Topological methods for the prescribed Webster Scalar Curvature problem on CR manifolds

Hichem Chtioui[a], Mohameden Ould Ahmedou [b*]and Ridha Yacoub[c]

a: Département de Mathématiques, Faculté des Sciences de Sfax, Route de Soukra, Sfax, Tunisie.

b: Mathematisches Institut der Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany

c: Institut préparatoire aux Etudes d'Ingénieur de Monastir, Avenue Ibn Al Jazzar, 5019 Monastir, Tunisie.

Abstract. We consider the existence of contact forms of prescribed Webster scalar curvature on a $2n+1$ dimensional CR compact manifold locally conformally CR equivalent to the standard unit sphere $\mathbb{S}^{2n+1}$ of $\mathbb{C}^{n+1}$. We give some existence results, using dynamical and topological methods involving the study of the *critical points at infinity* of the associated noncompact variational problem.

Mathematics Subject Classification (2000) : 53C15, 53C21, 35J65, 18G35.

Key words : Contact Geometry, Webster scalar curvature, Critical point at infinity, Noncompact geometric variational problems, Morse index, Morse Lemma, Topological methods.

Contents

1 Introduction and statement of main results

The geometry of CR manifolds, the abstract models of real hypersurfaces in complex manifolds, has recently attracted much attention. This is in particular due to the fact that,

*Corresponding author, `ahmedou@analysis.mathematik.uni-tuebingen.de`

in the strictly pseudoconvex case, there are many parallels with Riemannian geometry. Indeed a CR manifold carries a natural hermitian metric on its holomorphic tangent bundle -the Levi form- which is, like a metric on a conformal manifold, determined only up to multiplication by a smooth function. The multiple is fixed by choosing a contact form (a real 1-form) annihilating the holomorphic tangent bundle. A CR manifold together with a choice of a contact form is called a pseudohermitian manifold.
The simplest scalar invariant for a pseudohermitian manifold is the pseudohermitian scalar curvature, which we denote by $\mathcal{R}_\theta$, defined independently by S. Webster [27] and N. Tanaka [26].
Let (M, θ) be a strictly pseudoconvex CR compact manifold of dimension $2n+1$ with a contact form θ, and $H : M \to \mathbb{R}$ be a smooth function. The prescribed Webster scalar curvature is to find a choice of a contact form for which the pseudohermitian scalar curvature is given by H. If we set $\tilde{\theta} = u^{2/n}\theta$, where u is a smooth positive function on M, then the above problem is equivalent to solve the following equation

$$(P) \quad \begin{cases} Lu = \frac{n}{2(n+1)} H u^{1+\frac{2}{n}} & \text{in } M \\ u > 0 & \text{in } M, \end{cases}$$

where

$$Lu = \Delta_\theta u + \frac{n}{2(n+1)} \mathcal{R}_\theta u$$

Δ_θ is the sublaplacian operator on (M, θ) and $\mathcal{R}_\theta$ is the Webster scalar curvature of (M, θ). Problem (P) is the analogue of the prescribed scalar curvature problem on Riemannian manifolds. While the scalar curvature problem in the Riemannian framework was extensively studied (see for example the monograph [2] and the references therein), only few results were established for problem (P) (see [13], [14], [17], [24] and [28]). On the contrary, the Yamabe problem on CR manifolds, that is when H is assumed to be constant, was widely studied by various authors (see among others [20], [21], [22], [16] and [18]).
The main difficulty one encounters in problem (P) appears when we consider it from a variational viewpoint. Indeed, the Euler functional associated to (P) does not satisfy the Palais-Smale condition, that is, there exist noncompact sequences along which the functional is bounded and its gradient goes to zero. Moreover, there are topological obstructions of Kazdan-Warner condition type to solve (P), see [19]. Hence, it is not expectable to solve problem (P) for all functions H, and so it is natural to ask : under which conditions on H has (P) have a positive solution? In [24], Malchiodi and Uguzzoni considered the case where $M = \mathbb{S}^{2n+1}$ the unit sphere of $\mathbb{C}^{n+1}$ and gave a perturbative result for problem (P), that is when H is assumed to be a small perturbation of a constant (see also [14]). Their approach uses a perturbation method due to Ambrosetti [1]. In [17], N. Gamara noticed, in analogy with the 4-dimensional Riemannian case, that there is a balance phenomenon between the self interactions and the mutual interactions of the functions failing to satisfy Palais-Smale condition in the 3-dimensional CR case (see [9] and [11] for the Riemannian case). In [17] the case where M is locally conformally CR

equivalent to the Sphere of $\mathbb{C}^2$ was considered (thus when $n=1$), and a Euler-Hopf type criterion for H was provided to find solutions for (P). The existence results of N. Gamara have been generalized by the authors see [13], where multiplicity results are also given.
In this paper we consider the prescribed Webster scalar curvature problem on strictly pseudoconvex CR manifolds which are locally CR equivalent to the unit sphere $\mathbb{S}^{2n+1}$ of $\mathbb{C}^{n+1}$. Our aim is to give new existence results through the use of topological methods.
To state our results, we set the following notations. Let $G(a,.)$ be the Green's function of L on M and A_a the value of the regular part of G at a. Throughout the whole of this paper, we assume that H has only nondegenerate critical points $y_0, y_1, ..., y_N$ such that

$$H(y_0) \geq H(y_1) \geq ... \geq H(y_N) \quad \text{and}$$

$$\frac{-\Delta_\theta H(y_i)}{3H(y_i)} - 2A_{y_i} \neq 0\ \forall i=0,1,...,N \text{ for } n=1, \text{ and } \Delta_\theta H(y_i) \neq 0\ \forall i=0,1,...,N \text{ for } n\geq 2.$$

For each y_i, we denote by $ind(H,y_i)$, the Morse index of H at y_i. Now, we introduce the following set

$$\mathcal{I}_+ = \Big\{ y_i \in \{y_0,...,y_N\} / \ \frac{-\Delta_\theta H(y_i)}{3H(y_i)} - 2A_{y_i} > 0 \text{ for } n=1, \text{ and } -\Delta_\theta H(y_i) > 0 \text{ for } n\geq 2 \Big\} \tag{1.1}$$

For $s \in \mathbb{N}^*$ and for any s-tuple $\tau_s = (y_{i_1},...,y_{i_s}) \in (\mathcal{I}_+)^s$ such that $y_{i_p} \neq y_{i_q}$ if $p \neq q$, we define a matrix $M(\tau_s) = (M_{pq})_{1\leq p,q\leq s}$, by

$$M_{pp} = \frac{-\Delta_\theta H(y_{i_p})}{3H(y_{i_p})^2} - 2\frac{A_{y_{i_p}}}{H(y_{i_p})}, \qquad M_{pq} = -\frac{2G(y_{i_p},y_{i_q})}{\big(H(y_{i_p})H(y_{i_q})\big)^{1/2}} \quad \text{for } p\neq q,$$

and we denote by $\rho(\tau_s)$ the least eigenvalue of $M(\tau_s)$. It was first pointed out by A. Bahri [5] (see also [9] and [11]), in his studies on Yamabe type problems on Riemannian manifolds, that when the self interactions and the mutual interactions between different bubbles are of the same size, the function similar to the above function ρ plays a fundamental role in the existence of solutions to problems like (P). As it is observed in [17], such a phenomenon appears for problem (P) when $n=1$. In the first part of this article, we revisit the three dimensional CR case to provide more existence results. Our approach goes along with the topological ideas and tools of the critical point at Infinity of A. Bahri [5]. The main idea is to compute the topological contribution of the critical points at infinity between the level sets of the associated Euler functional, and the main issue is, under which conditions on H, there is some difference of topology which is not due to the critical points at infinity, and can be only explained by the existence of solutions for (P).
let Z be a pseudo-gradient of H of Morse-Smale type (that is the intersections of stable and unstable manifolds of the critical points of H are transverse).
$(\mathbf{H_0})$ Assume that $W_s(y_i) \cap W_u(y_j) = \emptyset$ for each $y_i \in \mathcal{I}_+$ and $y_j \notin \mathcal{I}_+$, where $W_s(y_i)$ is the

stable manifold of y_i and $W_u(y_j)$ is the unstable manifold of y_j for Z.
For each $0 \le i \le N$ we denote by

$$X_i = \bigcup_{\substack{0 \le k \le i \\ y_k \in \mathcal{I}_+}} \overline{W_s}(y_k) \tag{1.2}$$

where $W_s(y_i)$ is the stable manifold of y_i and $W_u(y_j)$ is the unstable manifold of y_j for Z.
$(\mathbf{H_1})$ Assume that for each $y_i \neq y_j \in \mathcal{I}_+$, we have $M(y_i, y_j)$ is nondegenerate and $\rho(y_i, y_j) < 0$. We then have

Theorem 1.1 *Under the assumption* $(\mathbf{H_0})$ *and* $(\mathbf{H_1})$*, if there exist an index* $I \in \{0, \dots, N\}$ *satisfying the following conditions:*

$(\mathbf{H_2})$ X_I *is not contractible. We denote by* m_I *the dimension of the first non trivial reduced homology group.*

$(\mathbf{H_3})$ $\frac{1}{H(y_j)} > \frac{1}{H(y_0)} + \frac{1}{H(y_I)}$ *for each* $j \in \{I+1, \dots, N\}$ *and* $y_j \in \mathcal{I}_+$.

Then problem (P) *has a solution of Morse index* $\ge m_I$

In the case where the index $I = N$, we have the following interesting special case.

Corollary 1.2 *Under the assumption* $(\mathbf{H_0})$ *and* $(\mathbf{H_1})$*, if*

$$X_N = \bigcup_{y_k \in \mathcal{I}_+} \overline{W_s(y_k)} \tag{1.3}$$

is not contractible then (P) *has a solution.*

Corollary 1.3 *Under the assumption* $(\mathbf{H_1})$*, if*

$$\sum_{y_i \in \mathcal{I}_+} (-1)^{3 - ind(H, y_i)} \neq 1$$

then (P) *has a solution.*

In the second part of this work, we give some existence results of (P) in all dimensions $n \ge 1$. For this purpose, we introduce the following assumptions:
$(\mathbf{A_1})$ We assume that

$$H(y_0) \ge H(y_1) \ge \dots \ge H(y_h) > H(y_{h+1}) \ge \dots \ge H(y_N),$$

where $\mathcal{I}_+ = \{y_0, y_1, \dots, y_h\}$ and $0 \le h \le N$.
$(\mathbf{A'_1})$ We assume that $y_j \notin \mathcal{I}_+$ for all $j \in \{h+1, \dots, N\}$. In addition, we assume that for every $i \in \{1, \dots, h\}$, such that $y_i \notin \mathcal{I}_+$, we have

$$2n - m + 4 \le ind(H, y_i) \le 2n - 1,$$

where, $ind(H, y_i)$ is the Morse index of H at y_i and m is an integer defined in the following assumption $(\mathbf{A_2})$.
$(\mathbf{A_2})$ We assume that there exists a pseudo-gradient Z for H of Morse-Smale type, such that the set X is not contractible, where

$$X = \bigcup_{0 \leq i \leq h} \overline{W_s(y_i)} \tag{1.4}$$

and $W_s(y_i)$ is the stable manifold of y_i for Z. We denote by m the dimension of the first nontrivial reduced homology group of X.
$(\mathbf{A_3})$ We assume that there exists a positive constant $\bar{c}$ such that $\bar{c} < H(y_h)$ and such that X is deformable to a point in $H^{\bar{c}} = \{x \in M / \ H(x) \geq \bar{c}\}$.

We then have,

Theorem 1.4 *Let $n \geq 1$. There exists a positive constant c_0 independent of H such that if H satisfies $(\mathbf{A_1})$, $(\mathbf{A_2})$, $(\mathbf{A_3})$ and $H(y_0)/\bar{c} \leq 1 + c_0$, then problem (P) has a solution.*

Theorem 1.5 *Assume that $n \geq 2$. Then, there exists a positive constant c_0 independent of H such that if H satisfies $(\mathbf{A_1'})$, $(\mathbf{A_2})$, $(\mathbf{A_3})$ and $H(y_0)/\bar{c} \leq 1 + c_0$, then problem (P) has a solution.*

Remark 1.6
i) The assumption $n \geq 2$ in Theorem 1.5 is needed in order to make $(\mathbf{A_1'})$ meaningful.
ii) The assumption $H(y_0)/\bar{c} \leq 1 + c_0$ allows basically to perform a single-bubble analysis.
iii) To see how to construct an example of a function H satisfying our assumptions, we refer the reader to [4].

Please notice that the above theorems are the CR-analogue of existence results due to Aubin and Bahri in the Riemannian case, see please [4].

