *a priori* Estimate for a Fourth Order Operator

In this section, following a previous argument by Van der Vorst [40] (see also Djadli–Hebey–Ledoux [21]), we prove an *a priori* estimate for solutions of some fourth order equation. The result plays the same role as the Moser iteration scheme, which is used to gain integrability in second order equations.

Proposition 3.1. *Let (\mathbb{S}^n, h) be the standard n -sphere, let $\alpha > \frac{n}{4}$, $p \in [1; 2^\# - 1]$ and let $q \in L^\alpha(\mathbb{S}^n)$, $w \in L^\infty(\mathbb{S}^n)$. Assume that $u \in H_2^2(\mathbb{S}^n)$ is a weak solution of the equation*

$$\left(\Delta_h + \frac{c_n}{2}\right)^2 u = q|u|^{p-1}u + w.$$

Then there exist a positive constant δ^∞ depending only on n and α , and a positive constant C depending only on n , α and $\|q\|_\alpha$ such that if $q|u|^{p-1} \in L^{\frac{n}{4}}(\mathbb{S}^n)$ and $\|q|u|^{p-1}\|_{\frac{n}{4}} \leq \delta^\infty$ then

$$\begin{cases} u \in L^\infty(\mathbb{S}^n); \\ \|u\|_\infty \leq C(\|w\|_\infty + \|w\|_\infty^p). \end{cases}$$

Proof. Let $L_h = \Delta_h + \frac{c_n}{2}$. For all $r > 1$ and for all $g \in L^r(\mathbb{S}^n)$, there exists a unique function $\varphi \in H_4^r(\mathbb{S}^n)$ such that $L_h^2 \varphi = g$. Moreover for all $\varphi \in H_4^r(\mathbb{S}^n)$

$$\|L_h^2 \varphi\|_r \leq \|\Delta^2 \varphi\|_r + c_n \|\Delta \varphi\|_r + \frac{c_n^2}{4} \|\varphi\|_r \leq C \|\varphi\|_{H_4^r}.$$

This means that $L_h^2 : H_4^r(\mathbb{S}^n) \rightarrow L^r(\mathbb{S}^n)$ is continuous. Then according to the open mapping theorem there exists $C > 0$ depending only on n and r such that for all $g \in L^r(\mathbb{S}^n)$

$$\|L_h^{-2}g\|_{H_4^r} \leq C\|g\|_r.$$

Since

$$\begin{cases} H_4^r(\mathbb{S}^n) \hookrightarrow L^{\frac{nr}{n-4r}}(\mathbb{S}^n) & \text{if } n > 4r; \\ H_4^r(\mathbb{S}^n) \hookrightarrow L^\infty(\mathbb{S}^n) & \text{if } n < 4r, \end{cases}$$

it follows that for all $g \in L^r(\mathbb{S}^n)$

$$\begin{cases} \|L_h^{-2}g\|_{\frac{nr}{n-4r}} \leq C(n, r)\|g\|_r & \text{if } n > 4r; \\ \|L_h^{-2}g\|_\infty \leq C(n, r)\|g\|_r & \text{if } n < 4r. \end{cases} \tag{3.1}$$

Now, let $s > 1$ and let $v \in L^s(\mathbb{S}^n)$. It is clear that $q|u|^{p-1}v \in L^{\hat{s}}(\mathbb{S}^n)$ with $\hat{s} = \frac{ns}{n+4s}$. Let $u_{q,v}$ be the solution of

$$L_h^2 u_{q,v} = q|u|^{p-1}v.$$

Then, since $\hat{s} < \frac{n}{4}$, using the first estimate in (3.1) with $r = \hat{s}$ and the Hölder inequality, it turns out that

$$\|u_{q,v}\|_s \leq C(n, s)\|q|u|^{p-1}\|_{\frac{n}{4}}\|v\|_s.$$

Hence the operator $\mathcal{H}_q : v \rightarrow u_{q,v}$ acts continuously from $L^s(\mathbb{S}^n)$ to $L^s(\mathbb{S}^n)$ and its norm is less or equal than $C(n, s)\|q|u|^{p-1}\|_{\frac{n}{4}}$. In particular if $C(n, s)\|q|u|^{p-1}\|_{\frac{n}{4}} < \frac{1}{2}$, the operator $I - \mathcal{H}_q$ is invertible and the norm of its inverse is less than 2. The equation satisfied by u can be written as

$$(I - \mathcal{H}_q)u = L_h^{-2}(w).$$

Hence for s fixed and if $\|q|u|^{p-1}\|_{\frac{n}{4}} < \frac{1}{2}C(n, s)^{-1}$, we deduce

$$\|u\|_s \leq 2C'(n, s)\|w\|_s \leq 2C''(n, s)\|w\|_\infty.$$

Now we use the equation satisfied by u to deduce the estimate of the Proposition. First we take $s > \frac{\frac{n}{4}p\alpha}{\alpha - \frac{n}{4}}$ (this choice implies that $\frac{\alpha s}{s + \alpha p} > \frac{n}{4}$). Then for this s we have (note that $p \leq 2^\# - 1$ allows us to take a s independent of p) according to the previous estimate

$$\|u\|_s \leq C(n, s)\|w\|_\infty.$$

It is

$$\|q|u|^p\|_{\frac{\alpha s}{s + \alpha p}} \leq \|q\|_\alpha \|u\|_s^p \leq \tilde{C}(n, \alpha)\|q\|_\alpha \|w\|_\infty^p,$$

with $\frac{\alpha s}{s + \alpha p} > \frac{n}{4}$. It follows from the second estimate of (3.1) with $r = \frac{\alpha s}{s + \alpha p}$ that

$$\|u\|_\infty \leq C'(n, s, \|q\|_\alpha)(\|w\|_\infty + \|w\|_\infty^p).$$

To conclude the proof of the proposition it remains to remark that, since we just have to choose $s > \frac{\frac{n}{4}p\alpha}{\alpha - \frac{n}{4}}$ and since $p < 2^\# - 1$, in fact we find a constant in the last estimate depending only on $\|q\|_\alpha$, α and n . □

Remark 3.1. If $\alpha = +\infty$ then δ^∞ depends only on n . As a matter of fact, in this case we choose $s > p\frac{n}{4}$ and then we get

$$\|q|u|^p\|_{\frac{s}{p}} \leq C(n, \|q\|_\infty) \|u\|_s^p \leq C(n, \|q\|_\infty) \|w\|_\infty^p.$$

Remark 3.2. From the arguments in the proof of Proposition 3.1, it follows that we can obtain, for $\|q|u|^{p-1}\|_{\frac{n}{4}} \leq \delta^\infty$, the estimate

$$\|u\|_\infty \leq C(n, s, \alpha) (\|w\|_s + \|w\|_s^p),$$

for all $s > \frac{\frac{n}{4}p\alpha}{\alpha - \frac{n}{4}}$ (and if $\alpha = +\infty$ for all $s > p\frac{n}{4}$ and $C = C(n, s)$).

4. A Fourth Order Sobolev–Aubin Inequality

Following an original idea by Aubin [4], the Sobolev inequality in Theorem 2.1 can be improved when we restrict u to belong to the symmetric class $\mathcal{S}_{2^\#}$ (see notation below). For this new framework, such an improvement is given by Theorem 4.1, to which this section is devoted. In the last steps of the proof (see Lemmas 4.6 and 4.7) are needed some L^∞ estimates proved using Proposition 3.1. The aim of this section is to prove the following theorem.

Theorem 4.1. For $q \in (1; 2^\#)$ let $\mathcal{S}_q = \{u \in H_2^2(\mathbb{S}^n) \mid \int_{\mathbb{S}^n} |u|^q \xi = 0 \text{ for all } \xi \in \mathbb{H}\}$. Then $\exists a_n < 1$ and $\exists q_0 < 2^\#$ so that for all $q_0 \leq q \leq 2^\#$

$$\inf_{\{u \in \mathcal{S}_q, u \neq 0\}} \frac{a_n \left(\int_{\mathbb{S}^n} (\Delta_h u)^2 + c_n \int_{\mathbb{S}^n} |\nabla u|^2 \right) + d_n \int_{\mathbb{S}^n} u^2}{\left(\int_{\mathbb{S}^n} |u|^q \right)^{\frac{2}{q}}} \geq K_0^{-1} \omega_n^{-\frac{4}{n}}.$$

Before giving the proof of Theorem 4.1, we introduce some further notation.

Notation 4.1. For $a < 1$ and $u \in H_2^2(\mathbb{S}^n)$ set

$$\begin{aligned} E_a[u] &= a \left(\int_{\mathbb{S}^n} (\Delta_h u)^2 + c_n \int_{\mathbb{S}^n} |\nabla u|^2 \right) + d_n \int_{\mathbb{S}^n} u^2; \\ J_{a,q}[u] &= \frac{E_a[u]}{\left(\int_{\mathbb{S}^n} |u|^q \right)^{\frac{2}{q}}} \quad \forall u \in H_2^2(\mathbb{S}^n) \setminus \{0\}; \\ \mathcal{M}_{a,q} &= \inf_{\{u \in \mathcal{S}_q, u \neq 0\}} J_{a,q}[u] = \inf_{u \in \mathcal{S}_q^0} E_a[u]; \end{aligned}$$

where

$$\mathcal{S}_q^0 = \left\{ u \in \mathcal{S}_q \text{ such that } \int_{\mathbb{S}^n} |u|^q = 1 \right\}.$$

We split the proof of Theorem 4.1 in several lemmas.

Lemma 4.1. $\mathcal{M}_{a,q} \leq K_0^{-1} \omega_n^{-\frac{4}{n}}$ for all $0 \leq a < 1$ and for all $2 < q \leq 2^\#$. Moreover, for all $2 < q \leq 2^\#$, we have

$$\lim_{a \rightarrow 1} \mathcal{M}_{a,q} = K_0^{-1} \omega_n^{-\frac{4}{n}}. \tag{4.1}$$

Proof. We have $\mathcal{M}_{a,q} \leq J_{a,q}[1] = d_n$ and $d_n = K_0^{-1} \omega_n^{-\frac{4}{n}}$ by (1.3) and (2.1). Hence the first part of the lemma is proved. Now, according to Hölder’s inequality and to Theorem 2.1, for all $2 < q \leq 2^\#$

$$\mathcal{M}_{1,q} \geq K_0^{-1} \omega_n^{-\frac{4}{n}}. \tag{4.2}$$

Since $a \rightarrow \mathcal{M}_{a,q}$ is increasing, $\lim_{a \rightarrow 1} \mathcal{M}_{a,q}$ exists. Assume by contradiction that $\lim_{a \rightarrow 1} \mathcal{M}_{a,q} < K_0^{-1} \omega_n^{-\frac{4}{n}}$. Setting $\lambda = \lim_{a \rightarrow 1} \mathcal{M}_{a,q}$, we have

$$\exists a_0 < 1 \text{ such that } \forall a \in (a_0; 1) \exists u_a \in \mathcal{S}_q^0 \text{ satisfying } E_a[u_a] \leq \lambda + \frac{l}{4} < K_0^{-1} \omega_n^{-\frac{4}{n}},$$

where $l = K_0^{-1} \omega_n^{-\frac{4}{n}} - \lambda$. Hence, $\forall a \in (a_0; 1)$, there exists $u_a \in \mathcal{S}_q^0$ such that

$$E_1[u_a] \leq \lambda + \frac{l}{4} + (1 - a) \left(\int_{\mathbb{S}^n} (\Delta_h u_a)^2 + c_n \int_{\mathbb{S}^n} |\nabla u_a|^2 \right).$$

Since the quantities $\int_{\mathbb{S}^n} (\Delta_h u_a)^2 + c_n \int_{\mathbb{S}^n} |\nabla u_a|^2$ are uniformly bounded, for a close to 1,

$$E_1[u_a] \leq \lambda + \frac{l}{2} < K_0^{-1} \omega_n^{-\frac{4}{n}},$$

contradicting (4.2). The second part of the lemma follows. This concludes the proof of Lemma 4.1. \square

Lemma 4.2. Let $r \in (2; 2^\#]$. For all $\varepsilon > 0$, there exists a C_ε depending only on ε , so that $\forall u \in \mathcal{S}_r$

$$\left(\int_{\mathbb{S}^n} |u|^r \right)^{\frac{2}{r}} \leq K_0 \omega_n^{\frac{4}{n}} (2^{-\frac{4}{n}} + \varepsilon) \int_{\mathbb{S}^n} (\Delta_h u)^2 + C_\varepsilon \left(\int_{\mathbb{S}^n} \{ |\nabla u|^2 + u^2 \} \right).$$

Proof. This lemma (analogous to a result of Aubin [4]) is due to Jourdain [29]. \square

Let $a_0 \in (2^{-\frac{4}{n}}; 1)$ and fix it in the sequel.