The remainder of the paper is organized as follows. In section two, we set up the variational structure and we recall some known facts. In section three, we perform an expansion of the Euler functional near the sets of its potential critical points at infinity consisting of one single mass. Then, we prove a Morse lemma at infinity, which allows us to refine the expansion of the functional in section four. While, section five is devoted to proof our results. The proofs require some technical lemmas, which, for the convenience of the reader, are established in the Appendix.

2 Variational structure and some known facts

In this section we recall the functional setting and the variational problem associated to (P). We will also recall some useful previous results.

Problem (P) has a variational structure, the functional being

$$J(u) = \frac{\int_M Lu\, u\, \theta \wedge d\theta^n}{\left(\int_M H\, u^{\frac{2(n+1)}{n}}\, \theta \wedge d\theta^n\right)^{\frac{n}{n+1}}},$$

defined on the unit sphere of $\mathcal{S}_1^2(M)$ equipped with the norm

$$||u||^2 = \int_M Lu\, u\, \theta \wedge d\theta^n, \tag{2.1}$$

where $\mathcal{S}_1^2(M)$ is the Folland-Stein space (see [15] for the definition).

Problem (P) is equivalent to finding the critical points of J subjected to the constraint $u \in \Sigma^+$, where

$$\Sigma^+ = \{u \in \Sigma \,/\, u \geq 0\}, \quad \Sigma = \{u \in \mathcal{S}_1^2(M)/\ ||u|| = 1\} \tag{2.2}$$

The Palais-Smale condition fails to be satisfied for J on Σ^+. To characterize the sequences failing the Palais-Smale condition, we need to fix some notations and constructions.

Since M is compact and locally CR equivalent to $\mathbb{S}^{2n+1}$, any point a in M has a neighborhood $U_a \supset B(a,r)$, r is independent of a, where CR normal coordinates are defined, and such that the contact form of M is conformal to the standard contact form θ_0 of the Heisenberg group $\mathbb{H}^n$; that is there exists a positive function $\tilde{u}_a$ on $B(a,r)$ such that $\theta_0 = \tilde{u}_a^{\frac{2}{n}}\theta$, ($\tilde{u}_a$ smoothly dependent on a). Let $u_a(x) = w_a(x)\tilde{u}_a(x)$, where $w_a(x) = \chi(|x|)$, χ is a cut-off function $\chi : \mathbb{R} \to [0,1]$ defined by

$$\chi(t) = 1 \quad \text{if} \quad 0 \leq t \leq r/2; \quad \chi(t) = 0 \quad \text{if} \quad t \geq r$$

and $|x| = |\exp_a^{-1}(x)|_{\mathbb{H}^n}$, where, letting $(z,t) = \exp_a^{-1}(x)$, $\exp_a$ being the parabolic exponential map based at a, then $|(z,t)|_{\mathbb{H}^n} = (|z|^4 + t^2)^{\frac{1}{4}}$ is the norm of the Heisenberg group $\mathbb{H}^n$ (one can see [20], [21]).

Let λ be a large positive parameter. We introduce on $B(a,r)$ the function

$$\delta_{(a,\lambda)}(x) = c_n \lambda^n |1 + \lambda^2(|z|^2 - it)|^{-n}, \tag{2.3}$$

and the constant c_n is chosen such that the following equation is satisfied

$$L_{\theta_0}\delta_{(a,\lambda)} = \delta_{(a,\lambda)}^{1+2/n} \quad \text{on} \quad B(a,r).$$

Let

$$\delta'_{(a,\lambda)}(x) = \begin{cases} u_a \delta_{(a,\lambda)}(x) & \text{in} \quad B(a,r) \\ 0 \quad \text{in} \quad B(a,r)^c. \end{cases} \tag{2.4}$$

We define a family of "almost solutions" $\tilde{\delta}_{(a,\lambda)}$ to be the unique solution of

$$L\tilde{\delta}_{(a,\lambda)}(x) = \left(\delta'_{(a,\lambda)}(x)\right)^{1+2/n} \quad \text{in } M.$$

Now, for $\varepsilon > 0$ and $p \in \mathbb{N}^*$, let us define

$$V(p,\varepsilon) = \Big\{ u \in \Sigma / \exists a_1, ..., a_p \in M, \exists \lambda_1, ..., \lambda_p > 0, \exists \alpha_1, ..., \alpha_p > 0 \text{ s.t. } \|u - \sum_{i=1}^{p} \alpha_i \tilde{\delta}_{(a_i,\lambda_i)}\| < \varepsilon,$$
$$|\frac{\alpha_i^{2/n} H(a_i)}{\alpha_j^{2/n} H(a_j)} - 1| < \varepsilon, \quad \varepsilon_{ij} < \varepsilon \quad \lambda_i > \varepsilon^{-1} \Big\},$$

where $\varepsilon_{ij}^{-1} = \Big(\lambda_i/\lambda_j + \lambda_j/\lambda_i + \lambda_i\lambda_j d(a_i,a_j)^2\Big)^n$, and $d(x,y) = |exp_x^{-1}(y)|_{\mathbb{H}^n}$ if x and y are in a small ball of M of radius r, and $d(x,y)$ is equal to $\frac{r}{2}$ otherwise.

The failure of Palais-Smale condition can be described, following the ideas introduced in [12] [23] [25], as follows:

Proposition 2.1 *Assume that J has no critical point in Σ^+ and let $(u_k) \in \Sigma^+$ be a sequence such that $J(u_k)$ is bounded and $J'(u_k) \to 0$. Then, there exist an integer $p \in \mathbb{N}^*$, a sequence $\varepsilon_k > 0$ ($\varepsilon_k \to 0$) and an extracted subsequence of u_k, again denoted (u_k), such that $u_k \in V(p, \varepsilon_k)$.*

If a function u belongs to $V(p,\varepsilon)$, we consider the following minimization problem for $u \in V(p,\varepsilon)$ with ε small

$$\min\{||u - \sum_{i=1}^{p} \alpha_i \tilde{\delta}_{(a_i,\lambda_i)}||, \ \alpha_i > 0, \ \lambda_i > 0, \ a_i \in M\}. \tag{2.5}$$

We then have the following proposition which defines a parameterization of the set $V(p,\varepsilon)$. It follows from corresponding statements in [6], [8].

Proposition 2.2 *For any $p \in \mathbb{N}^*$, there is $\varepsilon_p > 0$ such that if $\varepsilon < \varepsilon_p$ and $u \in V(p,\varepsilon)$, the minimization problem (2.5) has a unique solution (up to permutation). In particular, we can write $u \in V(p,\varepsilon)$ as follows*

$$u = \textstyle\sum_{i=1}^{p} \bar{\alpha}_i \tilde{\delta}_{\bar{a}_i,\bar{\lambda}_i} + v,$$

where $(\bar{\alpha}_1, ..., \bar{\alpha}_p, \bar{a}_1, ..., \bar{a}_p, \bar{\lambda}_1, ..., \bar{\lambda}_p)$ is the solution of (2.5) and $v \in \mathcal{S}_1^2(M)$ such that

$$(V_0) \qquad < v, \psi >= 0 \text{ for all } \psi \in \left\{ \tilde{\delta}_i, \frac{\partial \tilde{\delta}_i}{\partial \lambda_i}, \frac{\partial \tilde{\delta}_i}{\partial a_i}, \text{ for } i = 1, \dots, p \right\}.$$

Here, $< \, , \, >$ denotes the L-scalar product defined on $\mathcal{S}_1^2(M)$ by

$$< u,\, v >= \int_M \Delta_\theta u\, v\, \theta \wedge d\theta^n + \frac{n}{2(n+1)} \int_M \mathcal{R}_\theta u\, v\, \theta\wedge d\theta^n = \int_M Lu\, v\, \theta \wedge d\theta^n. \tag{2.6}$$

We will also use the CR gradient ∇_θ (or the subelliptic gradient) which can be defined by

$$\int_M \nabla_\theta u\, \nabla_\theta v\, \theta\wedge d\theta^n = \int_M \Delta_\theta u\, v\, \theta \wedge d\theta^n. \tag{2.7}$$

In the sequel, ∂J designates the gradient of J with respect to the L-scalar product $< \, , \, >$, that is $\forall\, u,\, v \in \mathcal{S}_1^2(M)$, we have $< \partial J(u),\, v >= J'(u)\, v$.

3 Expansion of the Functional at infinity

In this section, we perform a useful expansion of the functional J near a single potential critical point at infinity, that is, when we are in a $V(p, \varepsilon)$ with $p = 1$.

Proposition 3.1 *Let $n \geq 2$. There exists $\varepsilon_0 > 0$ such that for any $u = \alpha\tilde{\delta}_{(a,\lambda)} + v$ in $V(1, \varepsilon)$, $\varepsilon \leq \varepsilon_0$, v satisfying the condition*

$$(V_0) \;\; < v, \tilde{\delta}_1 >=< v, \partial\tilde{\delta}_1/\partial\lambda_1 >=< v, \partial\tilde{\delta}_1/\partial a_1 >= 0\,,$$

we have

$$J(u) = \frac{S_n^{\frac{n}{n+1}}}{H(a)^{\frac{n}{n+1}}} \Big[1 - \frac{n}{n+1} \frac{\bar{c}}{\alpha^{2\frac{n+1}{n}} H(a) S_n^n} \frac{\Delta_\theta H(a)}{\lambda^2} + O\Big(\frac{1}{\lambda^2}\Big) - f(v) + Q(v) + O\Big(\|v\|^{\inf(3, 2\frac{(n+1)}{n})}\Big)\Big]$$

where

$$Q(v) = \frac{\|v\|^2}{\alpha^2 S_n^n} - (1 + \frac{2}{n}) \frac{1}{\alpha^{2\frac{n+1}{n}} H(a) S_n^n} \int_M H(\alpha\tilde{\delta})^{\frac{2}{n}}.v^2$$

$$f(v) = \frac{2}{\alpha^{2\frac{n+1}{n}} H(a) S_n^n} \int_M H(\alpha\tilde{\delta})^{(1+2/n)}\, v.$$

Here, $\bar{c}$ is a positive constant defined in (6.9) in the Appendix, and S_n is the Sobolev constant given by the formulae

$$S_n^n = \int_{\mathbb{H}^n} \frac{1}{|1 + |z|^2 - it|^{2n+2}} \theta_0 \wedge d\theta_0^n.$$

Proof. Let

$$J(u) = \frac{\int_M Lu\, u\, \theta \wedge d\theta^n}{\left(\int_M H\, u^{\frac{2(n+1)}{n}}\, \theta \wedge d\theta^n\right)^{\frac{n}{n+1}}} = \frac{N}{D}$$

where $u = \alpha\tilde{\delta}_{(a,\lambda)} + v$.

In order to simplify the notations, we will write in the sequel $\tilde{\delta}$ instead of $\tilde{\delta}_{a,\lambda}$. Since v satisfies (V_0), we have

$$N = \|u\|^2 = \alpha^2\|\tilde{\delta}\|^2 + \|v\|^2$$

Observe that, using (ii) of Lemma 4 in [17]

$$\|\tilde{\delta}_{(a,\lambda)}\|^2 = \int_M L\tilde{\delta}_{(a,\lambda)}\,\tilde{\delta}_{(a,\lambda)}\,\theta\wedge d\theta^n = S_n^n + O(\frac{1}{\lambda^{2n}}).$$

Thus,

$$N = \alpha^2 S_n^n\Big[1 + \frac{1}{\alpha^2 S_n^n}\|v\|^2 + O(\frac{1}{\lambda^{2n}})\Big]. \tag{3.1}$$

For the denominator, we write

$$\begin{aligned} D^{\frac{n+1}{n}} = \int_M H\Big(\alpha\tilde{\delta}+v\Big)^{2\frac{(n+1)}{n}}\theta\wedge d\theta^n &= \int_M H(\alpha\tilde{\delta})^{2\frac{(n+1)}{n}} + 2\frac{(n+1)}{n}\int_M H(\alpha\tilde{\delta})^{1+2/n}.v \\ &+ 2(n+1)\frac{(n+2)}{n^2}\int_M H(\alpha\tilde{\delta})^{\frac{2}{n}}.v^2 \\ &+ O\Big(\int_M(\alpha\tilde{\delta})^{\frac{2-n}{n}}\inf\big[(\alpha\tilde{\delta}),\|v\|\big]^3 + \int_M|v|^{2\frac{n+1}{n}}\Big). \end{aligned}$$

It is easy to check that

$$\int_M(\alpha\tilde{\delta})^{\frac{2-n}{n}}\inf\big[(\alpha\tilde{\delta}),\|v\|\big]^3 + \int_M|v|^{2\frac{n+1}{n}} = O\Big(\|v\|^{\inf(3,2\frac{n+1}{n})}\Big).$$

Using then Lemma 6.1, we have

$$\int_M H\tilde{\delta}^{2\frac{n+1}{n}} = H(a)S_n^n + \bar{c}\frac{\Delta_\theta H(a)}{\lambda^2} + O(\frac{1}{\lambda^2})$$

Thus,

$$\begin{aligned} D^{\frac{n+1}{n}} = \alpha^{2\frac{(n+1)}{n}}H(a)S_n^n\Big[1 &+ \frac{\bar{c}\Delta_\theta H(a)}{S_n^n\alpha^{2\frac{(n+1)}{n}}H(a)\lambda^2} + \frac{2(n+1)}{nS_n^n\alpha^{2\frac{(n+1)}{n}}H(a)}\int_M H(\alpha\tilde{\delta})^{1+2/n}.v \\ &+ \frac{2(n+1)(n+2)}{n^2S_n^n\alpha^{2\frac{(n+1)}{n}}H(a)}\int_M H(\alpha\tilde{\delta})^{\frac{2}{n}}.v^2 + O\Big(\|v\|^{\inf(3,2\frac{n+1}{n})}\Big) + O\Big(\frac{1}{\lambda^2}\Big)\Big]. \end{aligned} \tag{3.2}$$

Combining (3.1) and (3.2), we easily derive our proposition. □

One of the basic phenomenon displayed by the above expansion is the behavior of the functional J with respect to v. We will prove the existence of a unique $\bar{v}$ which minimizes $J(\alpha\tilde{\delta}_{(a,\lambda)}+v)$ with respect to $v \in E_\varepsilon(a,\lambda)$, where

$$E_\varepsilon(a,\lambda)=\Big\{v\in\mathcal{S}_1^2(M)/\ \|v\|<\varepsilon \text{ and } v \text{ satisfies } (V_0)\Big\}.$$

Notice that, (see [17]), for $\varepsilon>0$ very small, there exists $\alpha_0>0$ such that, for all $v\in E_\varepsilon(a,\lambda)$

$$Q(v)\geq\alpha_0\|v\|^2\,.$$

Proposition 3.2 *There exists a C^1 map that associates to each $\alpha\tilde{\delta}_{(a,\lambda)}\in V(1,\varepsilon)$, with small ε, $\bar{v}=\bar{v}(\alpha,a,\lambda)$ such that $\bar{v}$ is unique and minimizes $J(\alpha\tilde{\delta}_{(a,\lambda)}+v)$ with respect to $v\in E_\varepsilon(a,\lambda)$. Moreover, we have the following estimate*

$$\|\bar{v}\|\leq c\left(\frac{|\nabla_\theta H(a)|}{\lambda}+\frac{1}{\lambda^2}\right).$$

Proof. We expand ∂J along a variation h in the v-space $E_\varepsilon(a,\lambda)$ (that is h is a variation with respect to v with fixed (α,a,λ)). Since Q is definite, positive, and lower bounded on $E_\varepsilon(a,\lambda)$, there exists a continuous self adjoint, positive and invertible operator A such that $Q(v)=\frac{1}{2}<Av,\,v>$ on $E_\varepsilon(a,\lambda)$. Therefore, as in [17], we derive that there exists a unique $\bar{v}$ which minimizes $J(\alpha\tilde{\delta}_{(a,\lambda)}+v)$, i.e $-f+A\bar{v}+o(\|\bar{v}\|^{\inf(3,1+2/n)})=0$. Setting then $\bar{v}=A^{-1}(f)+o(1)$, we obtain

$$\|\bar{v}\|\leq c\|A^{-1}(f)\|\leq c\|f\|.$$

Thus, it is sufficient to estimate $\|f\|$ where f is defined in Proposition 3.1 We have

$$f(v)=\frac{2}{\alpha^{2\frac{n+1}{n}}H(a)S_n^n}\int_M H(\alpha\tilde{\delta})^{1+2/n}\,v.$$

Expanding H around a (see Lemma 6.1) and using Hölder's inequality, we obtain

$$\|f(v)\|\leq c\|v\|\left(\frac{|\nabla_\theta H(a)|}{\lambda}+\frac{1}{\lambda^2}\right).$$

Thus, the estimate on $\|f\|$ follows. □

Now, since $\bar{v}$ is a minimizer, we have

$$-<f,\bar{v}>+Q(\bar{v})+o(\|\bar{v}\|^{\inf(3,2\frac{n+1}{n})})=0.$$

Hence,

$$-<f,v>+Q(v)+o(\|v\|^{\inf(3,2\frac{n+1}{n})})=Q(v-\bar{v})+o(\|\bar{v}\|^{\inf(3,2\frac{n+1}{n})}).$$

We then have,

Proposition 3.3 *Let $n \geq 2$. There exists $\varepsilon_0 > 0$ such that for any $u = \alpha\tilde{\delta}_{(a,\lambda)} + v$, where $v \in E_\varepsilon(a,\lambda)$, $(\varepsilon \leq \varepsilon_0)$, we have*

$$J(u) = \frac{S_n^{\frac{n}{n+1}}}{H(a)^{\frac{n}{n+1}}}\left(1 - \frac{n}{n+1}\frac{\bar{c}}{\alpha^{2\frac{n+1}{n}}H(a)S_n^n}\frac{\Delta_\theta H(a)}{\lambda^2} + Q(v-\bar{v}) + O\left(\|\bar{v}\|^{\inf(3,2\frac{n+1}{n})}\right) + O(\frac{1}{\lambda^2})\right).$$

4 Morse lemma at infinity

The following Morse lemma at infinity establishes in $V(1,\varepsilon)$ a change of the variables (α, a, λ, v) into $(\tilde{\alpha}, \tilde{a}, \tilde{\lambda}, V)$, $(\tilde{\alpha} = \alpha)$, where V is a variable completely independent of $\tilde{a}$ and $\tilde{\lambda}$, and such that $J(\alpha\tilde{\delta}_{(a,\lambda)} + v)$ behaves like $J(\alpha\tilde{\delta}_{(\tilde{a},\tilde{\lambda})}) + \|V\|^2$. Namely, we prove the following result

Proposition 4.1 *For $\varepsilon > 0$ small enough, there is a diffeomorphism $\xi : V(1,\varepsilon) \longrightarrow V(1,\varepsilon')$ for some $\varepsilon' > 0$ with $\xi\left(\alpha\tilde{\delta}_{(a,\lambda)} + \bar{v}\right) = \alpha\tilde{\delta}_{(\tilde{a},\tilde{\lambda})}$, such that*

$$J\left(\alpha\tilde{\delta}_{(a,\lambda)} + v\right) = J\left(\alpha\tilde{\delta}_{(\tilde{a},\tilde{\lambda})}\right) + \|V\|^2,$$

where V is a variable independent of $\tilde{a}$ and $\tilde{\lambda}$, belonging to a neighborhood of zero in a fixed Hilbert space, and orthogonal to $\tilde{\delta}_{\tilde{a},\tilde{\lambda}}$, $\frac{\partial\tilde{\delta}_{\tilde{a},\tilde{\lambda}}}{\partial\tilde{\lambda}}$, $\frac{\partial\tilde{\delta}_{\tilde{a},\tilde{\lambda}}}{\partial\tilde{a}}$.

The proof of Proposition 4.1 requires some technical results that will be established later on. We begin the proof of the Morse lemma at infinity by isolating the contribution of $v - \bar{v}$.

Lemma 4.2 *For any $\alpha_0\tilde{\delta}_{(a_0,\lambda_0)} \in V(1,\varepsilon)$, there is a neighborhood U of $(\alpha_0, a_0, \lambda_0)$ such that,*

$$J\left(\alpha\tilde{\delta}_{(a,\lambda)} + v\right) = J\left(\alpha\tilde{\delta}_{(a,\lambda)} + \bar{v}(\alpha,a,\lambda)\right) + \frac{1}{2}J''\left(\alpha_0\tilde{\delta}_{(a_0,\lambda_0)} + \bar{v}(\alpha_0,a_0,\lambda_0)\right)V.V$$

for any $\alpha\tilde{\delta}_{(a,\lambda)} + v \in V(1,\varepsilon)$ with $(\alpha,a,\lambda) \in U$, where $V = V(\alpha,a,\lambda,v)$ is a C^1-diffeomorphism whose range is orthogonal to $\left\{\tilde{\delta}_{a',\lambda'}, \frac{\partial\tilde{\delta}_{a',\lambda'}}{\partial\lambda'}, \frac{\partial\tilde{\delta}_{a',\lambda'}}{\partial a'}, \quad (\alpha',a',\lambda') \in U\right\}$ and $\|V\| = O\left(\|v\|\right)$.

The proof is similar to the one given for the Scalar curvature problem on closed manifolds (one can see [11] for the sake of completeness).
We introduce now the following proposition

Proposition 4.3 *Let $n \geq 2$. There exists a pseudo-gradient Z so that the following holds: there is a constant $c > 0$ independent of $u = \alpha\tilde{\delta}_{(a,\lambda)} \in V(1,\varepsilon)$ such that,*

1. $< -\partial J(u), Z >\geq c(\frac{|\nabla_\theta H(a)|}{\lambda} + \frac{1}{\lambda^2})$

2. $< -\partial J(u+\overline{v}), Z + \frac{\partial \overline{v}}{\partial(\alpha,a,\lambda)}(Z) >\geq c(\frac{|\nabla_\theta H(a)|}{\lambda} + \frac{1}{\lambda^2})$

3. *Z is bounded*

4. *the only region where λ increases along the flow lines of Z is the region where a is near a critical point y of H with, $-\Delta_\theta H(y) > 0$.*

To prove Proposition 4.3, we need the following lemma,

Lemma 4.4 *Let $n \geq 2$. For $\varepsilon > 0$ small enough and $u = \alpha\tilde{\delta}_{(a,\lambda)} \in V(1,\varepsilon)$, the following expansions holds,*

$$< \partial J(u), \lambda\frac{\partial\tilde{\delta}}{\partial\lambda} > = \frac{n}{n+1}\bar{c}J(u)^{\frac{2n+1}{n}}\alpha^{1+2/n}\frac{\Delta_\theta H(a)}{\lambda^2} + O\Big(\frac{1}{\lambda^2}\Big)$$
$$< \partial J(u), \frac{1}{\lambda}\frac{\partial\tilde{\delta}}{\partial a} > = -c_1\bar{c}J(u)^{\frac{2n+1}{n}}\alpha^{1+2/n}\frac{|\nabla_\theta H(a)|}{\lambda} + O\Big(\frac{1}{\lambda^2}\Big)$$

Proof. We have,

$$< \partial J(u), \lambda\frac{\partial\tilde{\delta}}{\partial\lambda} >= J(u)\Big[< \alpha\tilde{\delta}, \lambda\frac{\partial\tilde{\delta}}{\partial\lambda} > -J(u)^{\frac{n+1}{n}}\int_M H(\alpha\tilde{\delta})^{1+2/n}\lambda\frac{\partial\tilde{\delta}}{\partial\lambda}\Big].$$

Using the estimates

$$< \tilde{\delta}, \lambda\frac{\partial\tilde{\delta}}{\partial\lambda} >= o(1/\lambda^2)$$
$$\int_M H\tilde{\delta}^{1+2/n}\lambda\frac{\partial\tilde{\delta}}{\partial\lambda} = -\frac{n}{n+1}\bar{c}\frac{\Delta_\theta H(a)}{\lambda^2} + O\Big(\frac{1}{\lambda^2}\Big)$$

we derive that,

$$< \partial J(u), \lambda\frac{\partial\tilde{\delta}}{\partial\lambda} >= J(u)\Big[J(u)^{\frac{n+1}{n}}\Big(\frac{n}{n+1}\bar{c}\frac{\Delta_\theta H(a)}{\lambda^2}\alpha^{1+2/n}\Big) + O\Big(\frac{1}{\lambda^2}\Big)\Big].$$

Thus, the first expansion follows. For the second, we have

$$< \partial J(u), \frac{1}{\lambda}\frac{\partial\tilde{\delta}}{\partial a} >= J(u)\Big[< \alpha\tilde{\delta}, \frac{1}{\lambda}\frac{\partial\tilde{\delta}}{\partial a} > -J(u)^{\frac{n+1}{n}}\int_M H(\alpha\tilde{\delta})^{1+2/n}\frac{1}{\lambda}\frac{\partial\tilde{\delta}}{\partial a}\Big].$$

Using the following estimates

$$< \tilde{\delta}, \frac{1}{\lambda}\frac{\partial\tilde{\delta}}{\partial a} >= o(1/\lambda^2).$$

$$\int_M H\tilde{\delta}^{1+2/n}\frac{1}{\lambda}\frac{\partial\tilde{\delta}}{\partial a}=c_1\bar{c}\frac{|\nabla_\theta H(a)|}{\lambda}+O\Big(\frac{1}{\lambda^2}\Big)$$

we derive that,

$$<\partial J(u),\frac{1}{\lambda}\frac{\partial\tilde{\delta}}{\partial a}>=J(u)\Big[-J(u)^{\frac{n+1}{n}}\Big(c_1\bar{c}\alpha^{1+2/n}\frac{|\nabla_\theta H(a)|}{\lambda}\Big)+O\Big(\frac{1}{\lambda^2}\Big)\Big].$$

The second expansion follows and the proof of Lemma 4.4 is thereby completed. □

We are now able to prove Proposition 4.3.