Lemma 4.3. For each $a \in [a_0; 1)$ and each $q \in (2; 2^\#)$, $\inf_{u \in \mathcal{S}_q^0} J_{a,q}[u]$ is achieved by some function $u_{a,q} \in \mathcal{S}_q^0 \cap C^\infty(\mathbb{S}^n)$. Moreover, there exists $\Lambda_{a,q} \in \mathbb{R}^+$ and $\xi_{a,q} \in \mathbb{H}$, $\|\xi_{a,q}\|_\infty = 1$, such that $u_{a,q}$ satisfies

$$a(\Delta_h^2 u_{a,q} + c_n \Delta_h u_{a,q}) + d_n u_{a,q} = \mathcal{M}_{a,q} |u_{a,q}|^{q-2} u_{a,q} + \Lambda_{a,q} \xi_{a,q} |u_{a,q}|^{q-2} u_{a,q} \tag{4.3}$$

and there exists $C > 0$ depending only on n such that for all $a \in [a_0; 1)$ and for all $q \in (2; 2^\#)$, $\|u_{a,q}\|_2^2 \geq C$.

Proof. Using Lemma 4.2 we find the existence of $u_{a,q} \in \mathcal{S}_q^0$, $\Lambda_{a,q} \in \mathbb{R}^+$ and $\xi_{a,q} \in \mathbb{H}$ such that $u_{a,q}$ minimizes $J_{a,q}$ and satisfies the equation

$$a(\Delta_h^2 u_{a,q} + c_n \Delta_h u_{a,q}) + d_n u_{a,q} = \mathcal{M}_{a,q} |u_{a,q}|^{q-2} u_{a,q} + \Lambda_{a,q} \xi_{a,q} |u_{a,q}|^{q-2} u_{a,q}.$$

Concerning the regularity of $u_{a,q}$, reasoning as in Djadli–Hebey–Ledoux [21, Lemma 2.1], we can prove by a bootstrap argument that for all $s \geq 1$, $u_{a,q} \in L^s(\mathbb{S}^n)$. Then it easily follows that $u_{a,q} \in C^\infty(\mathbb{S}^n)$. Now, since $u_{a,q} \in \mathcal{S}_q^0$, according to Lemma 4.2,

$$1 \leq K_0 \omega_n^{\frac{4}{n}} (2^{-\frac{4}{n}} + \varepsilon) \int_{\mathbb{S}^n} (\Delta_h u_{a,q})^2 + C_\varepsilon \int_{\mathbb{S}^n} \{|\nabla u_{a,q}|^2 + u_{a,q}^2\}.$$

On the other hand by Lemma 4.1, since $u_{a,q}$ is a minimizer

$$a \left(\int_{\mathbb{S}^n} (\Delta_h u_{a,q})^2 + c_n \int_{\mathbb{S}^n} |\nabla u_{a,q}|^2 \right) + d_n \int_{\mathbb{S}^n} u_{a,q}^2 \leq K_0^{-1} \omega_n^{-\frac{4}{n}}.$$

Hence

$$\begin{aligned} C_\varepsilon \|u_{a,q}\|_{H_1^2}^2 &\geq 1 - K_0 \omega_n^{\frac{4}{n}} (2^{-\frac{4}{n}} + \varepsilon) \int_{\mathbb{S}^n} (\Delta_h u_{a,q})^2 \\ &\geq 1 - K_0 \omega_n^{\frac{4}{n}} \left(\frac{2^{-\frac{4}{n}} + \varepsilon}{a} \right) \{K_0^{-1} \omega_n^{-\frac{4}{n}} - c_n a \|u_{a,q}\|_{H_1^2}^2\} \end{aligned}$$

(note that when $n \geq 5$, $c_n \leq d_n$). Then

$$(C_\varepsilon - K_0 c_n \omega_n^{\frac{4}{n}} (2^{-\frac{4}{n}} + \varepsilon)) \|u_{a,q}\|_{H_1^2}^2 \geq 1 - \left(\frac{2^{-\frac{4}{n}} + \varepsilon}{a_0} \right) > 0$$

for ε small. Now, since $H_2^2 \hookrightarrow H_1^2 \hookrightarrow L^2$, the first embedding being compact and the second one being continuous, and using an interpolation type inequality of Lions [36], we get: for all $\delta > 0 \exists C_\delta > 0$ such that for all $u \in H_2^2(\mathbb{S}^n)$

$$\|u_{a,q}\|_{H_1^2}^2 \leq \delta \|u_{a,q}\|_{H_2^2}^2 + C_\delta \|u_{a,q}\|_2^2.$$

It follows that for δ small enough

$$\begin{aligned} C_\delta \|u_{a,q}\|_2^2 &\geq C(n) - \delta \|u_{a,q}\|_{H_2^2}^2 \geq C(n) - \frac{\delta}{a} K_0^{-1} \omega_n^{-\frac{4}{n}} \\ &\geq C(n) - \delta 2^{\frac{4}{n}} K_0^{-1} \omega_n^{-\frac{4}{n}} \geq C'(n) > 0, \end{aligned}$$

and we get the result. This concludes the proof of Lemma 4.3. □

Lemma 4.4. For all $a \in [a_0; 1]$ and all $q \in (2; 2^\#)$, $\Lambda_{a,q} \leq C(n)$, where $C(n)$ is a constant depending only on n .

Proof. Multiplying (4.3) by $u_{a,q} \xi_{a,q}$ and integrating over \mathbb{S}^n , we get

$$a \left(\int_{\mathbb{S}^n} \{\Delta_h^2 u_{a,q} + c_n \Delta_h u_{a,q}\} u_{a,q} \xi_{a,q} \right) + d_n \int_{\mathbb{S}^n} u_{a,q}^2 \xi_{a,q} = \Lambda_{a,q} \int_{\mathbb{S}^n} |u_{a,q}|^q \xi_{a,q}^2.$$

Integrating by parts and taking into account $\Delta_h \xi_{a,q} = n \xi_{a,q}$, we have

$$\begin{aligned} \int_{\mathbb{S}^n} (\Delta_h^2 u_{a,q}) u_{a,q} \xi_{a,q} &= \int_{\mathbb{S}^n} (\Delta_h u_{a,q})^2 \xi_{a,q} + n \int_{\mathbb{S}^n} (\Delta_h u_{a,q}) u_{a,q} \xi_{a,q} \\ &\quad - 2 \int_{\mathbb{S}^n} (\Delta_h u_{a,q}) \nabla u_{a,q} \nabla \xi_{a,q}; \end{aligned}$$

moreover

$$\int_{\mathbb{S}^n} \Delta_h u_{a,q} \nabla u_{a,q} \nabla \xi_{a,q} = \left(\frac{n}{2} - 1\right) \int_{\mathbb{S}^n} |\nabla u_{a,q}|^2 \xi_{a,q}.$$

It follows that

$$\begin{aligned} \Lambda_{a,q} \int_{\mathbb{S}^n} |u_{a,q}|^q \xi_{a,q}^2 &= a \left(\int_{\mathbb{S}^n} (\Delta_h u_{a,q})^2 \xi_{a,q} + (c_n + n) \int_{\mathbb{S}^n} (\Delta_h u_{a,q}) u_{a,q} \xi_{a,q} \right) \\ &\quad - (n - 2) a \int_{\mathbb{S}^n} |\nabla u_{a,q}|^2 \xi_{a,q} + d_n \int_{\mathbb{S}^n} u_{a,q}^2 \xi_{a,q}. \end{aligned}$$

Independently

$$\int_{\mathbb{S}^n} (\Delta_h u_{a,q}) u_{a,q} \xi_{a,q} = \int_{\mathbb{S}^n} |\nabla u_{a,q}|^2 \xi_{a,q} + \frac{n}{2} \int_{\mathbb{S}^n} (u_{a,q})^2 \xi_{a,q}$$

and hence,

$$\begin{aligned} \Lambda_{a,q} \int_{\mathbb{S}^n} |u_{a,q}|^q \xi_{a,q}^2 &= a \left(\int_{\mathbb{S}^n} (\Delta_h u_{a,q})^2 \xi_{a,q} + (c_n + 2) \int_{\mathbb{S}^n} (|\nabla u_{a,q}|^2 \xi_{a,q}) \right) \\ &\quad + \left(\frac{n^2 a}{2} + c_n a \frac{n}{2} + d_n \right) \int_{\mathbb{S}^n} u_{a,q}^2 \xi_{a,q}. \end{aligned}$$

It follows that

$$\Lambda_{a,q} \int_{\mathbb{S}^n} |u_{a,q}|^q \xi_{a,q}^2 \leq C(n) E_a[u_{a,q}] \leq C(n).$$

Using Lemma 4.3 we have $\|u_{a,q}\|_2^2 \geq C$, hence by the Hölder inequality there exists a constant $C' > 0$ such that for all a and q , $\int_{\mathbb{S}^n} |u_{a,q}|^q \xi_{a,q}^2 \geq C'$. It follows that $\Lambda_{a,q} \leq C(n)$. This concludes the proof of Lemma 4.4. \square

We now consider increasing sequences (a_k) and (q_k) such that

$$a_k < 1, \quad a_k \rightarrow 1; \quad 2 < q_k < 2^\#, \quad q_k \rightarrow 2^\#.$$

By Lemma 4.3 u_k , the solution of (4.3) associated to a_k and q_k , satisfies

$$a_k (\Delta_h^2 u_k + c_n \Delta_h u_k) + d_n u_k = \mathcal{M}_k |u_k|^{q_k-2} u_k + \Lambda_k \xi_k |u_k|^{q_k-2} u_k.$$

Since (\mathcal{M}_k) is a bounded sequence (see Lemma 4.1), we can extract a subsequence such that $\lim_{k \rightarrow +\infty} \mathcal{M}_k$ exists and clearly $\lim_{k \rightarrow +\infty} \mathcal{M}_k = K_0^{-1} \omega_n^{-\frac{4}{n}}$ (see the proof of Lemma 4.1). Clearly (u_k) is bounded in $H_2^2(\mathbb{S}^n)$; hence, some subsequence of (u_k)

converges weakly in $H_2^2(\mathbb{S}^n)$ and almost everywhere on \mathbb{S}^n to a function $u \in C^\infty(\mathbb{S}^n)$ which satisfies

$$\Delta_h^2 u + c_n \Delta_h u + d_n u = K_0^{-1} \omega_n^{-\frac{4}{n}} |u|^{\frac{8}{n-4}} u + \Lambda \xi_\infty |u|^{\frac{8}{n-4}} u. \tag{4.4}$$

Here $\Lambda \geq 0$ is the limit of (Λ_k) , and $\xi_\infty \in \mathbb{H}$, $\|\xi_\infty\|_\infty = 1$, is the limit of (ξ_k) (recall that \mathbb{H} is finite dimensional). We now show some properties of the sequence $(u_k)_k$.

Lemma 4.5. $u_k \rightarrow 1$ weakly in $H_2^2(\mathbb{S}^n)$ and strongly in $L^{2^\#}(\mathbb{S}^n)$.

Proof. First we remark that according to Lemma 4.3, u is not identically 0. Using Theorem 1.1, we can write

$$\Lambda \int_{\mathbb{S}^n} |\nabla \xi_\infty|^2 |u|^{2^\#} = 0,$$

so it follows that $\Lambda = 0$. We multiply (4.4) by u and we integrate to obtain

$$\int_{\mathbb{S}^n} (\Delta u)^2 + c_n \int_{\mathbb{S}^n} |\nabla u|^2 + d_n \int_{\mathbb{S}^n} u^2 = K_0^{-1} \omega_n^{-\frac{4}{n}} \int_{\mathbb{S}^n} |u|^{2^\#}.$$

According to Theorem 2.1, this gives

$$K_0^{-1} \omega_n^{-\frac{4}{n}} \int_{\mathbb{S}^n} |u|^{2^\#} \geq K_0^{-1} \omega_n^{-\frac{4}{n}} \left(\int_{\mathbb{S}^n} |u|^{2^\#} \right)^{\frac{2}{2^\#}},$$

so it follows that $\int_{\mathbb{S}^n} |u|^{2^\#} \geq 1$. Independently, since (u_k) is bounded in $H_2^2(\mathbb{S}^n)$

$$\begin{aligned} K_0^{-1} \omega_n^{-\frac{4}{n}} \left(\int_{\mathbb{S}^n} |u_k|^{2^\#} \right)^{\frac{2}{2^\#}} &\leq E_1[u_k] \leq \mathcal{M}_k + (1 - a_k) \\ &\times \left(\int_{\mathbb{S}^n} (\Delta u_k)^2 + c_n \int_{\mathbb{S}^n} |\nabla u_k|^2 \right) \rightarrow K_0^{-1} \omega_n^{-\frac{4}{n}}. \end{aligned}$$

Then,

$$1 \leq \int_{\mathbb{S}^n} |u|^{2^\#} \leq \liminf_{k \rightarrow +\infty} \int_{\mathbb{S}^n} |u_k|^{2^\#} \leq \limsup_{k \rightarrow +\infty} \int_{\mathbb{S}^n} |u_k|^{2^\#} \leq 1.$$

Hence, $\int_{\mathbb{S}^n} |u|^{2^\#} = 1$ and $\lim_{k \rightarrow +\infty} \int_{\mathbb{S}^n} |u_k|^{2^\#} = 1$. Then, applying a result of Brézis-Lieb [12] (see also Kavian [30]), we have $\lim_{k \rightarrow +\infty} \|u - u_k\|_{2^\#}^{2^\#} = \lim_{k \rightarrow +\infty} \|u_k\|_{2^\#}^{2^\#} - \|u\|_{2^\#}^{2^\#} = 0$. This means that $u_k \rightarrow u$ in $L^{2^\#}(\mathbb{S}^n)$. It remains to prove that $u \equiv 1$. First we show that $\int_{\mathbb{S}^n} |u|^{2^\#} \xi = 0$ for all $\xi \in \mathbb{H}$. We have $\int_{\mathbb{S}^n} |u_k|^{q_k-1} |u| \xi \rightarrow \int_{\mathbb{S}^n} |u|^{2^\#} \xi$ as $k \rightarrow +\infty$ since $|u_k|^{q_k-1} \rightarrow |u|^{2^\#-1}$ weakly in $L^{\frac{2^\#}{2^\#-1}}(\mathbb{S}^n)$. On the other hand, since $u_k \in \mathcal{S}_{q_k}$

$$\left| \int_{\mathbb{S}^n} |u_k|^{q_k-1} |u| \xi \right| = \left| \int_{\mathbb{S}^n} |u_k|^{q_k-1} (|u| - |u_k|) \xi \right| \leq \|u_k\|_{2^\#}^{q_k-1} \|u - u_k\|_r \leq C \|u - u_k\|_r,$$

where $r = \frac{2^\#}{2^\# - q_k + 1} \leq 2^\#$. Thus

$$\int_{\mathbb{S}^n} |u|^{2^\#} \xi = \lim_{k \rightarrow +\infty} \int_{\mathbb{S}^n} |u_k|^{q_k-1} |u| \xi = 0.$$

This proves that $u \in \mathcal{S}_{2^\#}^0$. Hence u is a solution of (4.4) with $u \in \mathcal{S}_{2^\#}^0$. Since such a solution is unique, see Edmunds–Fortunato–Janelli [25] and Lions [37], we have $u \equiv 1$ and this concludes the proof. \square

Lemma 4.6. (u_k) is bounded in $L^\infty(\mathbb{S}^n)$.

Proof. For $k_0 \in \mathbb{N}$, let

$$\Omega_{k_0} = \{x \in \mathbb{S}^n \mid |u_k(x)| < 2 \text{ for all } k \geq k_0\}.$$

Since $u_k \rightarrow 1$ in $L^{2^\#}(\mathbb{S}^n)$ it is clear that $|\mathbb{S}^n \setminus \Omega_{k_0}| = \text{measure}(\mathbb{S}^n \setminus \Omega_{k_0}) \rightarrow 0$ as $k_0 \rightarrow +\infty$. We can write

$$\left(\Delta_h + \frac{c_n}{2}\right)^2 u_k = b_k |u_k|^{q_k-2} + w_k,$$

where

$$b_k = \left(\left(\frac{\mathcal{M}_k + \Lambda_k \xi_k}{a_k} \right) + \left(\frac{c_n^2}{4} - \frac{d_n}{a_k} \right) \frac{1}{|u_k|^{q_k-2}} \right) \mathbf{1}_{\mathbb{S}^n \setminus \Omega_{k_0}},$$

$$w_k = \left(\left(\frac{\mathcal{M}_k + \Lambda_k \xi_k}{a_k} \right) |u_k|^{q_k-2} + \left(\frac{c_n^2}{4} - \frac{d_n}{a_k} \right) \right) u_k \cdot \mathbf{1}_{\Omega_{k_0}}.$$

According to Lemmas 4.1 and 4.4, there exists a constant $C > 0$ depending only on n such that $\forall k \in \mathbb{N}$

$$\|b_k\|_\infty \leq C \quad \text{and} \quad \|w_k\|_\infty \leq C, \tag{4.5}$$

and

$$\|b_k |u_k|^{q_k-2}\|_{\frac{q}{4}} \leq C (\|u_k - 1\|_{2^\#} + |\mathbb{S}^n \setminus \Omega_{k_0}|^{\frac{4}{n}} + |\mathbb{S}^n \setminus \Omega_{k_0}|^{\frac{n-4}{2n}}). \tag{4.6}$$

Using (4.5), the fact that $u_k \rightarrow 1$ in $L^{2^\#}(\mathbb{S}^n)$ as $k \rightarrow +\infty$ and the fact that $|\mathbb{S}^n \setminus \Omega_{k_0}| \rightarrow 0$ as $k_0 \rightarrow +\infty$, it follows that

$$\|b_k |u_k|^{q_k-2}\|_{\frac{q}{4}} \rightarrow 0 \text{ as } k \rightarrow +\infty.$$

Then, according to Proposition 3.1, Remark 3.1 and (4.5), there exists a constant C depending only on n such that for all k large

$$\|u_k\|_\infty \leq C.$$

This concludes the proof of the lemma. \square

Lemma 4.7. $\|u_k - 1\|_\infty = o(1)$ as $k \rightarrow +\infty$.

Proof. Fix $\eta > 0$ and for $k_0 \in \mathbb{N}$, let

$$\Omega_{k_0} = \{x \in \mathbb{S}^n \mid |u_k(x) - 1| < \eta \text{ for all } k \geq k_0\}.$$

As in the proof of the previous lemma, $|\mathbb{S}^n \setminus \Omega_{k_0}| \rightarrow 0$ as $k_0 \rightarrow +\infty$. We have

$$\left(\Delta_h + \frac{c_n}{2}\right)^2 (u_k - 1) = b_k (u_k - 1) + w_k,$$

where

$$\begin{aligned}
 b_k &= \left(\left(\frac{\mathcal{M}_k + \Lambda_k \xi_k}{a_k} \right) |u_k|^{q_k-2} + \left(\frac{c_n^2}{4} - \frac{d_n}{a_k} \right) \right) 1_{\mathbb{S}^n \setminus \Omega_{k_0}}, \\
 w_k &= \left(\left(\frac{\mathcal{M}_k + \Lambda_k \xi_k}{a_k} \right) |u_k|^{q_k-2} + \left(\frac{c_n^2}{4} - \frac{d_n}{a_k} \right) \right) (u_k - 1) 1_{\Omega_{k_0}} \\
 &\quad + \left(\left(\frac{\mathcal{M}_k + \Lambda_k \xi_k}{a_k} \right) |u_k|^{q_k-2} - \frac{d_n}{a_k} \right).
 \end{aligned}$$

As before there exists $C > 0$ depending only on n such that for all $k \in \mathbb{N}$

$$\begin{aligned}
 \|b_k\|_\infty &\leq C \quad \text{and} \quad \|w_k\|_\infty \leq C, \\
 \|b_k\|_{\frac{n}{4}} &\leq C |\mathbb{S}^n \setminus \Omega_{k_0}|^{\frac{4}{n}}.
 \end{aligned}$$

Consider δ^∞ given by Remark 3.1, and $s > \frac{n}{4}$; since $|\mathbb{S}^n \setminus \Omega_{k_0}| \rightarrow 0$ as $k_0 \rightarrow +\infty$, using Remark 3.2, there exists C , depending only on n , and there exists k_1 such that for all $k \geq k_1$

$$\|u_k - 1\|_\infty \leq C \|w_k\|_s.$$

Clearly, using Lemma 4.6 and the fact that $u_k \rightarrow 1$ almost everywhere, there exists $k_2 \in \mathbb{N}$ such that for all $k \geq k_2$

$$\|w_k\|_s \leq C\eta,$$

where C is independent of k and η . Then we have the lemma. □

We can now prove Theorem 4.1.