Proof of Proposition 4.3. Let $\rho>0$ be such that, for any critical point y of H, if $d(x,y)\leq 2\rho$ then $|\Delta_\theta H(x)|>c>0$. Three cases then may occur,

case 1: $d(a,y)>\rho$ for any critical point y. In this case we have, $|\nabla_\theta H(a)|>c>0$. Set

$$Z_1=\frac{1}{\lambda}\frac{\partial\tilde{\delta}}{\partial a}\frac{\nabla_\theta H(a)}{|\nabla_\theta H(a)|}.$$

From Lemma 4.4, we have

$$<-\partial J(u),Z_1>\geq c\frac{|\nabla_\theta H(a)|}{\lambda}+O\Big(\frac{1}{\lambda^2}\Big)\geq c\frac{|\nabla_\theta H(a)|}{\lambda}+\frac{c}{\lambda^2}.$$

case 2: $d(a,y)\leq 2\rho$ where y is a critical point of H with $-\Delta_\theta H(y)<0$. Set

$$Z_2=-\lambda\frac{\partial\tilde{\delta}}{\partial\lambda}+m\varphi(\lambda|\nabla_\theta H(a)|)Z_1$$

where, m is a small constant and φ is a C^∞ function which satisfies $\varphi(t)=1$ if $t\geq 2$ and $\varphi(t)=0$ if $t\leq 1$. Using Lemma 4.4, we derive that,

$$\begin{aligned}<-\partial J(u),Z_2>&\geq\frac{c}{\lambda^2}+cm\Big(\frac{|\nabla_\theta H(a)|}{\lambda}+O(\frac{1}{\lambda^2})\Big)\\&\geq c\frac{|\nabla_\theta H(a)|}{\lambda}+\frac{c}{\lambda^2}.\end{aligned}$$

case 3: $d(a,y)\leq 2\rho$ where y is a critical point of H with $-\Delta_\theta H(y)>0$. Set

$$Z_3=\lambda\frac{\partial\tilde{\delta}}{\partial\lambda}+m\varphi(\lambda|\nabla_\theta H(a)|)Z_1$$

We obtain the same equality as in case 2.
Hence, Z will be built as a convex combination of Z_1, Z_2 and Z_3. The proof of (1) is thereby completed. Claims (3) and (4) can be derived from the definition of Z. The claim (2) can be obtained using the claim (1) and arguing as in [6] and [11]. □

We will now give the following result which establishes our Morse lemma at infinity.

Lemma 4.5 *For any* $u = \alpha\tilde{\delta}_{(a,\lambda)} \in V(1,\varepsilon)$, *there is a change of variable*

$$(a,\lambda) \longrightarrow (\tilde{a},\tilde{\lambda})$$

such that,

$$J\Big(\alpha\tilde{\delta}_{(a,\lambda)} + \bar{v}(\alpha,a,\lambda)\Big) = J\Big(\alpha\tilde{\delta}_{(\tilde{a},\tilde{\lambda})}\Big),$$

with

(∗) $\frac{1}{\tilde{\lambda}^2} \longrightarrow 0 \Longleftrightarrow \frac{1}{\lambda^2} \longrightarrow 0$

(∗∗) $d(a,\,\tilde{a}) \longrightarrow 0$ *as* $\frac{1}{\lambda} \longrightarrow 0.$

Proof. Since $\bar{v}$ is a minimizer, we obtain

$$J\Big(\alpha\tilde{\delta}_{(a,\lambda)} + \bar{v}\Big) \leq J\Big(\alpha\tilde{\delta}_{(a,\lambda)}\Big).$$

Let h_s the 1-parameter group generated by Z, we have

$$\begin{cases} \frac{\partial}{\partial s} h_s\Big(\alpha\tilde{\delta}_{(a,\lambda)}\Big) & = & Z\Big(h_s(\alpha\tilde{\delta}_{(a,\lambda)})\Big), \\ h_0\Big(\alpha\tilde{\delta}_{(a,\lambda)}\Big) & = & \alpha\tilde{\delta}_{(a,\lambda)}. \end{cases}$$

By Proposition 4.3, $J(h_s)$ is a decreasing function of s. Using the fact that

$$J\Big(\alpha\tilde{\delta}_{(a,\lambda)} + \bar{v}\Big) \simeq J\Big(\alpha\tilde{\delta}_{(a,\lambda)}\Big) + o(1)$$

and since the flow line started from $\alpha\tilde{\delta}_{(a,\lambda)}$ which is not a critical point at infinity (a critical point at infinity occurs only when $\lambda = +\infty$, for more precision see definition below), the flow line should have been down the level $J\Big(\alpha\tilde{\delta}_{(a,\lambda)} + \bar{v}\Big)$.
Thus, there is at most one solution to the equation

$$J\Big(h_s\Big(\alpha\tilde{\delta}_{(a,\lambda)}\Big)\Big) = J\Big(\alpha\tilde{\delta}_{(a,\lambda)} + \bar{v}\Big) \tag{4.1}$$

The only case where there could be no solution to (4.1) is when the flow line exits from $V(1,\varepsilon_0)$ (ε_0 is defined in Proposition 3.1). We assume that $\alpha\tilde{\delta}_{(a,\lambda)}$ is in $V(1,\varepsilon)$ with $\varepsilon < \frac{\varepsilon_0}{2}$. Then the flow line will move from $\partial V(1,\frac{\varepsilon_0}{2})$ to $\partial V(1,\varepsilon_0)$, and during this traveling we have

$$-J'\Big(h_s(\alpha\tilde{\delta}_{(a,\lambda)})\Big) \geq c\Big(\frac{|\nabla_\theta H(a)|}{\lambda} + \frac{1}{\lambda^2}\Big) \geq c(\varepsilon_0) > 0$$

$$\text{distance}\Big(\partial V(1,\frac{\varepsilon_0}{2}),\ \partial V(1,\varepsilon_0)\Big) = a(\varepsilon_0) > 0\ , \quad \text{and} \quad |Z| \leq C.$$

If we denote by $\triangle s$ the time spent to travel from $\partial V(1, \frac{\varepsilon_0}{2})$ to $\partial V(1, \varepsilon_0)$, then we have $a(\varepsilon_0) \leq C \triangle s$. If we let $\gamma(\varepsilon_0) = \frac{c(\varepsilon_0)a(\varepsilon_0)}{C}$, then $J\Big(h_s(\alpha\tilde{\delta}_{(a,\lambda)})\Big)$ should have decreased at least of $\gamma(\varepsilon_0)$ during this crossing. Using Proposition 3.2 and Proposition 3.3, we derive that

$$J\Big(\alpha\tilde{\delta}_{(a,\lambda)}\Big) - J\Big(\alpha\tilde{\delta}_{(a,\lambda)} + \bar{v}\Big) \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0.$$

Hence, choosing ε small enough, we have

$$J\Big(\alpha\tilde{\delta}_{(a,\lambda)} + \bar{v}\Big) > J\Big(\alpha\tilde{\delta}_{(a,\lambda)}\Big) - \gamma(\varepsilon_0)$$

and therefore, equation (4.1) possesses a unique solution. Next, we are going to show that $\frac{1}{\tilde{\lambda}^2} \longrightarrow 0$ as $\frac{1}{\lambda^2} \longrightarrow 0$ and conversely, and $d(a, \tilde{a}) \longrightarrow 0$ under the above condition. Set

$$\alpha\tilde{\delta}(s) = \alpha\tilde{\delta}_{(a(s),\lambda(s))} = h_s\Big(\alpha\tilde{\delta}_{(a,\lambda)}\Big).$$

Since Z has no action on the variable α, we have

$$Z = \alpha\frac{1}{\lambda(s)}\frac{\partial\tilde{\delta}(s)}{\partial a}\Big(\lambda(s)\dot{a}(s)\Big) + \alpha\lambda(s)\frac{\partial\tilde{\delta}(s)}{\partial\lambda}\Big(\frac{\dot{\lambda}(s)}{\lambda(s)}\Big),$$

where $\dot{a}(s)$ and $\dot{\lambda}(s)$ denote the action of Z on the variables a and λ. We have $\frac{1}{\lambda}\frac{\partial\tilde{\delta}(s)}{\partial a}$ and $\lambda\frac{\partial\tilde{\delta}(s)}{\partial\lambda}$ are nearly orthogonal, are both of order $O(\tilde{\delta}_{(a,\lambda)})$, and such that

$$0 < C_1 \leq \Big\|\tfrac{1}{\lambda}\tfrac{\partial\tilde{\delta}(s)}{\partial a}\Big\| \leq C\,, \quad \text{and} \quad 0 < C_1 \leq \Big\|\lambda\tfrac{\partial\tilde{\delta}(s)}{\partial\lambda}\Big\| \leq C.$$

Since Z is bounded, it yields that $|\lambda\dot{a}| + |\frac{\dot{\lambda}}{\lambda}| \leq C'$, which establishes $(*)$.
On the other hand, we have $|\dot{a}(s)| \leq \frac{C'}{\lambda(s)} < \frac{C'e^{cs}}{\lambda(0)}$, thus $d(a, a(s)) \leq C's\frac{C'e^{cs}}{\lambda(0)}$. Since s satisfies equation (4.1), it is bounded. We derive then $(**)$ and the proof of Lemma 4.5 is thereby complete. □

Proof of Proposition 4.1. The proof of Proposition 4.1 follows from Lemma 4.2 and Lemma 4.5. □

Moreover, arguing as in [6] and [11], the expansion of J given by Proposition 4.1 can be improved when the concentration point is near a critical point y of H, with $-\Delta_\theta H(y) > 0$, leading to the following normal form.

Proposition 4.6 *Let $n \geq 2$. There is another change of variable*

$$(\tilde{a}, \tilde{\lambda}) \longrightarrow (\bar{a}, \bar{\lambda})$$

such that,

$$J(\alpha\tilde{\delta}_{(\tilde{a},\tilde{\lambda})}) = \frac{S_n^{\frac{n}{n+1}}}{\Big(H(\bar{a})^{\frac{n}{n+1}}\Big)}\Big(1 - \frac{(\bar{c} - \eta)}{\bar{\lambda}^2}\Delta_\theta H(y)\Big),$$

where η is a small positive constant.

Next, we derive from the above results, the characterization of the critical points at infinity in $V(1,\varepsilon)$. We recall that critical points at infinity are orbits of the gradient flow that remain in $V(p,\varepsilon(s))$, where $\varepsilon(s)$ is some function which tends to zero when the flow parameter s tends to $+\infty$ (see [5]).

Proposition 4.7 *Let $n \geq 1$. Assume that J does not have any critical point. Then, the only critical points at infinity of J in $V(1,\varepsilon)$ for ε small enough, correspond to $\tilde{\delta}_{(y,\infty)}$, where y is a critical point of H in $\mathcal{I}_+$.* ($\mathcal{I}_+$ is defined in (1.1)).

Proof. The proof is completed for $n = 1$ see [17]. For $n \geq 2$, using Propositions 4.3, we know that the only region where λ increases along the pseudo-gradient Z defined in Proposition 4.3, is the region where the concentration point a is near a critical point y of H such that $-\Delta_\theta H(y) > 0$. Proposition 4.6 yields a splitting of the variables a and λ, thus it is easy to see that if $a = y$, only λ can move. To decrease the functional J, we have to increase λ, thus we obtain a critical point at infinity only in this case and our result follows. □

5 Proofs of main results

Proof of Theorem 1.1. We argue by contradiction. We assume that J has no critical points in $V_\eta(\Sigma^+)$, where

$$V_\eta(\Sigma^+) = \{u \in \Sigma / \ \|u^-\| \leq \eta\} \tag{5.1}$$

where η is a small positive constant and $u^- = \max(0, -u)$ denotes the negative part of u. Let

$$c_\infty(y_0, y_I) = S_1^{\frac{1}{2}} \left(\frac{1}{H(y_0)} + \frac{1}{H(y_I)} \right)^{\frac{1}{2}}$$

We observe that under the assumption (H_1) of Theorem 1.1, the flow lines of the pseudo-gradient W defined in section 4 and Lemma 5.2 of [17] satisfies the Palais-Smale condition in $V(p,\varepsilon)$ for $p \geq 2$. Thus, the critical points at infinity of our variational problem are in $V(1,\varepsilon)$.