Proof of Theorem 4.1. We write $u_k = 1 + \alpha_k + h_k + \Psi_k$ where $\alpha_k \in \mathbb{R}$, $h_k \in \mathbb{H}$ and where Ψ_k is orthogonal to $\mathbb{R} \oplus \mathbb{H}$. Clearly,

$$E_{a_k}[u_k] \geq d_n + a_k \left(\int_{\mathbb{S}^n} (\Delta_h \Psi_k)^2 + c_n \int_{\mathbb{S}^n} |\nabla \Psi_k|^2 \right) + 2\alpha_k d_n + d_n \int_{\mathbb{S}^n} \Psi_k^2.$$

We want to estimate α_k . For this, we know that (using Lemma 4.7)

$$\begin{aligned}
 0 &= \int_{\mathbb{S}^n} (1 + \alpha_k + h_k + \Psi_k)^{q_k} h_k = q_k \int_{\mathbb{S}^n} h_k^2 + o(1)\alpha_k \left(\int_{\mathbb{S}^n} h_k^2 \right)^{\frac{1}{2}} \\
 &\quad + o(1) \int_{\mathbb{S}^n} h_k^2 + o(1) \left(\int_{\mathbb{S}^n} h_k^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{S}^n} \Psi_k^2 \right)^{\frac{1}{2}}
 \end{aligned}$$

and this implies, since $\alpha_k \leq 0$ (recall that $u_k \in \mathcal{S}_{q_k}^0$)

$$\int_{\mathbb{S}^n} h_k^2 = o(1) \left(\alpha_k^2 + \int_{\mathbb{S}^n} \Psi_k^2 \right).$$

Secondly, we have

$$1 = \int_{\mathbb{S}^n} (1 + \alpha_k + h_k + \Psi_k)^{q_k} = 1 + q_k \alpha_k + \frac{q_k(q_k - 1)}{2} \alpha_k^2 + \frac{q_k(q_k - 1)}{2} \int_{\mathbb{S}^n} \Psi_k^2 + o(1) \alpha_k^2 + o(1) \int_{\mathbb{S}^n} \Psi_k^2,$$

and this gives (using Lemma 4.5 to see that $\alpha_k \rightarrow 0$)

$$\alpha_k = -\frac{q_k - 1}{2} \int_{\mathbb{S}^n} \Psi_k^2 + o(1) \int_{\mathbb{S}^n} \Psi_k^2.$$

Now, using the Courant–Fischer characterization of the eigenvalues and Theorem 2.2

$$\begin{aligned} E_{a_k}[u_k] &\geq d_n + a_k \left(\int_{\mathbb{S}^n} (\Delta_h \Psi_k)^2 + c_n \int_{\mathbb{S}^n} |\nabla \Psi_k|^2 \right) - (d_n(q_k - 2) + o(1)) \int_{\mathbb{S}^n} \Psi_k^2 \\ &\geq d_n + \left(4(n + 1)^2 + 2c_n(n + 1) - \frac{8d_n}{n - 4} \right. \\ &\quad \left. + (a_k - 1)(n + 1)(4(n + 1) + 2c_n + o(1)) \right) \int_{\mathbb{S}^n} \Psi_k^2 \\ &\geq d_n + \frac{1}{2}(n^3 + 6n^2 + 8n + \beta_k) \int_{\mathbb{S}^n} \Psi_k^2 \end{aligned}$$

where $\beta_k \rightarrow 0$ when k goes to $+\infty$. This implies that for k large

$$E_{a_k}[u_k] \geq d_n$$

and this proves that $\exists a_n < 1$ and $\exists q_0 < 2^\#$ so that

$$\inf_{\{u \in \mathcal{S}_q, u \neq 0\}} \frac{a_n \left(\int_{\mathbb{S}^n} (\Delta_h u)^2 + c_n \int_{\mathbb{S}^n} |\nabla u|^2 \right) + d_n \int_{\mathbb{S}^n} u^2}{\left(\int_{\mathbb{S}^n} |u|^q \right)^{\frac{2}{q}}} \geq K_0^{-1} \omega_n^{-\frac{4}{n}}$$

for all $q_0 \leq q < 2^\#$. It remains to prove that this is also true for $q = 2^\#$. Denote

$$\mathcal{M}_{2^\#} = \inf_{\{u \in \mathcal{S}_{2^\#}, u \neq 0\}} \frac{a_n \left(\int_{\mathbb{S}^n} (\Delta_h u)^2 + c_n \int_{\mathbb{S}^n} |\nabla u|^2 \right) + d_n \int_{\mathbb{S}^n} u^2}{\left(\int_{\mathbb{S}^n} |u|^{2^\#} \right)^{\frac{2}{2^\#}}}$$

and consider $\varepsilon > 0$; then there exists $u \in \mathcal{S}_{2^\#}$ such that $J_{a_n, 2^\#}[u] \leq \mathcal{M}_{2^\#} + \varepsilon$. Since

$$\lim_{q \rightarrow 2^\#} \left(\int_{\mathbb{S}^n} |u|^q \right)^{\frac{2}{q}} = \left(\int_{\mathbb{S}^n} |u|^{2^\#} \right)^{\frac{2}{2^\#}},$$

it is clear that

$$K_0^{-1} \omega_n^{-\frac{4}{n}} \leq \limsup_{q \rightarrow 2^\#} J_{a_n, q}[u] = J_{a_n, 2^\#}[u] \leq \mathcal{M}_{2^\#} + \varepsilon,$$

and then $K_0^{-1} \omega_n^{-\frac{4}{n}} \leq \mathcal{M}_{2^\#}$. This concludes the proof of the theorem. □

5. Basic Estimates

Given $P \in \mathbb{S}^n$ and $t \in [1; +\infty)$ let $\varphi_{P,t}$ be the conformal transformation on \mathbb{S}^n defined in Sec. 1. For brevity we will denote $\varphi_{P,t}$ as φ_P and $f_{P,t} = f \circ \varphi_{P,t}$ as f_P . We set $T_\varphi u = (u \circ \varphi) |\det d\varphi|^{\frac{n-4}{2n}}$. Clearly, $\varphi_{P,t}$ acts on the set of conformal metrics $g = u^{\frac{4}{n-4}} h$ by

$$\varphi_{P,t}^* g = (T_{\varphi_{P,t}} u)^{\frac{4}{n-4}} h.$$

Consequently, equation (E) is transformed by $\varphi_{P,t}$ into

$$\Delta_h^2(T_{\varphi_{P,t}} u) + c_n \Delta_h(T_{\varphi_{P,t}} u) + d_n(T_{\varphi_{P,t}} u) = \frac{n-4}{2} (f \circ \varphi_{P,t})(T_{\varphi_{P,t}} u)^{\frac{n+4}{n-4}}.$$

Roughly, by means of the improved Sobolev inequality in Theorem 4.1, for f close to a constant, it is possible to find minima of $\frac{E_1[u]}{(f_{\mathbb{S}^n} f_P |u|^{2^\#})^{\frac{2}{2^\#}}}$ constrained on $\mathcal{S}_{2^\#}$, leading to solutions of

$$(E_p) \quad P_h^n u = \left(\frac{n-4}{2} f_P - \Lambda_p \xi_p \right) |u|^{2^\#-2} u,$$

where Λ_p is a Lagrange multiplier. Using this finite dimensional reduction, solving problem (E) amounts to finding p_0 for which $\Lambda_{p_0} = 0$. The same kind of strategy has been used by Chang–Yang [19].

Let $t > 1$ and $P \in \mathbb{S}^n$. Considering f_P and $2 < q \leq 2^\#$, we define

$$\mathcal{M}_q = \inf_{\{u \in \mathcal{S}_q | f_{\mathbb{S}^n} f_P |u|^q = 1\}} E_1[u];$$

it is now classical that for $q < 2^\#$ there exist $u_q \in C^\infty(\mathbb{S}^n) \cap \mathcal{S}_q$, $\Lambda_q \in \mathbb{R}^+$ and $\xi_q \in \mathbb{H}$, $\|\xi_q\|_\infty = 1$, which satisfy the equation

$$\Delta_h^2 u_q + c_n \Delta_h u_q + d_n u_q = (\mathcal{M}_q f_P - \Lambda_q \xi_q) |u_q|^{q-2} u_q.$$

Clearly (u_q) is bounded in $H_2^2(\mathbb{S}^n)$ and it follows that (u_q) converges weakly in $H_2^2(\mathbb{S}^n)$ to some $\hat{u}_p \in H_2^2(\mathbb{S}^n)$ satisfying

$$\Delta_h^2 \hat{u}_p + c_n \Delta_h \hat{u}_p + d_n \hat{u}_p = (\mathcal{M}_p f_P - \hat{\Lambda}_p \hat{\xi}_p) |\hat{u}_p|^{2^\#-2} \hat{u}_p,$$

where $\hat{\Lambda}_p \in \mathbb{R}^+$ and $\hat{\xi}_p \in \mathbb{H}$, $\|\hat{\xi}_p\|_\infty = 1$. We claim that \hat{u}_p is not identically 0, and that $\hat{u}_p \in \mathcal{S}_{2^\#}$. The proof of this fact relies on the following claim: if $\varepsilon_f = \|f - \frac{n(n^2-4)}{8}\|_\infty$ is small enough (this smallness being independent of t and P),

then $(u_q)_q$ is bounded in $L^\infty(\mathbb{S}^n)$. To show this, one can prove that

$$\left\{ \begin{array}{l} (1) \int_{\mathbb{S}^n} (\Delta_h u_q)^2 + c_n \int_{\mathbb{S}^n} |\nabla u_q|^2 = O(\|f_p - f(P)\|_\infty), \\ \int_{\mathbb{S}^n} u_q^2 - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{4}} = O(\|f_p - f(P)\|_\infty); \\ (2) \left\| u_q - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{8}} \right\|_{H_2^2}^2 = O(\|f_p - f(P)\|_\infty); \\ (3) (\Lambda_q) \text{ is bounded} \end{array} \right.$$

reasoning as in Lemmas 5.2, 5.3 and 5.4.

Up to a subsequence, $(u_q)_q$ converges to some \hat{u}_p almost everywhere and $(u_q)_q$ is bounded in $L^\infty(\mathbb{S}^n)$. It follows easily that (u_q) converges to \hat{u}_p in $L^{2^\#}(\mathbb{S}^n)$, where $\hat{u}_p \neq 0$, $\hat{u}_p \in \mathcal{S}_{2^\#}$ and \hat{u}_p realizes $\mathcal{M}_{2^\#}$.

Regarding equation (E_p) , we provide uniform estimates on the solutions u_p , uniform with respect to p . These estimates will be needed later to study the reduced finite dimensional problem.

Lemma 5.1. *Denote*

$$\mathcal{M}_p = \inf_{\{u \in \mathcal{S}_{2^\#} | \int_{\mathbb{S}^n} f_p |u|^{2^\#} = 1\}} E_1[u].$$

Then for $\varepsilon > 0$ small enough, if f satisfies

$$\varepsilon_f = \left\| f - \frac{n(n^2 - 4)}{8} \right\|_\infty \leq \varepsilon,$$

there exist $u_p \in C^\infty(\mathbb{S}^n) \cap \mathcal{S}_{2^\#}$ with $\int_{\mathbb{S}^n} f_p u_p \geq 0$, $\Lambda_p \in \mathbb{R}^+$ and $\xi_p \in \mathbb{H}$, $\|\xi_p\|_\infty = 1$, which satisfy the equation

$$\Delta_h^2 u_p + c_n \Delta_h u_p + d_n u_p = \left(\frac{n-4}{2} f_p - \Lambda_p \xi_p \right) |u_p|^{2^\#-2} u_p. \tag{5.1}$$

Furthermore, $(u_p)_p$ is bounded in $H_2^2(\mathbb{S}^n)$ and we have

$$\frac{n-4}{2} \left(\int_{\mathbb{S}^n} f_p |u_p|^{2^\#} \right)^{\frac{4}{n}} = \mathcal{M}_p \leq d_n \left(\int_{\mathbb{S}^n} f_p \right)^{-\frac{2}{2^\#}}. \tag{5.2}$$

Proof. As we have just seen, there exist $\hat{u}_p \in C^\infty(\mathbb{S}^n) \cap \mathcal{S}_{2^\#}$, $\hat{\Lambda}_p \in \mathbb{R}^+$ and $\hat{\xi}_p \in \mathbb{H}$, $\|\hat{\xi}_p\|_\infty = 1$, which satisfy the equation

$$\Delta_h^2 \hat{u}_p + c_n \Delta_h \hat{u}_p + d_n \hat{u}_p = (\mathcal{M}_p f_p - \hat{\Lambda}_p \hat{\xi}_p) |\hat{u}_p|^{2^\#-2} \hat{u}_p.$$

Hence, by a simple renormalization there exist $u_p \in C^\infty(\mathbb{S}^n) \cap \mathcal{S}_{2^\#}$ with $\int_{\mathbb{S}^n} f_p u_p \geq 0$, $\Lambda_p \in \mathbb{R}^+$ and $\xi_p \in \mathbb{H}$, $\|\xi_p\|_\infty = 1$, which satisfy equation (5.1). Clearly, with this renormalization it is $\frac{n-4}{2} (\int_{\mathbb{S}^n} f_p |u_p|^{2^\#})^{\frac{4}{n}} = \mathcal{M}_p$. Moreover, we have

$$\mathcal{M}_p \leq E_1 \left[\left(\int_{\mathbb{S}^n} f_p \right)^{-\frac{1}{2^\#}} \right] = d_n \left(\int_{\mathbb{S}^n} f_p \right)^{-\frac{2}{2^\#}}. \tag{5.3}$$

Now, using the estimate $(\int_{\mathbb{S}^n} f_p |u_p|^{2^\#})^{\frac{4}{n}} \leq \frac{2}{n-4} d_n (\int_{\mathbb{S}^n} f_p)^{-\frac{2}{2^\#}}$, it is clear that for ε small enough, (u_p) is bounded in $L^{2^\#}(\mathbb{S}^n)$. Since

$$\int_{\mathbb{S}^n} (\Delta_h u_p)^2 + c_n \int_{\mathbb{S}^n} |\nabla u_p|^2 + d_n \int_{\mathbb{S}^n} u_p^2 = \frac{n-4}{2} \int_{\mathbb{S}^n} f_p |u_p|^{2^\#},$$

$(u_p)_p$ is also bounded in $H_2^2(\mathbb{S}^n)$. This concludes the proof of Lemma 5.1. □

Lemma 5.2. *The following estimates hold, uniformly in p*

$$\begin{cases} \int_{\mathbb{S}^n} (\Delta_h u_p)^2 + c_n \int_{\mathbb{S}^n} |\nabla u_p|^2 = O(\|f_p - f(P)\|_\infty); \\ \int_{\mathbb{S}^n} u_p^2 - \left(\frac{n(n^2-4)}{8f(P)}\right)^{\frac{n-4}{4}} = O(\|f_p - f(P)\|_\infty). \end{cases}$$

Proof. We have by (5.2)

$$\begin{aligned} E_1[u_p] &= \frac{n-4}{2} \int_{\mathbb{S}^n} f_p |u_p|^{2^\#} = \left(\frac{2}{n-4}\right)^{\frac{n-1}{4}} \mathcal{M}_p^{\frac{n}{22^\#}} \mathcal{M}_p \\ &\leq \left(\frac{2}{n-4}\right)^{\frac{n-1}{4}} \mathcal{M}_p^{\frac{n}{22^\#}} d_n \left(\int_{\mathbb{S}^n} f_p\right)^{-\frac{2}{2^\#}}. \end{aligned}$$

By Theorem 4.1 applied with $q = 2^\#$, $d_n (\int_{\mathbb{S}^n} |u_p|^{2^\#})^{\frac{2}{2^\#}} \leq a_n (\int_{\mathbb{S}^n} (\Delta_h u_p)^2 + c_n \int_{\mathbb{S}^n} |\nabla u_p|^2) + d_n \int_{\mathbb{S}^n} u_p^2$. It follows that

$$\begin{aligned} &(1 - a_n) \left(\int_{\mathbb{S}^n} (\Delta_h u_p)^2 + c_n \int_{\mathbb{S}^n} |\nabla u_p|^2 \right) \\ &\leq \left(\frac{2}{n-4}\right)^{\frac{n-1}{4}} \mathcal{M}_p^{\frac{n}{22^\#}} d_n \left(\int_{\mathbb{S}^n} f_p\right)^{-\frac{2}{2^\#}} - d_n \left(f(P)^{-1} \int_{\mathbb{S}^n} f(P) |u_p|^{2^\#} \right)^{\frac{2}{2^\#}} \\ &\leq \left(\frac{2}{n-4}\right)^{\frac{n-1}{4}} \mathcal{M}_p^{\frac{n}{22^\#}} \left(d_n \left(\int_{\mathbb{S}^n} f_p\right)^{-\frac{2}{2^\#}} - d_n (f(P))^{-\frac{2}{2^\#}} \right) \\ &\quad + \left(\frac{2}{n-4}\right)^{\frac{n-1}{4}} \mathcal{M}_p^{\frac{n}{22^\#}} d_n (f(P))^{-\frac{2}{2^\#}} - d_n \left(f(P)^{-1} \int_{\mathbb{S}^n} f(P) |u_p|^{2^\#} \right)^{\frac{2}{2^\#}} \\ &\leq \left(\frac{2}{n-4}\right)^{\frac{n-1}{4}} \mathcal{M}_p^{\frac{n}{22^\#}} d_n \left(\left(\int_{\mathbb{S}^n} f_p\right)^{-\frac{2}{2^\#}} - (f(P))^{-\frac{2}{2^\#}} \right) \\ &\quad + \frac{d_n}{(f(P))^{\frac{2}{2^\#}}} \left(\left(\int_{\mathbb{S}^n} f_p |u_p|^{2^\#}\right)^{\frac{2}{2^\#}} - \left(\int_{\mathbb{S}^n} f(P) |u_p|^{2^\#}\right)^{\frac{2}{2^\#}} \right). \end{aligned}$$

With some computations using Lemma 5.1, we get $\int_{\mathbb{S}^n} (\Delta_h u_p)^2 + c_n \int_{\mathbb{S}^n} |\nabla u_p|^2 = O(\|f_p - f(P)\|_\infty)$. Now

$$\begin{aligned} E_1[u_p] - d_n \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{4}} &= \left(\frac{2}{n-4} \right)^{\frac{n}{4}-1} \mathcal{M}_p^{\frac{n}{4}} - d_n \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{4}} \\ &= \left(\frac{2}{n-4} \right)^{\frac{n}{4}-1} \left(\mathcal{M}_p^{\frac{n}{4}} - d_n \left(\frac{d_n}{f(P)} \right)^{\frac{n-4}{4}} \right) \\ &\leq \left(\frac{2}{n-4} \right)^{\frac{n}{4}-1} d_n^{\frac{n}{4}} \left(\left(\int_{\mathbb{S}^n} f_p \right)^{-\frac{n-4}{4}} - (f(P))^{-\frac{n-4}{4}} \right) \\ &= O(\|f_p - f(P)\|_\infty). \end{aligned}$$

On the other hand, by Theorem 2.1, (1.3), (2.1) and our renormalization

$$\begin{aligned} \left(\frac{2}{n-4} \right)^{\frac{n}{4}-1} \mathcal{M}_p^{\frac{n}{4}} = E_1[u_p] &\geq d_n \left(\int_{\mathbb{S}^n} |u_p|^{2^\#} \right)^{\frac{2}{2^\#}} \\ &= d_n (f(P))^{-\frac{2}{2^\#}} \left(\int_{\mathbb{S}^n} f(P) |u_p|^{2^\#} \right)^{\frac{2}{2^\#}} \\ &= d_n (f(P))^{-\frac{2}{2^\#}} \left(\int_{\mathbb{S}^n} (f(P) - f_p) |u_p|^{2^\#} + \int_{\mathbb{S}^n} f_p |u_p|^{2^\#} \right)^{\frac{2}{2^\#}} \\ &= d_n (f(P))^{-\frac{2}{2^\#}} \left(O(\|f_p - f(P)\|_\infty) + \left(\frac{2}{n-4} \right)^{\frac{n}{4}} \mathcal{M}_p^{\frac{n}{4}} \right)^{\frac{2}{2^\#}}. \end{aligned}$$

It follows that $\mathcal{M}_p \geq d_n (f(P))^{-\frac{2}{2^\#}} + O(\|f_p - f(P)\|_\infty)$, and thus we obtain

$$\int_{\mathbb{S}^n} u_p^2 - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{4}} = O(\|f_p - f(P)\|_\infty).$$

This concludes the proof of Lemma 5.2. □

Lemma 5.3. *There holds*

$$\left\| u_p - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{8}} \right\|_{H_2^2}^2 = O(\|f_p - f(P)\|_\infty).$$

Proof. According to Lemma 5.2, it remains to prove that $\int_{\mathbb{S}^n} |u_p - (\frac{n(n^2-4)}{8f(P)})^{\frac{n-4}{8}}|^2 = O(\|f_p - f(P)\|_\infty)$. We already know from Lemma 5.2 that $\int_{\mathbb{S}^n} u_p^2 - (\frac{n(n^2-4)}{8f(P)})^{\frac{n-4}{4}} = O(\|f_p - f(P)\|_\infty)$; using the Sobolev–Poincaré inequality, we have

$$\left| \|u_p\|_2 - \left\| \int_{\mathbb{S}^n} u_p \right\|_2 \right| \leq \left\| u_p - \int_{\mathbb{S}^n} u_p \right\|_2 \leq C \left(\int_{\mathbb{S}^n} |\nabla u_p|^2 \right)^{\frac{1}{2}},$$

and since $\int_{\mathbb{S}^n} u_p \geq 0$

$$-C \left(\int_{\mathbb{S}^n} |\nabla u_p|^2 \right)^{\frac{1}{2}} + \left(\int_{\mathbb{S}^n} u_p^2 \right)^{\frac{1}{2}} \leq \int_{\mathbb{S}^n} u_p \leq C \left(\int_{\mathbb{S}^n} |\nabla u_p|^2 \right)^{\frac{1}{2}} + \left(\int_{\mathbb{S}^n} u_p^2 \right)^{\frac{1}{2}} .$$

Hence,

$$-C \|f_p - f(P)\|_{\infty}^{\frac{1}{2}} \leq \int_{\mathbb{S}^n} u_p - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{8}} \leq C \|f_p - f(P)\|_{\infty}^{\frac{1}{2}} .$$

Using once again the Sobolev–Poincaré inequality, it is

$$\begin{aligned} & \left(\int_{\mathbb{S}^n} \left| u_p - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{8}} - \left(\int_{\mathbb{S}^n} u_p - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{8}} \right)^2 \right|^2 \right)^{\frac{1}{2}} \\ & \leq C \left(\int_{\mathbb{S}^n} |\nabla u_p|^2 \right)^{\frac{1}{2}} , \end{aligned}$$

and this implies

$$\begin{aligned} \left(\int_{\mathbb{S}^n} \left| u_p - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{8}} \right|^2 \right)^{\frac{1}{2}} & \leq C \left(\int_{\mathbb{S}^n} |\nabla u_p|^2 \right)^{\frac{1}{2}} \\ & + \left| \int_{\mathbb{S}^n} u_p - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{8}} \right| . \end{aligned}$$

It follows that

$$\int_{\mathbb{S}^n} \left| u_p - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{8}} \right|^2 = O(\|f_p - f(P)\|_{\infty}) .$$

This concludes the proof of Lemma 5.3. □

Lemma 5.4. $(\Lambda_p)_p$ is bounded.

Proof. We have, multiplying (5.1) by $u_p \xi_p$ and integrating,

$$\begin{aligned} \Lambda_p \int_{\mathbb{S}^n} |u_p|^{2^{\#}} \xi_p^2 & = \frac{n-4}{2} \int_{\mathbb{S}^n} f_p |u_p|^{2^{\#}} \xi_p - \int_{\mathbb{S}^n} (\Delta_h u_p)^2 \xi_p - c_n \int_{\mathbb{S}^n} |\nabla u_p|^2 \xi_p \\ & - \left(\frac{n^2}{2} - c_n \frac{n}{2} + d_n \right) \int_{\mathbb{S}^n} u_p^2 \xi_p , \end{aligned}$$

and (u_p) is bounded in $H_2^2(\mathbb{S}^n)$; so, as in the proof of Lemma 4.4, we obtain the lemma. □

Lemma 5.5. $(u_p)_p$ is bounded in $L^\infty(\mathbb{S}^n)$.