Using Proposition 4.7 and the assumption (H_3) of Theorem 1.1, it follows that the only critical points at infinity of J under the level $c_I = c_\infty(y_0, y_I) + \varepsilon$, for ε small enough, are $\tilde{\delta}_{(y_j,\infty)}$, $0 \leq j \leq I$ and $y_j \in \mathcal{I}_+$. The unstable manifolds at infinity of such critical points at infinity, $W_u(y_j)_\infty$, can be described using the expansion given by Lemma 5.3 of [17], as the product of $W_s(y_j)$, (for a pseudogradient of H) by $[A, +\infty[$ domain of the variable λ, for some positive number A large enough.

Since J has no critical point, it follows that $J_{c_I} = \{u \in \Sigma^+ / J(u) \leq c_I\}$ retracts by deformation on

$$X_{I\infty} = \bigcup_{\substack{0\leq j\leq I \\ y_j\in\mathcal{I}_+}} \overline{W_u(y_j)}$$

(see Sections 7 and 8 of [10]) which can be parameterized by $X_I \times [A, +\infty[$ where X_I is defined by (1.2).
We now claim that $X_{I\infty}$ is contractible in J_{c_I}. Indeed, let a_1, $a_2 \in M$, α_1, $\alpha_2 > 0$ and λ large enough. For $u = \alpha_1\tilde{\delta}_{(a_1,\lambda)} + \alpha_2\tilde{\delta}_{(a_2,\lambda)}$, we have the following expansion

$$J\Big(\frac{u}{||u||}\Big) \leq S_1^{\frac{1}{2}}\Big(\frac{1}{H(a_1)} + \frac{1}{H(a_2)}\Big)^{\frac{1}{2}}(1 + o(1)). \tag{5.2}$$

Let $h : [0, 1] \times X_I \times [A, +\infty[\longrightarrow \Sigma^+$ defined by

$$(t, x, \lambda) \longmapsto \frac{t\tilde{\delta}_{(y_0,\lambda)} + (1-t)\tilde{\delta}_{(x,\lambda)}}{||t\tilde{\delta}_{(y_0,\lambda)} + (1-t)\tilde{\delta}_{(x,\lambda)}||}$$

h is continuous and satisfies

$$h(0, x, \lambda) = \frac{\tilde{\delta}_{(x,\lambda)}}{||\tilde{\delta}_{(x,\lambda)}||} \quad \text{and} \quad h(1, x, \lambda) = \frac{\tilde{\delta}_{(y_0,\lambda)}}{||\tilde{\delta}_{(y_0,\lambda)}||}.$$

In addition, since $H(x) \geq H(y_I)$ for any $x \in X_I$, it follows from (5.2) that $J(h(t, x, \lambda)) < c_I$, for each $(t, x, \lambda) \in [0, 1]\times X_I\times[A, +\infty[$. Thus, the contraction h is performed under the level c_I. We derive that $X_{I\infty}$ is contractible in J_{c_I}, which retracts by deformation on $X_{I\infty}$, therefore $X_{I\infty}$ is contractible leading to the contractibility of X_I, which is a contradiction with the assumption $(\mathbf{H_2})$. Hence there exists a critical point of J in $V_\eta(\Sigma^+)$. Arguing as in [17], we prove that such critical point is positive. Now, we are going to show that such a critical point has a Morse index $\geq m_I$.

Arguing by contradiction, we assume that the Morse index is $\leq m_I - 1$. Perturbing, if necessary J, we may assume that all the critical points of J are nondegenerate and have their Morse index $\leq m_I - 1$. Such critical points do not change the homological group in dimension m_I of level sets of J.
Now, let $c_\infty(y_I) = S_1^{\frac{1}{2}} H(y_I)^{-\frac{1}{2}}$ and let ε be a small positive real. Since $X_{I\infty}$ defines a homological class in dimension m_I which is trivial in J_{c_I}, but not trivial in $J_{c_\infty(y_I)+\varepsilon}$, our result follows. □

Proof of Corollary 1.3. We recall that H has only non degenerate critical points $y_0, y_1, \dots, y_N$ such that $H(y_0) \geq H(y_1) \geq \cdots \geq H(y_N)$. For $i = N$ we have $X_i = \bigcup_{y_j\in\mathcal{I}_+} W_s(y_j)$.
Let $\chi(X_i)$ be the Euler-poincaré characteristic of X_i. We have $\chi(X_i) = \sum_{y_j\in\mathcal{I}_+} (-1)^{3-ind(H,y_i)}$.
Since $\chi(X_i) \neq 1$, we derive that X_i is not contractible. Hence, the result follow from Theorem 1.1. □

Proof of Theorem 1.4. Arguing by contradiction, we may assume that J has no critical points in $V_\eta(\Sigma^+)$.
Under the assumptions of Theorem 1.4 and according to Proposition 4.7, we see that the critical points at infinity of J under the level $c_1 = \frac{S_n^{\frac{n}{n+1}}}{H(y_h)^{\frac{n}{n+1}}} + \varepsilon$, for ε small enough, are in one to one correspondence with the critical points of H in $\mathcal{I}_+$, i.e. $y_0, \dots, y_h$. The unstable manifold of such critical points at infinity, $W_u(y_0)_\infty, \dots, W_u(y_h)_\infty$ can be described, using Proposition 4.6 for $n \geq 2$ and Lemma 5.3 of [17] for $n = 1$, as a product of $W_s(y_0), \dots, W_s(y_h)$ by $[A, +\infty[$, domain of the variable λ, for some positive number A large enough.
Since J has no critical points in $V_\eta(\Sigma^+)$, it follows that

$$J_{c_1} = \{u \in V_\eta(\Sigma^+)/\ J(u) < c_1\}$$

retracts by deformation onto $X_\infty = \cup_{0 \leq j \leq h} W_u(y_j)_\infty$, (see section 7 and 8 of [10]) which can be parameterized by $X \times [A, +\infty[$ where X is defined by (1.4).
Furthermore, we claim that X_∞ is contractible in $J_{c_2+\varepsilon}$, where $c_2 = \frac{S_n^{\frac{n}{n+1}}}{(\bar{c})^{\frac{n}{n+1}}}$ and $\bar{c}$ is given in assumption $(\mathbf{A_3})$ of the Theorem. Indeed, from the assumption $(\mathbf{A_3})$, it follows that there exists a continuous contraction $h : [0,1] \times X \longrightarrow H^{\bar{c}}$, such that for any $a \in X$, we have $h(0,a) = a$ and $h(1,a) = a_0$, a point of X. Such a contraction gives rise to the following contraction,

$$\begin{aligned} \tilde{h} :\ X_\infty &\longrightarrow V_\eta(\Sigma^+) \\ [0,1] \times X \times [A, \infty[\ni (t,a,\lambda) &\longmapsto \tilde{\delta}_{(h(t,a),\lambda)} + \bar{v}. \end{aligned}$$

For $t = 0$, we have $\tilde{\delta}_{(h(0,a),\lambda)} + \bar{v} = \tilde{\delta}_{(a,\lambda)} + \bar{v} \in X_\infty$. Also, $\tilde{h}$ is continuous and $\tilde{h}(1,a,\lambda) = \tilde{\delta}_{(a_0,\lambda)} + \bar{v}$, hence our claim follows.
From Proposition 4.6, we deduce that

$$J(\tilde{\delta}_{(h(t,a),\lambda)} + \bar{v}) \sim \frac{S_n^{\frac{n}{n+1}}}{\Big(H(h(t,a))\Big)^{\frac{n}{n+1}}} \Big(1 + O(A^{-2})\Big),$$

where $H(h(t,a)) \geq \bar{c}$ by construction. Therefore, such a contraction is performed below the level $c_2 + \varepsilon$ (for A large enough), so X_∞ is contractible in $J_{c_2+\varepsilon}$. Furthermore, choosing c_0 small enough, we see that there is no critical point at infinity for J between the levels $c_2 + \varepsilon$ and c_1. Thus, $Jc_2 + \varepsilon$ retracts by deformation onto J_{c_1}, which in turn retracts by deformation onto X_∞. Therefore, X_∞ is contractible leading to the contractibility of X, which is in contradiction with our assumption.

Therefore J has a critical point u_0 in $V_\eta(\Sigma^+)$. Now, we claim that such critical point is a positive function, when η is small enough. Otherwise, let $u_0 = u_0^+ - u_0^-$. Multiplying equation (P) by u_0^- and integrating, we obtain

$$\|u_0^-\|^2 \le c|u_0^-|^{2\frac{n+1}{n}}_{L^{2\frac{n+1}{n}}(M)} \le c'\|u_0^-\|^{2\frac{n+1}{n}}$$

(since H is bounded on M by a positive constant). Hence, either $u_0^- = 0$ or $\|u_0^-\| \ge c''$, where $c'' > 0$ and this case cannot occur since by the definition of the neighborhood of Σ^+, this norm is small. Therefore, $u_0^- = 0$ and $u_0 > 0$. This completes the proof of Theorem 1.4. □

Proof of Theorem 1.5. Arguing by contradiction, we assume that J has no critical points in the set $V_\eta(\Sigma^+)$ (defined in (5.1)). Let $\{z_1, \dots, z_r\} \subset \{y_1, \dots, y_h\}$ be the critical points of H with,

$$-\Delta_\theta H(z_j) \le 0, \qquad \forall\, 1 \le j \le r.$$

The idea of the proof of Theorem 1.5 is to perturb the function H in the C^1 sense in some neighborhoods of $z_1, \dots, z_r$ such that the new function $\tilde{H}$ has the same critical points than H with the same Morse index but satisfying that $-\Delta_\theta \tilde{H}(z_j) > 0, \forall\, 1 \le j \le r$.
The new set $\tilde{X}$ associated to $\tilde{H}$, defined in the assumption $(\mathbf{A_2})$, is also not contractible and its homology group in dimension m is nontrivial.
Under the level $c_2 + \varepsilon$, where c_2 is defined in the proof of Theorem 1.4. The functional $\tilde{J}$ may have other critical points, however a careful choice of $\tilde{H}$ ensures that all these critical points have their Morse indexes less than $m-2$, and so they do not change the homology in dimension m. Therefore, the arguments used in the proof of Theorem 1.4, lead to a contradiction. It follows that Theorem 1.5 will be as a consequence of the following proposition. □

Proposition 5.1 *There exists a function $\tilde{H}$ close to H in the C^1 sense such that $\tilde{H}$ has the same critical points than H with the same Morse indexes and such that,*

(i) $-\Delta_\theta \tilde{H}(z_i) > 0$ for $1 \le j \le r$
(ii) $-\Delta_\theta \tilde{H}(y_i) > 0$ for $i \in \{0, \dots, h\} \setminus \{1, \dots, r\}$
(iii) $-\Delta_\theta \tilde{H}(y_i) < 0$ for $h+1 \le j \le s$
(iv) *if $\tilde{J}$ has critical points under the level $c_2 + \varepsilon$, then their Morse indices are less than $m-2$, where m is defined in assumption $(\mathbf{A_2})$*
(v) *the new set $\tilde{X}$ associated to $\tilde{H}$, defined in analogy to assumption $(\mathbf{A_2})$, is not contractible and its homology group in dimension m is nontrivial.*

In order to prove Proposition 5.1, we need the following Lemmas. Their proofs are given in the Appendix.

Lemma 5.2 *Let $P = P(z, \lambda)$ be the orthogonal projection from $\mathcal{S}_1^2(M)$ equipped with the scalar product defined in (2.6) onto the vector subspace generated by $\tilde{\delta}_{(z,\lambda)}$, $\frac{\partial \tilde{\delta}_{(z,\lambda)}}{\partial \lambda}$ and $\frac{\partial \tilde{\delta}_{(z,\lambda)}}{\partial z}$. Then, we have the following estimates*

(i) $\|J'(\tilde{\delta}_{(z,\lambda)})\| = O(\frac{1}{\lambda})$
(ii) $\|\frac{\partial P}{\partial z}\| = O(\lambda)$
(iii) $\|\frac{\partial^2 P}{\partial z^2}\| = O(\lambda^2)$.