Proof. Consider for $p = \frac{t-1}{t}P \in B$

$$\Omega_p = \left\{ x \in \mathbb{S}^n \mid \left| u_p(x) - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{8}} \right| < 1 \right\}.$$

According to Lemma 5.3, $|\mathbb{S}^n \setminus \Omega_p| \rightarrow 0$ uniformly in p as $\varepsilon_f \rightarrow 0$. Now, we write

$$\left(\Delta_h + \frac{c_n}{2} \right)^2 u_p = b_p u_p + w_p,$$

where

$$\begin{aligned} b_p &= \left(\frac{n-4}{2} f_p - \Lambda_p \xi_p \right) \left(|u_p|^{2^\#-2} - \left(\frac{n(n^2 - 4)}{8f(P)} \right) \right) \\ &\quad + \left(\frac{c_n^2}{4} - d_n + \left(\frac{n-4}{2} f_p - \Lambda_p \xi_p \right) \left(\frac{n(n^2 - 4)}{8f(P)} \right) \right) 1_{\mathbb{S}^n \setminus \Omega_p}, \\ w_p &= \left(\frac{c_n^2}{4} - d_n + \left(\frac{n-4}{2} f_p - \Lambda_p \xi_p \right) \left(\frac{n(n^2 - 4)}{8f(P)} \right) \right) u_p 1_{\Omega_p}. \end{aligned}$$

Using Lemmas 5.3 and 5.4, we have

$$\begin{aligned} \|b_p\|_{\frac{n}{4}} &\leq C \left(\left\| u_p - \frac{n(n^2 - 4)}{8f(P)} \right\|_{2^\#} + |\mathbb{S}^n \setminus \Omega_p|^{\frac{4}{n}} \right) \\ &\leq C (\|f_p - f(P)\|_\infty^{\frac{1}{2}} + \|f_p - f(P)\|_\infty^{\frac{2}{n}}), \end{aligned}$$

where C is independent of p . It follows, using Remark 3.1 and Proposition 3.1, since $b_p \in L^\infty(\mathbb{S}^n)$ and $|\mathbb{S}^n \setminus \Omega_p| \rightarrow 0$ as $\varepsilon_f \rightarrow 0$ uniformly in p , that for ε_f small enough, we have for all $p \in B$

$$\|u_p\|_\infty \leq C \|w_p\|_\infty,$$

where C is independent of p . Thanks to Lemma 5.4 (w_p) is bounded in $L^\infty(\mathbb{S}^n)$ uniformly in p . This concludes the proof of the lemma. \square

Remark 5.1. From the previous lemma, the proof of Lemma 5.2, and the proof of Lemma 5.3, we can deduce the following estimate

$$\left\| u_p - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{8}} \right\|_{H^2_\mathbb{S}^2}^2 = O(\|f_p - f(P)\|_1).$$

Lemma 5.6. *There holds*

$$\Lambda_p = O(\|f_p - f(P)\|_\infty).$$

Proof. From Theorem 1.1, we deduce

$$\Lambda_p \int_{\mathbb{S}^n} |\nabla \xi_p|^2 |u_p|^{2^\#} = \frac{n-4}{8} \int_{\mathbb{S}^n} \langle \nabla f_p, \nabla \xi_p \rangle |u_p|^{2^\#}.$$

Writing

$$\begin{aligned} \int_{\mathbb{S}^n} \langle \nabla f_p, \nabla \xi_p \rangle |u_p|^{2^\#} &= \int_{\mathbb{S}^n} \langle \nabla(f_p - f(P)), \nabla \xi_p \rangle |u_p|^{2^\#} \\ &= n \int_{\mathbb{S}^n} (f_p - f(P)) \xi_p |u_p|^{2^\#} \\ &\quad - 2^\# \int_{\mathbb{S}^n} (f_p - f(P)) \langle \nabla u_p, \nabla \xi_p \rangle |u_p|^{2^\#-2} u_p, \end{aligned}$$

and using Hölder inequality, Lemmas 5.3 and 5.5, we get

$$\Lambda_p \int_{\mathbb{S}^n} |\nabla \xi_p|^2 |u_p|^{2^\#} = O(\|f_p - f(P)\|_\infty).$$

So, as in the proof of Lemma 4.4, we derive Lemma 5.6. □

Lemma 5.7. *There holds*

$$\left\| u_p - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{8}} \right\|_{C^2} = O(\|f_p - f(P)\|_\infty).$$

Proof. Consider for C large, and for $p \in B$

$$\Omega_p = \left\{ x \in \mathbb{S}^n \mid \left| u_p(x) - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{8}} \right| < C \|f_p - f(P)\|_\infty \right\}.$$

According to Lemma 5.3, $|\mathbb{S}^n \setminus \Omega_p| \rightarrow 0$ as $C \rightarrow +\infty$ uniformly in p . We have

$$\left(\Delta_h + \frac{c_n}{2} \right)^2 \left(u_p - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{8}} \right) = b_p \left(u_p - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{8}} \right) + w_p, \tag{5.4}$$

where

$$\begin{aligned} b_p &= \left(\frac{n-4}{2} f_p |u_p|^{2^\#-2} - d_n + \frac{c_n^2}{4} \right) 1_{\mathbb{S}^n \setminus \Omega_p}, \\ w_p &= \left(\frac{n-4}{2} f_p |u_p|^{2^\#-2} - d_n + \frac{c_n^2}{4} \right) \left(u_p - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{8}} \right) \\ &\quad \times 1_{\Omega_p} - \Lambda_p \xi_p |u_p|^{2^\#-2} u_p \\ &\quad + \left(\frac{n-4}{2} f_p |u_p|^{2^\#-2} - d_n \right) \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{8}}, \end{aligned}$$

we clearly get, using Lemma 5.5 (C_1 being independent of p)

$$\|b_p\|_{\frac{n}{4}} \leq C_1 |\mathbb{S}^n \setminus \Omega_p|^{\frac{4}{n}}.$$

It follows that for C large enough (independent of p), $\|b_p\|_{\frac{n}{4}} \leq \delta^\infty$ for all $p \in B$ (here δ^∞ is given by Remark 3.1). From this we deduce, using Remark 3.2, Proposition 3.1, Lemma 5.5, Lemma 5.6 and (5.4), that for all $p \in B$

$$\left\| u_p - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{8}} \right\|_\infty \leq C \|w_p\|_{\frac{n}{4}} \leq C_1 \|f_p - f(P)\|_\infty, \tag{5.5}$$

where C_1 is independent of p . Then using (5.4) and (5.5), it easily follows that for all $p \in B$

$$\left\| u_p - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n-4}{8}} \right\|_{C^2} \leq C \|f_p - f(P)\|_\infty,$$

where C is independent of p . This concludes the proof of the lemma. □

Remark 5.2. It follows from the previous lemma that if $\|f_p - f(P)\|_\infty$ is small enough (this smallness being independent of t and P), $u_p > 0$ on \mathbb{S}^n . So, if for some t_0 it is $\Lambda_{t_0} = 0$, this gives rise, according to our transformation rules, to a positive solution of the original equation (E).

6. Continuous Dependence on the Parameter

In this section we prove that, given $p \in B$, $p = (P, t)$, the function $u_p = u_{P,t}$ given by Lemma 5.1 is uniquely determined and that u_p , as well as the Lagrange multiplier Λ_p , vary continuously in B . For $p = \frac{t-1}{t}P$, $P \in \mathbb{S}^n$, we consider the functional

$$\bar{J}_p[u] = \frac{\int_{\mathbb{S}^n} (\Delta_h u)^2 + c_n \int_{\mathbb{S}^n} |\nabla u|^2 + d_n \int_{\mathbb{S}^n} u^2}{\left(\int_{\mathbb{S}^n} f_p |u|^{2^\#} \right)^{\frac{2}{2^\#}}},$$

where f_p stands for $f_{P,t}$, and we let

$$\mathcal{M}_p = \inf_{u \in \mathcal{S}_{2^\#}} \bar{J}_p[u].$$

Proposition 6.1. *For $\varepsilon_f = \|f - \frac{n(n^2-4)}{8}\|_\infty$ sufficiently small, the functional \bar{J}_p has a unique minimum u_p in the class $\mathcal{S}_{2^\#}^0$. The map $p \rightarrow u_p$ is continuous from B into $\mathcal{S}_{2^\#}^0$.*

Proof. To verify the uniqueness assertion, and thanks to the homogeneity of the functional, we assume that there exists $p \in B$ for which \bar{J}_p has two distinct minima in $\{u \in \mathcal{S}_{2^\#} \text{ such that } \int_{\mathbb{S}^n} f_p |u|^{2^\#} = 1\}$, u_0 and u_1 . According to Remark 5.2 we can choose ε_f small enough such that u_0 and u_1 are positive. For convenience reason, by a simple renormalization, (and without loss of generality) we assume that u_0 and u_1 are solutions respectively of

$$\Delta_h^2 u_0 + c_n \Delta_h u_0 + d_n u_0 = \frac{n-4}{2} f_p u_0^{2^\#-1} - \Lambda_0 \xi_0 u_0^{2^\#-1}$$

and

$$\Delta_h^2 u_1 + c_n \Delta_h u_1 + d_n u_1 = \frac{n-4}{2} f_p u_1^{2^\#-1} - \Lambda_1 \xi_1 u_1^{2^\#-1},$$

where $\Lambda_0, \Lambda_1 \in \mathbb{R}^+, \xi_0, \xi_1 \in \mathbb{H}, \|\xi_0\|_\infty = \|\xi_1\|_\infty = 1$. We set, for $\lambda \in [0; 1]$,

$$u_\lambda^{2^\#} = \lambda u_0^{2^\#} + (1-\lambda) u_1^{2^\#}.$$

For each λ there holds

$$\begin{cases} \dot{u}_\lambda = \frac{1}{2^\#} u_\lambda^{1-2^\#} (u_0^{2^\#} - u_1^{2^\#}); \\ \ddot{u}_\lambda = -(2^\# - 1) u_\lambda^{-1} (\dot{u}_\lambda)^2. \end{cases}$$

Also, differentiating with respect to λ , we deduce

$$u_\lambda \in \mathcal{S}_{2^\#} \Rightarrow \int_{\mathbb{S}^n} u_\lambda^{2^\#-1} \dot{u}_\lambda \xi = 0 \quad \text{for all } \xi \in \mathbb{H}.$$

Hence, by Hölder’s inequality and Lemma 5.7, for all $\xi \in \mathbb{H}$

$$\begin{aligned} \left(\frac{n(n^2-4)}{8f(P)} \right)^{\frac{n+4}{8}} \left| \int_{\mathbb{S}^n} \dot{u}_\lambda \xi \right| &= \left| \int_{\mathbb{S}^n} \left(\left(\frac{n(n^2-4)}{8f(P)} \right)^{\frac{n+4}{8}} - u_\lambda^{2^\#-1} \right) \dot{u}_\lambda \xi \right| \\ &= O(\varepsilon_f) \|\dot{u}_\lambda\|_2 \|\xi_\lambda\|_2. \end{aligned}$$

Decomposing \dot{u}_λ into $\alpha_\lambda + \Psi_\lambda + \xi_\lambda$, where $\alpha_\lambda \in \mathbb{R}, \xi_\lambda$ is the orthogonal projection of \dot{u}_λ onto \mathbb{H} and Ψ_λ is orthogonal to $\mathbb{R} \oplus \mathbb{H}$, we have

$$\int_{\mathbb{S}^n} |\nabla \dot{u}_\lambda|^2 = \int_{\mathbb{S}^n} |\nabla \Psi_\lambda|^2 + \int_{\mathbb{S}^n} |\nabla \xi_\lambda|^2 + 2 \int_{\mathbb{S}^n} \nabla \Psi_\lambda \nabla \xi_\lambda = \int_{\mathbb{S}^n} |\nabla \Psi_\lambda|^2 + n \int_{\mathbb{S}^n} \xi_\lambda^2.$$

On the other hand

$$\int_{\mathbb{S}^n} |\nabla \dot{u}_\lambda|^2 = \int_{\mathbb{S}^n} \dot{u}_\lambda (\Delta \Psi_\lambda + n \xi_\lambda) = \int_{\mathbb{S}^n} |\nabla \Psi_\lambda|^2 + n \int_{\mathbb{S}^n} \xi_\lambda \dot{u}_\lambda.$$

Taking into account that $\int_{\mathbb{S}^n} \dot{u}_\lambda \xi_\lambda = O(\varepsilon_f) \|\dot{u}_\lambda\|_2 \|\xi_\lambda\|_2$, it is

$$\int_{\mathbb{S}^n} \xi_\lambda^2 = O(\varepsilon_f) \|\dot{u}_\lambda\|_2^2.$$

It follows, from the Courant–Fischer characterization of the eigenvalues and from Theorem 2.2, that

$$\int_{\mathbb{S}^n} (\Delta_h \Psi_\lambda)^2 + c_n \int_{\mathbb{S}^n} |\nabla \Psi_\lambda|^2 \geq (4(n+1)^2 + c_n 2(n+1)) \int_{\mathbb{S}^n} \Psi_\lambda^2. \tag{6.1}$$

Moreover, setting $f_0 = \frac{n(n^2-4)}{8}$, we have the following estimates. By Lemma 5.3

$$\int_{\mathbb{S}^n} f_p u_\lambda^{2^\#} = \int_{\mathbb{S}^n} (f_p - f_0) u_\lambda^{2^\#} + \int_{\mathbb{S}^n} f_0 u_\lambda^{2^\#} = f_0 (1 + O(\varepsilon_f)); \tag{6.2}$$

from $\int_{\mathbb{S}^n} f_p u_\lambda^{2^\#} = \text{constant}$ we deduce

$$\int_{\mathbb{S}^n} f_p u_\lambda^{2^\#-1} \dot{u}_\lambda = 0; \tag{6.3}$$

and using Lemmas 5.3 and 5.7,

$$\int_{\mathbb{S}^n} (\Delta_h u_\lambda)^2 + c_n \int_{\mathbb{S}^n} |\nabla u_\lambda|^2 + d_n \int_{\mathbb{S}^n} u_\lambda^2 = \frac{n-4}{2} f_0 (1 + O(\varepsilon_f)); \tag{6.4}$$

$$\begin{aligned} \int_{\mathbb{S}^n} f_p u_\lambda^{2^\#-2} \dot{u}_\lambda^2 &= \int_{\mathbb{S}^n} (f_p - f_0) u_\lambda^{2^\#-2} \dot{u}_\lambda^2 + \int_{\mathbb{S}^n} f_0 u_\lambda^{2^\#-2} \dot{u}_\lambda^2 \\ &= f_0 (1 + O(\varepsilon_f)) \left(\int_{\mathbb{S}^n} \dot{u}_\lambda^2 \right). \end{aligned} \tag{6.5}$$

We have also

$$\begin{aligned} d^2 \bar{J}_p[u] \cdot (\varphi, \Psi) &= 2 \left(\int_{\mathbb{S}^n} f_p u^{2^\#} \right)^{-\frac{2}{2^\#}-1} \\ &\times \left\{ \int_{\mathbb{S}^n} (f_p u^{2^\#}) \cdot \left(\int_{\mathbb{S}^n} \Delta_h \varphi \Delta_h \Psi + c_n \int_{\mathbb{S}^n} \nabla \varphi \nabla \Psi + d_n \int_{\mathbb{S}^n} \varphi \Psi \right) \right. \\ &- 2 \left(\int_{\mathbb{S}^n} f_p u^{2^\#-1} \Psi \right) \left(\int_{\mathbb{S}^n} \Delta u \Delta \varphi + c_n \int_{\mathbb{S}^n} \nabla u \nabla \varphi + d_n \int_{\mathbb{S}^n} u \varphi \right) \\ &- 2 \left(\int_{\mathbb{S}^n} f_p u^{2^\#-1} \varphi \right) \left(\int_{\mathbb{S}^n} \Delta u \Delta \Psi + c_n \int_{\mathbb{S}^n} \nabla u \nabla \Psi + d_n \int_{\mathbb{S}^n} u \Psi \right) \\ &- (2^\# - 1) \left(\int_{\mathbb{S}^n} (\Delta_h u)^2 + c_n \int_{\mathbb{S}^n} |\nabla u|^2 + d_n \int_{\mathbb{S}^n} u^2 \right) \\ &\times \left(\int_{\mathbb{S}^n} f_p u^{2^\#-2} \varphi \Psi \right) \\ &+ \left(\frac{2}{2^\#} + 1 \right) \left(\int_{\mathbb{S}^n} (\Delta_h u)^2 + c_n \int_{\mathbb{S}^n} |\nabla u|^2 + d_n \int_{\mathbb{S}^n} u^2 \right) \\ &\times \left. \left(\int_{\mathbb{S}^n} f_p u^{2^\#-1} \varphi \right) \left(\int_{\mathbb{S}^n} f_p u^{2^\#-1} \Psi \right) \left(\int_{\mathbb{S}^n} f_p u^{2^\#} \right)^{-1} \right\}. \end{aligned}$$

It is clear, using the Sobolev inequality, that $u \rightarrow D^2 \bar{J}_p[u](\cdot, \cdot)$ is a continuous map from $H^2_2(\mathbb{S}^n)$ to $L(H^2_2(\mathbb{S}^n) \times H^2_2(\mathbb{S}^n), \mathbb{R})$. Clearly

$$\frac{d^2}{d\lambda^2} \bar{J}_p[u_\lambda] = d^2 \bar{J}_p[u_\lambda] \cdot (\dot{u}_\lambda, \dot{u}_\lambda) + d \bar{J}_p[u_\lambda] \cdot \ddot{u}_\lambda,$$

and moreover from (6.2)–(6.5) we have

$$\begin{aligned} d^2 \bar{J}_p[u_\lambda] \cdot (\dot{u}_\lambda, \dot{u}_\lambda) &= 2 \left(\int_{\mathbb{S}^n} f_p u_\lambda^{2^\#} \right)^{-\frac{2}{2^\#}-1} \\ &\times \left\{ (1 + O(\varepsilon_f)) f_0 \int_{\mathbb{S}^n} (\Delta_h \dot{u}_\lambda)^2 + (1 + O(\varepsilon_f)) f_0 c_n \int_{\mathbb{S}^n} |\nabla \dot{u}_\lambda|^2 \right. \end{aligned}$$

$$\begin{aligned}
 & \left. + d_n f_0 (1 - (2^\# - 1)) \int_{\mathbb{S}^n} \dot{u}_\lambda^2 (1 + O(\varepsilon_f)) \right\} \\
 &= 2 \left(\int_{\mathbb{S}^n} f_p u_\lambda^{2^\#} \right)^{-\frac{2}{2^\#} - 1} f_0 (1 + O(\varepsilon_f)) \\
 & \times \left\{ \int_{\mathbb{S}^n} (\Delta_h \dot{u}_\lambda)^2 + c_n \int_{\mathbb{S}^n} |\nabla \dot{u}_\lambda|^2 - \frac{8}{n-4} d_n \int_{\mathbb{S}^n} \dot{u}_\lambda^2 \right\},
 \end{aligned}$$

and

$$\begin{aligned}
 d\bar{J}_p[u_\lambda] \cdot \ddot{u}_\lambda &= 2 \left(\int_{\mathbb{S}^n} f_p u_\lambda^{2^\#} \right)^{-\frac{2}{2^\#}} \times \left\{ \int_{\mathbb{S}^n} \Delta_h u_\lambda \Delta_h \ddot{u}_\lambda + c_n \int_{\mathbb{S}^n} \nabla u_\lambda \nabla \ddot{u}_\lambda \right. \\
 & \left. + d_n \int_{\mathbb{S}^n} u_\lambda \ddot{u}_\lambda - \left(\int_{\mathbb{S}^n} (\Delta_h u_\lambda)^2 + c_n \int_{\mathbb{S}^n} |\nabla u_\lambda|^2 + d_n \int_{\mathbb{S}^n} u_\lambda^2 \right) \right. \\
 & \left. \times \int_{\mathbb{S}^n} f_p u_\lambda^{2^\# - 1} \ddot{u}_\lambda \left(\int_{\mathbb{S}^n} f_p u_\lambda^{2^\#} \right)^{-1} \right\}
 \end{aligned}$$

One can check with some straightforward computations and using Lemma 5.7, that

$$\begin{aligned}
 d\bar{J}_p[u_\lambda] \cdot \ddot{u}_\lambda &= 2 \left(\int_{\mathbb{S}^n} f_p u_\lambda^{2^\#} \right)^{-\frac{2}{2^\#}} \left\{ (1 - 2^\#) \left(\int_{\mathbb{S}^n} \Delta_h u_\lambda \Delta_h \left(\frac{\dot{u}_\lambda^2}{u_\lambda} \right) \right) \right. \\
 & \left. + c_n \int_{\mathbb{S}^n} \nabla u_\lambda \nabla \left(\frac{\dot{u}_\lambda^2}{u_\lambda} \right) + d_n \int_{\mathbb{S}^n} u_\lambda \frac{\dot{u}_\lambda^2}{u_\lambda} \right) \\
 & - \left(\int_{\mathbb{S}^n} (\Delta_h u_\lambda)^2 + c_n \int_{\mathbb{S}^n} |\nabla u_\lambda|^2 + d_n \int_{\mathbb{S}^n} u_\lambda^2 \right) \\
 & \times \int_{\mathbb{S}^n} f_p u_\lambda^{2^\# - 1} \ddot{u}_\lambda \left(\int_{\mathbb{S}^n} f_p u_\lambda^{2^\#} \right)^{-1} \left. \right\} \\
 &= O(\varepsilon_f) \left(\int_{\mathbb{S}^n} (\Delta_h \dot{u}_\lambda)^2 + \int_{\mathbb{S}^n} |\nabla \dot{u}_\lambda|^2 + \int_{\mathbb{S}^n} \dot{u}_\lambda^2 \right).
 \end{aligned}$$

It follows that

$$\begin{aligned}
 \frac{d^2}{d\lambda^2} \bar{J}_p[u_\lambda] &= 2 \left(\int_{\mathbb{S}^n} f_p u_\lambda^{2^\#} \right)^{-\frac{2}{2^\#} - 1} f_0 (1 + O(\varepsilon_f)) \\
 & \times \left\{ \int_{\mathbb{S}^n} (\Delta_h \dot{u}_\lambda)^2 + c_n \int_{\mathbb{S}^n} |\nabla \dot{u}_\lambda|^2 - \frac{8}{n-4} d_n \int_{\mathbb{S}^n} \dot{u}_\lambda^2 \right\} \\
 & + O(\varepsilon_f) \left\{ \int_{\mathbb{S}^n} (\Delta_h \dot{u}_\lambda)^2 + c \int_{\mathbb{S}^n} |\nabla \dot{u}_\lambda|^2 + \int_{\mathbb{S}^n} \dot{u}_\lambda^2 \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= 2 \left(\int_{\mathbb{S}^n} f_p u_\lambda^{2^\#} \right)^{-\frac{2}{2^\#}-1} f_0 (1 + O(\varepsilon_f)) \\
 &\quad \times \left\{ \int_{\mathbb{S}^n} (\Delta_h \Psi_\lambda)^2 + c_n \int_{\mathbb{S}^n} |\nabla \Psi_\lambda|^2 - \frac{8d_n}{n-4} \int_{\mathbb{S}^n} \Psi_\lambda^2 \right. \\
 &\quad \left. - \frac{8d_n}{n-4} \alpha_\lambda^2 + O(\varepsilon_f) \int_{\mathbb{S}^n} \dot{u}_\lambda^2 \right\} \\
 &\quad + O(\varepsilon_f) \left\{ \int_{\mathbb{S}^n} (\Delta_h \dot{u}_\lambda)^2 + c \int_{\mathbb{S}^n} |\nabla \dot{u}_\lambda|^2 + \int_{\mathbb{S}^n} \dot{u}_\lambda^2 \right\}. \tag{6.6}
 \end{aligned}$$