Lemma 5.3 *Let z_0 be a point of M close to a critical point of H, and let $\bar{v} = \bar{v}(\alpha, z_0, \lambda) \in E_\varepsilon(z_0, \lambda)$ defined in Proposition 3.2. Then, we have the following estimates*

$$(i)\ \ \|\bar{v}\| = o(\frac{1}{\lambda}), \qquad (ii)\ \ \|\frac{\partial \bar{v}}{\partial z}\| = o(1).$$

$$(iii)\ \ \frac{\partial^2}{\partial z^2}\tilde{J}(\tilde{\delta}_{(z,\lambda)} + \bar{v}) = \frac{\partial^2}{\partial z^2}\tilde{J}(\tilde{\delta}_{(z,\lambda)}) + o(1).$$

Proof of Proposition 5.1. We suppose that J has no critical points in $V_\eta(\Sigma^+)$ and we perturb the function H only in some neighborhood of $z_1, \dots, z_r$. Therefore, claims **(ii)** and **(iii)** follow from the assumption $(\mathbf{A_1'})$. Let $u_0 = \tilde{\delta}_{(z_0,\lambda)} + v$ be a critical point of $\tilde{J}$. We notice that under the level $c_2 + \varepsilon$ and outside $V(1, \varepsilon_0)$, we have $\|\partial J\| > c > 0$. If $\tilde{H}$ is close to H in the C^1 sense, then $\tilde{J}$ is close to J in the C^1 sense and therefore $\|\partial \tilde{J}\| > c/2$ in this region. Since u_0 is critical, it is optimal in all directions, including the v-direction, thus we must have $u_0 = \tilde{\delta}_{(z_0,\lambda)} + \bar{v}$. Now, using Lemma 4.4, we derive that

$$0 =< \partial\tilde{J}(u_0), \frac{1}{\lambda}\frac{\partial \delta}{\partial z} >= -c\frac{|\nabla_\theta \tilde{H}(z_0)|}{\lambda} + O(\frac{1}{\lambda^2}), \tag{5.3}$$

where c is a positive constant. Thus, z_0 has to be close to y_i where $i \in \{0, \dots, \ell\}$. Using again Lemma 4.4, we have

$$0 =< \partial\tilde{J}(u_0), \lambda\frac{\partial \delta}{\partial \lambda} >= c\frac{\Delta_\theta \tilde{H}(z_0)}{\lambda^2} + O(\frac{1}{\lambda^2}). \tag{5.4}$$

In the neighborhood of y_i with $i \in \{\, k \,/\, -\Delta_\theta H(y_k) > 0\} \cup \{\ell+1, \dots, s\}$, we have $\tilde{H} \equiv H$ and therefore, $\Delta_\theta H(y_k) > c > 0$ in this neighborhood. Thus, (5.4) implies that z_0 has to be near z_i with $1 \le i \le r$, where z_i's are the critical points among $y_1, \dots, y_\ell$ with a nonnegative value of $\Delta_\theta H$.
In order to compute the Morse index of $\tilde{J}$ at u_0, we need to compute $\frac{\partial^2}{\partial z^2}\tilde{J}(\tilde{\delta}_{(z,\lambda)} + \bar{v})_{z=z_0}$. Using the third claim of Lemma 5.3 , it is sufficient to estimate $\frac{\partial^2}{\partial z^2}\tilde{J}(\tilde{\delta}_{(z,\lambda)})$. We have

$$\tilde{J}(\tilde{\delta}_{(z,\lambda)}) = \frac{\|\tilde{\delta}_{(z,\lambda)}\|^2}{\left(\int_M \tilde{H}(x)\tilde{\delta}_{(z,\lambda)}^{2\frac{n+1}{n}}(x)\theta \wedge d\theta^n\right)^{\frac{n}{n+1}}}.$$

Standard estimates provided by Lemma 6.1 in the Appendix below, yields

$$\|\tilde{\delta}_{(z,\lambda)}\|^2 = \int_M L\tilde{\delta}_{(z,\lambda)}\,\tilde{\delta}_{(z,\lambda)}\,\theta \wedge d\theta^n = S_n^n + O(\frac{1}{\lambda^{2n}})$$

and

$$\int_M H\tilde{\delta}^{2\frac{n+1}{n}}_{(z,\lambda)} = H(z)S_n^n + \bar{c}\frac{\Delta_\theta H(z)}{\lambda^2} + O(\frac{1}{\lambda^2}).$$

We have therefore

$$\tilde{J}(\tilde{\delta}_{(z,\lambda)}) = \frac{S_n^{\frac{n}{n+1}}}{\tilde{H}(z)^{\frac{n}{n+1}}}\big(1 + O(\frac{1}{\lambda^2})\big).$$

Then,

$$\frac{\partial}{\partial z}\tilde{J}(\tilde{\delta}_{(z,\lambda)}) = -\tfrac{n}{n+1}S_n^{\frac{n}{n+1}}\frac{D\tilde{H}(z)}{\tilde{H}(z)^{\frac{2n+1}{n+1}}}\big(1 + O(\frac{1}{\lambda^2})\big)$$

and then

$$\frac{\partial^2}{\partial z^2}\tilde{J}(\tilde{\delta}_{(z,\lambda)}) = -\tfrac{n}{n+1}\frac{S_n^{\frac{n}{n+1}}}{\tilde{H}(z)^{\frac{2n+1}{n+1}}}D^2\tilde{H}(z)\big(1 + O(\frac{1}{\lambda^2})\big) + O\bigg(\frac{\big|D\tilde{H}(z)\big|^2}{\tilde{H}(z)^{\frac{3n+2}{n+1}}}\bigg).$$

Thus, if z_0 is close to a critical point, the second term is $o(1)$, and then we have, with $c > 0$:

$$\frac{\partial^2}{\partial z^2}\tilde{J}(\tilde{\delta}_{(z_0,\lambda)}) = -c\, D^2\tilde{H}(z_0) + o(1).$$

Without loss of generality, we can assume that z_0 is close to z_1, and thus, that they are in the same CR normal coordinates chart. We can also assume that $D^2H = D^2H(z_1) + o(1)$ in $B(z_1,\rho)$ and $D^2H(z_1)$ is diagonal, where ρ is a small fixed positive constant. We notice that $D^2H(z_1)$ possesses some negative eigenvalues in $B(z_1,\rho)$. Using the diagonal form of $D^2H(z_1)$, we can obtain a function $\tilde{H}$, if we decrease the negative eigenvalues of $D^2H(z_1)$ in $B(z_1,\rho)$, such that $-\Delta_\theta\tilde{H}(z_1) > 0$ and that $\tilde{H}$ has only z_1 as a critical point in $B(z_1,\rho)$. Indeed let (z^α, t) be pseudohermitian normal coordinates for θ centered at z_0, $1 \leqslant \alpha \leqslant n$. We write $Z_\alpha := \frac{\partial}{\partial z^\alpha} + i\overline{z^\alpha}\frac{\partial}{\partial t}$ and $Z_{\overline{\alpha}} := \frac{\partial}{\partial z^{\overline{\alpha}}} - iz^\alpha\frac{\partial}{\partial t}$ in these coordinates, and we set $\mathcal{L}_0 := \frac{1}{2}(Z_\alpha Z_{\overline{\alpha}} + Z_{\overline{\alpha}} Z_\alpha)$. It holds then that

$$\Delta_\theta = \mathcal{L}_0 + \mathcal{O}_m,$$

where $\mathcal{O}_m$ is homogeneous of arbitrary order m. See please [21].

Using the above construction, we will bring back the negative eigenvalues of $D^2H(z_1)$ to their initial values on $\partial B(z_1,\rho)$. The Morse index of H at z_1 is greater than $2n - m + 4$. Since ρ is fixed, the Morse index of H at z_0 is equal to the number of negative eigenvalues of $D^2\tilde{H}(z_0)$ which is the same as that of $D^2\tilde{H}(z_1)$. Thus, the contribution of the variable z to the Morse index of $\tilde{J}$ is less than or equal to $m-3$. Taking into account the contribution of λ, we derive **(i)** and **(vi)**.

On the other hand, the assumption $(\mathbf{A_1'})$ implies that,

$$(2n+1) - (m-3) \leq ind(H, z_j) \leq ind(\tilde{H}, z_j) \qquad \text{for } 1 \leq j \leq r.$$

Thus, for any pseudo-gradient of $\tilde{H}$, the dimension of the stable manifolds of z_j is less than $m-3$. Note that, our perturbation changes the pseudo-gradient Z to $\tilde{Z}$, but only in some neighborhoods of $z_1, \dots, z_r$. Therefore, the stable manifolds of y_i, for $i \notin \{1, \dots, r\}$, remains unchanged. Since the dimension of X is greater than m, and its homology group in dimension m is nontrivial, we derive that the homology group of $\tilde{X}$ in dimension m is also nontrivial. This completes the proof of Proposition 5.2. □

6 Appendix

In this section, we collect some technical results used in the proof of the Theorems.

Proof of Lemma 5.2. The proof of **(i)** is easy, so we will omit it. In order to prove claim **(ii)**, let

$$\varphi \in \Big\{\tilde{\delta}_{(z,\lambda)}, \lambda\frac{\partial\tilde{\delta}_{(z,\lambda)}}{\partial\lambda}, \ \frac{1}{\lambda}\frac{\partial\tilde{\delta}_{(z,\lambda)}}{\partial z}\Big\}.$$

Observe that $P\varphi = \varphi$, then

$$\frac{\partial P}{\partial z}(\varphi) = \frac{\partial \varphi}{\partial z} - P\frac{\partial \varphi}{\partial z}$$

and so, $\|\frac{\partial P}{\partial z}(\varphi)\| = O(\lambda\|\varphi\|)$. Furthermore, for $v \in E_\varepsilon(z,\lambda)$, we have $Pv = 0$, then

$$\frac{\partial P}{\partial z}(v) = -P\frac{\partial v}{\partial z} = \sum_{i=1}^{2n+3} a_i\varphi_i$$

where, $\varphi_i = \frac{1}{\lambda}\frac{\partial\tilde{\delta}_{(z,\lambda)}}{\partial z^i}$ for $1\leqslant i\leqslant 2n+1$, $\varphi_{2n+2} = \tilde{\delta}_{(z,\lambda)}$, $\varphi_{2n+3} = \lambda\frac{\partial\tilde{\delta}_{(z,\lambda)}}{\partial\lambda}$, and z^i are the coordinates of z in a suitable local chart. But we have,

$$a_i\|\varphi_i\|^2 = - < \frac{\partial v}{\partial z}, \varphi_i >=< v, \frac{\partial \varphi_i}{\partial z} >= O(\lambda\|v\|).$$

Thus, the claim **(ii)** follows. In the same way, we can prove claim **(iii)** and thus the proof is competed. □

Proof of Lemma 5.3. The first estimate follows directly from Proposition 3.2, so we need to prove claim *(ii)*. Let $\bar{v} = \bar{v}(\alpha, z_0, \lambda) \in E_\varepsilon(z_0,\lambda)$ be defined in Proposition 3.2. Then, $\bar{v}$ satisfies,

$$A\bar{v} = f + O(\|v\|^{1+2/n}),$$

where A is the operator associated to the quadratic form Q defined on $E_\varepsilon(z_0,\lambda)$ (Q and f are defined in Proposition 3.1). Differentiating this equation, we obtain

$$\frac{\partial A}{\partial z}\bar{v} + A\frac{\partial \bar{v}}{\partial z} = \frac{\partial f}{\partial z} + O\Big(\|\bar{v}\|^{\frac{2}{n}}\frac{\partial \bar{v}}{\partial z}\Big).$$

Then,

$$A\Big(\frac{\partial\bar v}{\partial z}-P\frac{\partial\bar v}{\partial z}\Big)=\frac{\partial f}{\partial z}-\frac{\partial A}{\partial z}\bar v-AP\frac{\partial\bar v}{\partial z}+O\Big(\|v\|^{\frac{2}{n}}\frac{\partial\bar v}{\partial z}\Big).$$

Using the positivity of the quadratic form Q, we derive that

$$\|\frac{\partial\bar v}{\partial z}-P\frac{\partial\bar v}{\partial z}\|\leq C\Big(\|\frac{\partial f}{\partial z}\|-\|\frac{\partial A}{\partial z}\|\|\bar v\|-\|P\frac{\partial\bar v}{\partial z}\|+\|v\|^{\frac{2}{n}}\|\frac{\partial\bar v}{\partial z}\|\Big).$$

In order to obtain claim (ii), we need to estimate the right-hand side of the last inequality. First, we have

$$\begin{aligned}<\frac{\partial f}{\partial z},v> &= c\int H\tilde\delta^{\frac{2}{n}}_{(z_0,\lambda)}\frac{\partial\tilde\delta}{\partial z}v\\ &= c|\nabla_\theta H(z_0)|\int d(z_0,z)\tilde\delta^{\frac{2}{n}}\frac{\partial\tilde\delta}{\partial z}v+O\Big(\int d^2(z_0,z)\tilde\delta^{1+2/n}\lambda|v|\Big)\\ &\leq c\|v\|\big(|\nabla_\theta H(z_0)|+\frac{1}{\lambda}\big).\end{aligned}$$

By the assumptions of the Lemma, we have that z_0 is close to a critical point of H, so we deduce that $\frac{\partial f}{\partial z}=o(1)$. On the other hand, observe that, $\|\frac{\partial A}{\partial z}\|=O(\lambda)$. Using claim (i), we deduce that $\|\frac{\partial A}{\partial z}\|\|\bar v\|=o(1)$. Also, using the fact that $\bar v\in E_\varepsilon(z_0,\lambda)$, we derive that

$$\begin{aligned}&<\frac{\partial\bar v}{\partial z},\tilde\delta_{(z_0,\lambda)}>=-<\bar v,\frac{\partial\tilde\delta_{(z_0,\lambda)}}{\partial z}>=0\\ &<\frac{\partial\bar v}{\partial z},\lambda\frac{\partial\tilde\delta_{(z_0,\lambda)}}{\partial\lambda}>=-<\bar v,,\lambda\frac{\partial^2\tilde\delta_{(z_0,\lambda)}}{\partial z\partial\lambda}>=O(\lambda\|\bar v\|)=o(1)\\ &<\frac{\partial\bar v}{\partial z},\frac{1}{\lambda}\frac{\partial\tilde\delta_{(z_0,\lambda)}}{\partial z}>=-<\bar v,\frac{1}{\lambda}\frac{\partial^2\tilde\delta_{(z_0,\lambda)}}{\partial z^2}>=o(1).\end{aligned}$$

Collecting those estimates, we deduce that, $\|P(\frac{\partial\bar v}{\partial z})\|=o(1)$. Finally, using the inequality

$$\|\frac{\partial\bar v}{\partial z}\|\leq\|\frac{\partial\bar v}{\partial z}-P\frac{\partial\bar v}{\partial z}\|+\|P\frac{\partial\bar v}{\partial z}\|,$$

the second claim follows. For the third claim, observe that

$$\frac{\partial}{\partial z}\tilde J(\tilde\delta_{(z,\lambda)}+\bar v)=\tilde J'(\tilde\delta_{(z,\lambda)}+\bar v)\frac{\partial}{\partial z}(\tilde\delta_{(z,\lambda)}+\bar v)=\tilde J'(\tilde\delta_{(z,\lambda)}+\bar v)P\Big(\frac{\partial}{\partial z}(\tilde\delta_{(z,\lambda)}+\bar v)\Big)$$

and

$$\frac{\partial^2}{\partial z^2}\tilde J(\tilde\delta_{(z,\lambda)}+\bar v)=\tilde J''(\tilde\delta_{(z,\lambda)}+\bar v)\frac{\partial}{\partial z}(\tilde\delta_{(z,\lambda)}+\bar v)P\Big(\frac{\partial}{\partial z}(\tilde\delta_{(z,\lambda)}+\bar v)\Big)+\tilde J'(\tilde\delta_{(z,\lambda)}+\bar v)\frac{\partial}{\partial z}\Big(P(\tilde\delta_{(z,\lambda)}+\bar v)\Big).\tag{6.1}$$

For $z = z_0$, we have $\tilde{J}'(\tilde{\delta}_{(z_0,\lambda)} + \bar{v}) = 0$. We estimate each term of (6.1). First, using the two first claims of this lemma, we deduce that

$$\tilde{J}''(\tilde{\delta}_{(z,\lambda)} + \bar{v})\frac{\partial(\bar{v})}{\partial z}P(\frac{\partial \bar{v}}{\partial z}) = o(1).$$

Secondly, we compute

$$\begin{aligned} T &= \tilde{J}''(\tilde{\delta}_{(z,\lambda)} + \bar{v})\frac{\partial \tilde{\delta}}{\partial z}P(\frac{\partial \bar{v}}{\partial z}) \\ &= c\Big[< \frac{\partial \tilde{\delta}}{\partial z}, P(\frac{\partial \bar{v}}{\partial z}) > - \frac{n}{n-2}\tilde{J}(u_0)^{\frac{n+1}{n}} \int \tilde{H}(\tilde{\delta} + \bar{v})^{\frac{2}{n}} \frac{\partial \tilde{\delta}}{\partial z}P(\frac{\partial \bar{v}}{\partial z})\Big]. \end{aligned}$$

According to Proposition 3.1, we have

$$\tilde{J}(\tilde{\delta} + \bar{v}) = \frac{S_n^{\frac{n}{n+1}}}{\tilde{H}(z)^{\frac{n}{n+1}}} + O\Big(\frac{\|\bar{v}\|}{\lambda} + \frac{1}{\lambda^2}\Big).$$

Thus,

$$\begin{aligned} T &= c\Big[< \frac{\partial \tilde{\delta}}{\partial z}, P(\frac{\partial \bar{v}}{\partial z}) > o(1) - (1+2/n)S_n^{\frac{n}{n+1}} \int \frac{\tilde{H}}{\tilde{H}(z)}\Big(\tilde{\delta}^{\frac{2}{n}} + O(\tilde{\delta}^{\frac{2}{n-2}}|\bar{v}| + |\bar{v}|^{\frac{2}{n}}\chi_{\tilde{\delta}\le|\bar{v}|})\Big)\frac{\partial \tilde{\delta}}{\partial z}P(\frac{\partial \bar{v}}{\partial z})\Big] \\ &= c(1+2/n)S_n^{\frac{n}{n+1}} \int \Big(1 - \frac{\tilde{H}}{\tilde{H}(z)}\Big)\tilde{\delta}^{\frac{2}{n}}\frac{\partial \tilde{\delta}}{\partial z}P(\frac{\partial \bar{v}}{\partial z}) + O\Big(\lambda\|\bar{v}\|\|\frac{\partial \bar{v}}{\partial z}\| + \lambda\|\bar{v}\|^{\frac{n+1}{n}}\|\frac{\partial \bar{v}}{\partial z}\Big) + o(1) \\ &= o(1). \end{aligned}$$

Then, (6.1) becomes

$$\begin{aligned} \frac{\partial^2}{\partial z^2}\tilde{J}(\tilde{\delta}_{(z,\lambda)} + \bar{v}) &= \tilde{J}''(\tilde{\delta}_{(z,\lambda)} + \bar{v})\frac{\partial \tilde{\delta}}{\partial z}\frac{\partial \tilde{\delta}}{\partial z} + \tilde{J}''(\tilde{\delta}_{(z,\lambda)} + \bar{v})\frac{\partial \bar{v}}{\partial z}\frac{\partial \tilde{\delta}}{\partial z} + o(1) \\ &= \tilde{J}''(\tilde{\delta}_{(z,\lambda)} + \bar{v})\frac{\partial \tilde{\delta}}{\partial z}\frac{\partial \tilde{\delta}}{\partial z} + \tilde{J}''(\tilde{\delta}_{(z,\lambda)})\frac{\partial \bar{v}}{\partial z}\frac{\partial \tilde{\delta}}{\partial z} + o(1) \\ &= \tilde{J}''(\tilde{\delta}_{(z,\lambda)})\frac{\partial \tilde{\delta}}{\partial z}\frac{\partial \tilde{\delta}}{\partial z} + \tilde{J}^{(3)}(\tilde{\delta}_{(z,\lambda)})\bar{v}\frac{\partial \tilde{\delta}}{\partial z}\frac{\partial \tilde{\delta}}{\partial z} + \tilde{J}''(\tilde{\delta}_{(z,\lambda)})\frac{\partial \bar{v}}{\partial z}\frac{\partial \tilde{\delta}}{\partial z} + o(1) \end{aligned}$$

and,

$$\frac{\partial^2}{\partial z^2}\tilde{J}(\tilde{\delta}_{(z,\lambda)} + \bar{v}) = \tilde{J}''(\tilde{\delta}_{(z,\lambda)})\frac{\partial \tilde{\delta}}{\partial z}\frac{\partial \tilde{\delta}}{\partial z} + \frac{\partial}{\partial z}\Big(\tilde{J}''(\tilde{\delta}_{(z,\lambda)})\bar{v}\frac{\partial \tilde{\delta}}{\partial z}\Big) - \tilde{J}''(\tilde{\delta}_{(z,\lambda)})\bar{v}\frac{\partial^2 \tilde{\delta}}{\partial z^2} + o(1). \quad (6.2)$$

Since we have,

$$0 = \tilde{J}'(\tilde{\delta}_{(z,\lambda)} + \bar{v})\frac{\partial^2 \tilde{\delta}}{\partial z^2} = \tilde{J}'(\tilde{\delta}_{(z,\lambda)})\frac{\partial^2 \tilde{\delta}}{\partial z^2} + \tilde{J}''(\tilde{\delta}_{(z,\lambda)})\bar{v}\frac{\partial^2 \tilde{\delta}}{\partial z^2} + o(1)$$

(6.2) becomes,

$$\frac{\partial^2}{\partial z^2}\tilde{J}(\tilde{\delta}_{(z,\lambda)}+\bar{v}) = \tilde{J}''(\tilde{\delta}_{(z,\lambda)})\frac{\partial\tilde{\delta}}{\partial z}\frac{\partial\tilde{\delta}}{\partial z} + \frac{\partial}{\partial z}\Big(\tilde{J}''(\tilde{\delta}_{(z,\lambda)})\bar{v}\frac{\partial\tilde{\delta}}{\partial z}\Big) + \tilde{J}'(\tilde{\delta}_{(z,\lambda)})\frac{\partial^2\tilde{\delta}}{\partial z^2} + o(1)$$
$$= \frac{\partial^2\tilde{\delta}}{\partial z^2}\Big(\tilde{J}(\tilde{\delta})\Big) + \frac{\partial}{\partial z}\Big(\tilde{J}''(\tilde{\delta}_{(z,\lambda)})\bar{v}\frac{\partial\tilde{\delta}}{\partial z}\Big) + o(1).$$

Observe that,

$$\tilde{J}''(\tilde{\delta}_{(z,\lambda)})\bar{v}\frac{\partial\tilde{\delta}}{\partial z} = -c(1+2/n)S_n^{\frac{n}{n+1}}\int\Big(1-\frac{\tilde{H}}{\tilde{H}(z)}\Big)\tilde{\delta}^{\frac{2}{n}}\frac{\partial\tilde{\delta}}{\partial z}.$$

Thus,

$$\frac{\partial}{\partial z}\Big(\tilde{J}''(\tilde{\delta}_{(z,\lambda)})\bar{v}\frac{\partial\tilde{\delta}}{\partial z}\Big) = -c(1+2/n)S_n^{\frac{n}{n+1}}\Big[\int\frac{-\tilde{H}}{\tilde{H}(z)}D\tilde{H}(z)\tilde{\delta}^{\frac{2}{n}}\bar{v}\frac{\partial\tilde{\delta}}{\partial z}$$
$$+\int\Big(\frac{\tilde{H}}{\tilde{H}(z)}-1\Big)\tilde{\delta}^{\frac{2}{n}}\bar{v}\frac{\partial^2\tilde{\delta}}{\partial z^2} + \frac{2}{n}\int\Big(\frac{\tilde{H}}{\tilde{H}(z)}-1\Big)\tilde{\delta}^{\frac{2-n}{n}}\bar{v}(\frac{\partial\tilde{\delta}}{\partial z})^2$$
$$+\int\Big(\frac{\tilde{H}}{\tilde{H}(z)}-1\Big)\tilde{\delta}^{\frac{2}{n}}\frac{\partial\bar{v}}{\partial z}\frac{\partial\tilde{\delta}}{\partial z}\Big]$$
$$= O\Big(\lambda\|\bar{v}\| + \|\frac{\partial\bar{v}}{\partial z}\|\Big)$$
$$= o(1).$$

Collecting all the estimates, we finally obtain,

$$\frac{\partial^2}{\partial z^2}\tilde{J}(\tilde{\delta}_{(z,\lambda)}+\bar{v}) = \frac{\partial^2}{\partial z^2}\tilde{J}(\tilde{\delta}_{(z,\lambda)}) + o(1).$$

Our claim follows and the proof of Lemma 5.3 is thereby completed. □

Now we are going to prove the following technical Lemma:

Lemma 6.1 *Assume $n \geqslant 2$. Let $a \in M$ and $\lambda > 0$ very large. There exists a constant $\bar{c} > 0$ such that*

$$\int_M H\tilde{\delta}_{(a,\lambda)}^{2+\frac{2}{n}} = H(a)S_n^n + \bar{c}\frac{\Delta_\theta H(a)}{\lambda^2} + o\big(\frac{1}{\lambda^2}\big)+O\big(\frac{1}{\lambda^4}\big) + O\big(\frac{1}{\lambda^{2n}}\big) + O\big(\frac{1}{\lambda^{2n+2}}\big).$$