Independently, we have

$$0 = \int_{\mathbb{S}^n} f_p u_\lambda^{2^\#-1} \dot{u}_\lambda = \int_{\mathbb{S}^n} f_p u_\lambda^{2^\#-1} \Psi_\lambda + \int_{\mathbb{S}^n} f_p u_\lambda^{2^\#-1} \xi_\lambda + \alpha_\lambda \int_{\mathbb{S}^n} f_p u_\lambda^{2^\#-1},$$

and it follows using Hölder’s inequality and Lemma 5.7, that

$$\alpha_\lambda^2 \leq \int_{\mathbb{S}^n} \Psi_\lambda^2 + O(\varepsilon_f) \int_{\mathbb{S}^n} \dot{u}_\lambda^2. \tag{6.7}$$

We deduce from Lemma 5.7 and (6.1), (6.6) and (6.7),

$$\frac{d^2}{d\lambda^2} \bar{J}_p[u_\lambda] \geq C(n) \left(1 - \left(\frac{n^3 - 4n}{(n^3 + 3n^2 + 2n)} \right) + O(\varepsilon_f) \right) \int_{\mathbb{S}^n} \Psi_\lambda^2.$$

Then, if Ψ_λ is not identically 0, $\frac{d^2}{d\lambda^2} \bar{J}_p[u_\lambda] > 0$ and this can not occur since u_0 and u_1 are both minima. It follows that Ψ_λ is identically 0. Using (6.7) we then get

$$\alpha_\lambda^2 \leq O(\varepsilon_f) \int_{\mathbb{S}^n} \dot{u}_\lambda^2;$$

this is impossible since

$$\int_{\mathbb{S}^n} \dot{u}_\lambda^2 = \alpha_\lambda^2 + O(\varepsilon_f) \int_{\mathbb{S}^n} \dot{u}_\lambda^2,$$

unless $u_0 = u_1$.

Let us observe that, reasoning as above, one can prove that $d^2 \bar{J}_p[u_p]$ is positive definite on the tangent space of the constraint $\mathcal{S}_{2^\#}$, so using the Implicit Functions theorem, it follows that $p \rightarrow u_p$ is continuous. This concludes the proof of Proposition 6.1. □

7. Comparison of the Maps Λ and G

Consider the map $\Lambda : B \rightarrow \mathbb{R}$. We can associate to this map the following map $\mathbf{\Lambda} : B \rightarrow \mathbb{R}^{n+1}$ consisting in the components of $\Lambda_p \xi_p$ in the basis ξ . Namely, in the basis ξ of \mathbb{H} we can write

$$\Lambda_p \xi_p = \sum_{i=1}^{n+1} \Lambda_i \xi_i,$$

and then

$$\mathbf{\Lambda} = (\Lambda_1, \dots, \Lambda_{n+1}).$$

In this section we prove that $\mathbf{\Lambda}$ has the same degree as the map G defined in the introduction. We set

$$\begin{cases} C_{i,j}(p) = \int_{\mathbb{S}^n} \langle \nabla \xi_i, \nabla \xi_j \rangle u_p^{2^\#}, \\ A_j(p) = \int_{\mathbb{S}^n} \langle \nabla f, \nabla \xi_j \rangle u_p^{2^\#}. \end{cases}$$

We observe that, since u_p is close to 1 by Lemma 5.7, for ε_f small, $(C_{i,j}(p))$ is a positive definite matrix. Therefore the Kazdan–Warner condition can be rewritten as

$$\mathbf{\Lambda}(p) = C(p)^{-1}A(p).$$

It follows immediately that $\mathbf{\Lambda}$ is continuous (using Proposition 6.1) and

Lemma 7.1.

- (i) $\mathbf{\Lambda}(p) = 0 \Leftrightarrow A(p) = 0$;
- (ii) $\deg(\mathbf{\Lambda}|\{(P, t), t < t_0\}, 0) = \deg(A|\{(P, t), t < t_0\}, 0)$ for all $t_0 > 1$.

Now consider (using the notation of the introduction)

$$A(P, t) = \int_{\mathbb{S}^n} \langle \nabla(f \circ \varphi_{P,t}), \nabla \xi \rangle u_p^{2^\#},$$

$$G_1(P, t) = n \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n}{4}} \int_{\mathbb{S}^n} (f \circ \varphi_{P,t}) \xi = n \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n}{4}} G(P, t).$$

(this means that the components of $A(P, t)$ are $\int_{\mathbb{S}^n} \langle \nabla(f \circ \varphi_{P,t}), \nabla \xi_i \rangle u_p^{2^\#}$, for $i = 1, \dots, n+1$). We choose ε sufficiently small so that the error term in $f(y) = f(P) + \sum_{k=1}^\alpha f_k(y) + O(|y|^{\alpha+1})$ is small, say $O(|y|^{\alpha+1}) \leq C|y|^{\alpha+\frac{1}{2}}$ for $|y| < \varepsilon$. We have

$$A(P, t) = G_1(P, t) + I + II,$$

where

$$I = n \int_{\mathbb{S}^n} (f \circ \varphi_{P,t} - f(P)) \xi \left(u_p^{2^\#} - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n}{4}} \right);$$

$$II = \int_{\mathbb{S}^n} (f \circ \varphi_{P,t} - f(P)) \langle \nabla \xi, \nabla(u_p^{2^\#}) \rangle.$$

We set, using stereographic coordinates, $\Omega_t = \{y \in \mathbb{R}^n \mid |y| \leq \varepsilon t\}$ and we have $|\mathbb{S}^n \setminus \Omega_t| = O(\frac{1}{t^n})$. Clearly we have

$$\begin{aligned} & \left| \int_{\Omega_t} (f \circ \varphi_{P,t} - f(P)) \xi \left(u_p^{2^\#} - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n}{4}} \right) \right| \\ & \leq \left(\int_{\Omega_t} (f \circ \varphi_{P,t} - f(P))^2 \right)^{\frac{1}{2}} \left(\int_{\Omega_t} \left(u_p^{2^\#} - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n}{4}} \right)^2 \right)^{\frac{1}{2}}. \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{\Omega_t} (f \circ \varphi_{P,t} - f(P)) \langle \nabla x, \nabla(u_p^{2^\#}) \rangle \right| \\ & \leq \left(\int_{\Omega_t} (f \circ \varphi_{P,t} - f(P))^2 \right)^{\frac{1}{2}} \left(\int_{\Omega_t} |\nabla(u_p^{2^\#})|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

To have further control on the error terms we point out the following estimates.

Lemma 7.2. *If P is a critical point of f non degenerated of order α , we have the following estimates*

$$\int_{\mathbb{S}^n} (f_p - f(P))^2 = \begin{cases} O\left(\frac{1}{t^{2\alpha}}\right) & \text{if } 2\alpha < n, \\ O\left(\frac{1}{t^n} \log t\right) & \text{if } 2\alpha = n, \\ O\left(\frac{1}{t^n}\right) & \text{if } 2\alpha > n, \end{cases}$$

$$\int_{\mathbb{S}^n} |\nabla u_p|^2 = O(\|f_p - f(P)\|_1) = \begin{cases} O\left(\frac{1}{t^\alpha}\right) & \text{if } \alpha < n, \\ O\left(\frac{1}{t^n} \log t\right) & \text{if } \alpha = n. \end{cases}$$

$$\left\| u_p^{2^\#} - \left(\frac{n(n^2 - 4)}{8f(P)} \right)^{\frac{n}{4}} \right\|_2^2 = O(\|f_p - f(P)\|_1) = \begin{cases} O\left(\frac{1}{t^\alpha}\right) & \text{if } \alpha < n, \\ O\left(\frac{1}{t^n} \log t\right) & \text{if } \alpha = n. \end{cases}$$

The proof of the first estimate of Lemma 7.2 is obtained by a simple computation, the second and the third estimates are obtained using Remark 5.1 and a simple computation.

Proposition 7.1. *Assume that f is everywhere positive, uniformly non degenerated of order at most n if n is even, and of order at most $n - 1$ if n is odd. Then we*

have for all $P \in \mathbb{S}^n$ and for all t large (with the notation of this section)

$$G(P, t) \cdot A(P, t) > 0.$$

Proposition 7.1 is a direct consequence of Lemma 7.2.

One has to notice that in the case $n = 2k + 1 = \alpha$, Lemma 7.2 shows that the corrections terms I and II may in fact be the dominant one. For this reason we exclude this case from consideration and this accounts for the difference between the case where the dimension of the manifold is odd and the case where this dimension is even in the statement of Theorem 1.2.

8. Proof of Theorem 1.2 and Corollary 1.1

By a simple compactness argument and using Proposition 7.1, there exists t_0 such that for all $t \geq t_0$

$$G(P, t) \cdot A(P, t) > 0.$$

Then, for $0 \leq r \leq 1$,

$$\deg(rG(P, t) + (1 - r)A(P, t), \{p|t < t_0\}, 0) = \text{constant}.$$

It follows that, using Lemma 7.1 and under the notation of Sec. 7

$$\deg(\mathbf{\Lambda}(p), \{p|t < t_0\}, 0) = \deg(A(p), \{p|t < t_0\}, 0) = \deg(G(p), \{p|t < t_0\}, 0) \neq 0.$$

Hence, there exists $p \in B$ such that $\mathbf{\Lambda}(p) = 0$. This concludes the proof of Theorem 1.2. □

In order to show Corollary 1.1, we recall the formulas for the stereographic projection: the point $(x, x_{n+1}) \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$ is projected through the north pole on the point $y \in \mathbb{R}^n$ by the following formulas

$$x = \frac{2y}{|y|^2 + 1}; \quad x_{n+1} = \frac{|y|^2 - 1}{|y|^2 + 1}. \tag{8.1}$$

Consider the stereographic projection $\pi_P : \mathbb{S}^n \rightarrow \mathbb{R}^n$ through the point $-P$, and define $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R}$ in the following way

$$\tilde{f}(y) = f(\pi_P^{-1}(y)).$$

Then, using (8.1) one can prove that

$$G(P, t) = \int_{\mathbb{S}^n} \tilde{f}\left(\frac{1}{t}\right) (1 + |y|^2)^{-(n+1)} \begin{pmatrix} 2y \\ 1 - |y|^2 \end{pmatrix} dy.$$

Here the quantity $\begin{pmatrix} 2y \\ 1 - |y|^2 \end{pmatrix} dy$ is considered as an $(n + 1)$ -tuple. Hence expanding the above expression in powers of $\frac{1}{t}$, one finds

$$G(P, t) = \begin{pmatrix} a_1 \nabla \tilde{f}(P) \frac{1}{t} \\ a_2 \Delta \tilde{f}(P) \frac{1}{t^2} \end{pmatrix} + o\left(\frac{1}{t^2}\right),$$

where a_1 and a_2 are non-zero coefficients given by

$$a_1 = \frac{2}{n} \int_{\mathbb{S}^n} |y|^2 (1 + |y|^2)^{-(n+1)}; \quad a_2 = \frac{2}{n} \int_{\mathbb{S}^n} (|y|^2 - |y|^4) (1 + |y|^2)^{-(n+1)},$$

and where $o(\frac{1}{t^2})$ is uniform in P . So under the assumptions of the corollary, we deduce that f is uniformly non degenerated of order 2. Moreover, the arguments of Chang–Gursky–Yang [14] show that under the condition

$$\sum_{P \in \mathbb{S}^n \mid \nabla f(P)=0 \text{ and } \Delta_h f(P)>0} (-1)^{m(P,f)} \neq -1,$$

the degree is different from zero. This concludes the proof of Corollary 1.1. □

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