Proof. First, one have

$$\int_M H\tilde{\delta}_{(a,\lambda)}^{2+\frac{2}{n}} = H(a)\int_M \tilde{\delta}_{(a,\lambda)}^{2+\frac{2}{n}} + \int_M \big(H(x)-H(a)\big)\tilde{\delta}_{(a,\lambda)}^{2+\frac{2}{n}}. \tag{6.3}$$

Denoting then by B the ball $B(a,\frac{r}{2})$, we have

$$\int_M \big(H(x)-H(a)\big)\tilde{\delta}_{(a,\lambda)}^{2+\frac{2}{n}} = \int_B \big(H(x)-H(a)\big)\tilde{\delta}_{(a,\lambda)}^{2+\frac{2}{n}} + \int_{^cB} \big(H(x)-H(a)\big)\tilde{\delta}_{(a,\lambda)}^{2+\frac{2}{n}}.$$

Using (iii) of Lemma 4 and Lemma A.1 in [17], we get

$$\int_{^cB} \big(H(x)-H(a)\big)\tilde{\delta}_{(a,\lambda)}^{2+\frac{2}{n}} = O\big(\frac{1}{\lambda^{2n+2}}\big). \tag{6.4}$$

On the other hand, using Lemma 3 in [17], then (2.3)-(2.4), we derive that

$$\begin{aligned}\int_B \big(H(x)-H(a)\big)\tilde{\delta}_{(a,\lambda)}(x)^{2+\frac{2}{n}}\theta\wedge d\theta^n &= \int_B \big(H(x)-H(a)\big)\big(\delta'_{(a,\lambda)}(x)\big)^{2+\frac{2}{n}}\theta\wedge d\theta^n + O\big(\frac{1}{\lambda^{2n}}\big)\\ &= \int_{B(0,\rho')}\big(H(x)-H(a)\big)\delta_{(a,\lambda)}(x)^{2+\frac{2}{n}}\,\theta_0\wedge d\theta_0^n + O\big(\frac{1}{\lambda^{2n}}\big)\end{aligned}$$

where $x=\exp_a(z,t)$, $B(0,\rho')=\exp_a^{-1}B$, $\delta_{(a,\lambda)}(x)=\lambda^n|1+\lambda^2(|z|^2-it)|^{-n}$, and where $(z,t)=(z(x),\,t(x))$ are pseudohermitian normal coordinates centered at a, i.e. such that $z(a)=0$, and $t(a)=0$. Let us denote by $\{Z_j,\,\bar{Z}_j,\,T\}$ the standard CR structure of the Heisenberg group $\mathbb{H}^n$, where $Z_j=\dfrac{\partial}{\partial z_j}+i\bar{z}_j\dfrac{\partial}{\partial t}$, $\bar{Z}_j=\dfrac{\partial}{\partial \bar{z}_j}-iz_j\dfrac{\partial}{\partial t}$, $(1\le j\le n)$, and $T=\dfrac{\partial}{\partial t}$.
Buy virtue of Lemma 3.10 in [21], the Taylor expansion of the function H around a at the second order is:

$$H(x)=H(a)+H_{(1)}(x)+H_{(2)}(x)+o(\rho^2)$$

where $\rho=\sqrt[4]{|z|^4+t^2}$, and $H_{(1)}(x)$, resp. $H_{(2)}(x)$, is the homogeneous part (in terms of the Heisenberg norm) of order 1, resp. 2, of this expansion; more precisely:

$$H_{(1)}(x)=\sum_{j=1}^n Z_jH(a).z_j+\bar{Z}_jH(a).\bar{z}_j$$

and

$$H_{(2)}(x)=T\,H(a).t+\tfrac{1}{2}\sum_{j,\,k=1}^n Z_j\bar{Z}_kH(a)z_j\bar{z}_k+\bar{Z}_jZ_kH(a)\bar{z}_jz_k+\bar{Z}_j\bar{Z}_kH(a)\bar{z}_j\bar{z}_k+Z_jZ_kH(a)z_jz_k.$$

From this we derive that:

$$\int_{B(0,\rho')}\big(H(x)-H(a)\big)\delta_{(a,\lambda)}^{2+\frac{2}{n}} = \int_{B(0,\rho')}\big(H_{(1)}(x)+H_{(2)}(x)+o(\rho^2)\big)\delta_{(a,\lambda)}^{2+\frac{2}{n}}. \tag{6.5}$$

A straightforward reckoning, shows to us that most of the integrals vanish by oddness, so that (6.5) becomes:

$$\int_{B(0,\rho')}\big(H(x)-H(a)\big)\delta_{(a,\lambda)}^{2+\frac{2}{n}} = \tfrac{1}{2}\int_{B(0,\rho')}\sum_{j=1}^{n}\big(Z_j\bar{Z}_j+\bar{Z}_jZ_j\big)H(a)|z_j|^2\,\delta_{(a,\lambda)}^{2+\frac{2}{n}} + \int_{B(0,\rho')}o(\rho^2)\,\delta_{(a,\lambda)}^{2+\frac{2}{n}}. \tag{6.6}$$

Using the results of Theorem 3.1 and Lemma 3.5 in [21], we derive the existence of a choice of contact form θ', such that, in a pseudohermitian normal coordinates chart centered at a, we have:

$$\Delta_{\theta'} = \Delta_\theta + O(\rho^{m-2})$$

for an arbitrary integer $m \geqslant 2$. Notice that we have also in the same chart

$$\Delta_{\theta'} = \tfrac{1}{2}\sum_{j=1}^{n}(Z_j\bar{Z}_j+\bar{Z}_jZ_j) + O(\rho^2).$$

Taking then $m \geqslant 4$, we derive that

$$\Delta_{\theta} = \tfrac{1}{2}\sum_{j=1}^{n}(Z_j\bar{Z}_j+\bar{Z}_jZ_j) + O(\rho^2).$$

Hence, equation (6.6) becomes:

$$\int_{B(0,\rho')}\big(H(x)-H(a)\big)\delta_{(a,\lambda)}^{2+\frac{2}{n}} = \int_{B(0,\rho')}\big(\Delta_\theta + O(\rho^2)\big)H(a)|z_j|^2\,\delta_{(a,\lambda)}^{2+\frac{2}{n}} + \int_{B(0,\rho')}o(\rho^2)\,\delta_{(a,\lambda)}^{2+\frac{2}{n}}. \tag{6.7}$$

Using now the change of variable $(z,t) \longmapsto \lambda(z,t) = (\lambda z, \lambda^2 t)$, equation (6.7) becomes:

$$\int_{B(0,\rho')}\big(H(x)-H(a)\big)\delta_{(a,\lambda)}^{2+\frac{2}{n}} = \bar{c}\frac{\Delta_\theta H(a)}{\lambda^2} + o\big(\frac{1}{\lambda^2}\big) + O\big(\frac{1}{\lambda^4}\big) + O\big(\frac{1}{\lambda^{2n+2}}\big) \tag{6.8}$$

where

$$\bar{c} = \int_{\mathbb{H}^n}|z_j|^2\delta_{(a,1)}^{2+\frac{2}{n}} \tag{6.9}$$

and where we used the following estimates:

$$\int_{^cB(0,\rho')}|z_j|^2\delta_{(a,\lambda)}^{2+\frac{2}{n}} = O\big(\frac{1}{\lambda^{2n}}\big),$$

$$\int_{B(0,\rho')}o(\rho^2)\delta_{(a,\lambda)}^{2+\frac{2}{n}} = o\big(\frac{1}{\lambda^{2}}\big),$$

$$\int_{B(0,\rho')}O(\rho^2)H(a)|z_j|^2\delta_{(a,\lambda)}^{2+\frac{2}{n}} = O\big(\frac{1}{\lambda^{4}}\big).$$

From another side, using (iii) of Lemma 4 in [17], we have

$$H(a)\int_M \tilde{\delta}_{(a,\lambda)}^{2+\frac{2}{n}} = H(a)S_n^n + O\big(\frac{1}{\lambda^{2n}}\big)\,. \tag{6.10}$$

Collecting then our estimates in (6.4), (6.8), (6.10), and inserting in (6.3), Lemma 6.1 follows. □

References

[1] A. Ambrosetti and M. Badiale, *Homoclinics : Poincaré-Melnikov type results via a variational approach,* Ann. Inst. H. Poincarré Anal. Nonlinéaire **15** (1998), 233-252.

[2] T. Aubin, *Some nonlinear problem in differential geometry*, Springer-Verlag, New York 1997.

[3] T. Aubin and A. Bahri, *Méthode de topologie algébrique pour le problème de la courbure scalaire prescrite*, J. Math. Pures Appl., 76, 1997, 525–549.

[4] T. Aubin and A. Bahri, *Une hypothèse topologique pour le problème de la courbure scalaire prescrite*, J. Math. Pures Appl. **76** (1997), 843-850.

[5] A. Bahri, Critical points at infinity in some variational problems, Pitman Res. Notes Math, Ser **182**, Longman Sci. Tech. Harlow 1989.

[6] A. Bahri, *An invariant for Yamabe-type flows with applications to scalar-curvature problems in high dimension*, A celebration of John F. Nash, Jr. Duke Math. J. **81** (1996),323-466.

[7] A. Bahri, *The scalar curvature problem on sphere of dimension* $n \geq 7$, preprint 1996.

[8] A. Bahri and J. M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of the topology of the domain,* Comm. Pure Appl. Math. **41** (1988), 255-294.

[9] A. Bahri and J. M. Coron, *The scalar curvature problem on the standard three dimensional spheres*, J. Funct. Anal. **95** (1991), 106-172.

[10] A. Bahri and P. Rabinowitz, *Periodic orbits of hamiltonian systems of three body type*, Ann. Inst. H. Poincaré Anal. Non linéaire **8** (1991), 561-649.

[11] M. Ben Ayed, Y. Chen, H. Chtioui and M. Hammami, *On the prescribed scalar curvature problem on 4-manifolds*, Duke Math. J. **84** (1996), 633-677.

[12] H. Brezis and J. M. Coron, *Convergence of solutions of H-systems or how to blow bubbles,* Arch. Rational Mech. Anal. **89** (1985), 21-56.

[13] H. Chtioui, M. Ould Ahmedou and R. Yacoub, *Existence and Multiplicity Results for the prescribed Webster scalar curvature on 3-dimensional CR manifolds*, to appear.

[14] V. Felli and F. Uguzzoni, *Some existence results for the Webster scalar curvature problem in presence of symmetry,* Ann. Mat. Pura Appl. **183** (2004), 469-493.

[15] G.B. Folland and E. Stein, *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group,* Comm. Pure Appl. Math. **27** (1974), 429-522.

[16] N. Gamara, *The CR Yamabe conjecture, the case $n = 1$,* J. Eur. Math. Soc. **3** (2001), 105-137.

[17] N. Gamara, *The Prescribed scalar curvature on a* 3*-dimensional CR manifold,* Adv. Nonlinear Stud. **2** (2002), 193-235.

[18] N. Gamara and R. Yacoub, *CR Yamabe conjecture, the conformally flat case,* Pac. J. Math. **201** (2001), 121-175.

[19] N. Garofalo and E. Lanconelli, *Existence and nonexistence results for semilinear equations on the Heisenberg group,* Indiana Univ. Math. J. **41** (1992), 71-98.

[20] D. Jerison and J.M. Lee, *The Yamabe problem on CR manifolds,* J. Differential Geom. **25** (1987), 167-197.

[21] D. Jerison and J.M. Lee, *Intrinsic CR normal coordinates and the CR Yamabe problem,* J. Differential Geom. **29** (1989), 303-343.

[22] D. Jerison and J.M. Lee, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem,* J. Amer. Math. Soc. **1** (1988),1-13.

[23] P. L. Lions, *The concentration compactness principle in the calculus of variations. The limit case,* Rev. Mat. Iberoamericana **1** (1985), I: 145-201; II: 45-121.

[24] A. Malchiodi and F. Uguzzoni, *A perturbation result for the Webster scalar curvature problem on the CR sphere,* J. Math. Pures Appl. **81** (2002), 983-997.

[25] M. Struwe, *A global compactness result for elliptic boundary value problems involving nonlinearities,* Math. Z. **187** (1984), 511-517.

[26] N. Tanaka, A differentila geometric study on strongly pseuconvex manifolds, Kinokuniya, Tokyo, 1975.

[27] S. Webster, *Pseudohermitian structures on a real hypersurface*, J. Diff. Geometry **13** (1975), 25-41.

[28] R. Yacoub, *The Webster scalar curvature problem on high dimensional CR manifolds*, to appear.