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On an Elliptic Problem with Critical Nonlinearity in Expanding Annuli

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This paper is concerned with an elliptic problem with homogeneous boundary conditions and critical nonlinearity (P_ε): $-\Delta u = u^p$, $u > 0$ on A_ε, $u = 0$ on ∂A_ε, where $A_\varepsilon = \{x \in \mathbb{R}^n / \varepsilon < |x| < 1/\varepsilon\}$ are expanding annuli as $\varepsilon \to 0$, $n \geqslant 3$ and $p + 1 = 2n/(n-2)$ is the critical Sobolev exponent. We compute the difference of topology induced by the critical points at infinity between the level sets of the functional corresponding to (P_ε) for ε small enough.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In this paper, we consider the nonlinear elliptic problem

$$(\mathrm{P}) \begin{cases} -\Delta u = u^{(n+2)/(n-2)}, u > 0 & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded regular domain in $\mathbb{R}^n$ with $n \geqslant 3$.

(P) is interesting to study because of its resemblance to some variational problems in geometry and physics where some lack of compactness occurs. The most notorious example is the Yamabe problem in differential geometry, which consists of finding $u > 0$ satisfying

$$-\Delta u = u^{(n+2)/(n-2)} - \frac{n-2}{4(n-1)} R(x)\, u \qquad \text{on } M,$$

where M is a Riemannian manifold of dimension n without boundary and $R(x)$ is the scalar curvature (see, for example, [3, 10, 17]).

The problem (P) has a variational structure. The related functional is

$$J(u) = \frac{1}{\left(\int_\Omega |u|^{2n/(n-2)}\right)^{(n-2)/n}} \tag{1}$$

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defined on

$$\sum(\Omega)=\left\{u\in H_0^1(\Omega)\Big/\int_\Omega |\nabla u|^2=1\right\}. \tag{2}$$

The gradient of J is

$$\partial J(u)=2J(u)[u+(J(u))^{n/(n-2)}\,\Delta^{-1}\,(|u|^{4/(n-2)}\,u)]. \tag{3}$$

The positive critical points of J are solutions of (P) (up to a multiplicative constant).

Since the embedding of $H_0^1(\Omega)$ into $L^{2n/(n-2)}(\Omega)$ is not compact, the functional J does not satisfy the Palais–Smale condition (PS), that is, there exist sequences along which J remains bounded, its gradient goes to zero, and which do not converge.

The failure of the (PS) condition has been studied by many authors, including M. Struwe [16] and H. Brezis and J. M. Coron [8].

These studies lead to the following characterization of the sequences which in turn lead to the failure of the Palais–Smale condition. To describe this failure, we need some notations.

Let $P\delta(a,\lambda)$ be the orthogonal projection onto $H_0^1(\Omega)$ of the function

$$\delta(a,\lambda)(y)=\frac{\lambda^{(n-2)/2}}{(1+\lambda^2\,|y-a|^2)^{(n-2)/2}},\qquad a\in\Omega,\quad y\in\mathbb{R}^n,\quad \lambda>0 \tag{4}$$

defined by

$$\Delta P\delta(a,\lambda)=\Delta\delta(a,\lambda)\quad\text{on }\Omega;\qquad P\delta(a,\lambda)=0\quad\text{on }\partial\Omega. \tag{5}$$

Let, for $\eta>0$, $p\in\mathbb{N}^*$ and w either a solution of (P) or equal to zero,

$$\begin{aligned} V_\Omega(p,\eta,w)=\Big\{u\in\textstyle\sum/\exists(a_1,\dots,a_p)\in\Omega^p;\ \exists(\lambda_1,\dots,\lambda_p)\in\left]\frac{1}{\eta},+\infty\right[^p\\ \exists(\alpha_0,\dots,\alpha_p)\in\,]0,+\infty[^{p+1}\\ \text{s.t.}\left\|u-\alpha_0 w-\sum_{i=1}^p\alpha_i P\delta(a_i,\lambda_i)\right\|_{H_0^1(\Omega)}<\eta;\\ \lambda_i d(x_i,\partial\Omega)>\frac{1}{\eta}\ \forall i;\ \varepsilon_{ij}<\eta;\ \left|\frac{\alpha_i}{\alpha_j}-1\right|<\eta\Big\}, \end{aligned} \tag{6}$$

where

$$\varepsilon_{ij}=\left(\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j\,|a_i-a_j|^2\right)^{-(n-2)/2},\qquad i\neq j. \tag{7}$$

Let us denote

$$V_\Omega(p,\eta)=V_\Omega(p,\eta,0). \tag{8}$$

Observe that the functions of $V_\Omega(p,\eta,w)$ need not be positive; however, they are close to positive functions.

Let

$$S=\frac{1}{\int_{\mathbb{R}^n}\delta(a,\lambda)^{2n/(n-2)}} \tag{9}$$

and

$$b_p=(pS)^{2/n}; \qquad \Sigma^+=\{u\in\Sigma/u\geqslant 0\}. \tag{10}$$

The failure of the Palais–Smale condition can be described as follows.

PROPOSITION 1.1. *Let $u_k\in\Sigma^+$ be a sequence such that $\partial J(u_k)$ tends to zero and $J(u_k)$ is bounded. Then, after possibly extracting a subsequence, there exist $p\in\mathbb{N}$, a sequence $(\eta_k)_k$, $\eta_k>0$ and $\eta_k\to 0$ and w (either a solution of (P) or zero) such that $u_k\in V_\Omega(p,\eta_k,w)$.*

Conversely, let p be a positive integer, let (η_k) be a sequence in $(0,+\infty)$ with $\eta_k\to 0$, and let (u_k) be a sequence in Σ^+ such that $u_k\in V_\Omega(p,\eta_k)$ then $\partial J(u_k)\to 0$ and $J(u_k)\to b_p$.

The problem we are interested in is difficult from a variational viewpoint because of the failure of the (PS) condition, more precisely because of the possible existence of critical points at infinity, which are orbits of J along which J remains bounded, the gradient goes to zero, and the orbits do not converge (see [4, 5]).

Besides the characterization above, we have more precise information about the location of critical points and critical points at infinity.

PROPOSITION 1.2 ([7, 15]). *There are no critical points at infinity and no critical points in a neighborhood of $V_\Omega(p,\eta,w)$ when $w\neq 0$.*

A strategy to prove existence or nonexistence of solutions to (P) is to compute the difference of topology between the level sets of the functional J. However, as we said, critical points at infinity occur, so it becomes crucial to know the exact contribution of critical points at infinity to the relative topology between the level sets. This program was performed in [6] and [14].

To state this result, we need some notations.

We denote by G_Ω the Green's function of the Laplace operator defined by

$$\forall x \in \Omega \begin{cases} -\Delta G_\Omega(x, \cdot) = c_n \delta_x & \text{on } \Omega \\ G_\Omega(x, \cdot) = 0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

where δ_x is the Dirac mass at x and $c_n = (n-2)$ meas (S^{n-1}).

We denote by H_Ω the regular part of G_Ω, that is,

$$H_\Omega(x_1, x_2) = \frac{1}{|x_1 - x_2|^{n-2}} - G_\Omega(x_1, x_2) \qquad \text{for} \quad (x_1, x_2) \in \Omega^2. \tag{12}$$

For $p \in \mathbb{N}^*$ and $x = (x_1, ..., x_p) \in \Omega^P$ we set

$$M_\Omega(x) = (m_{i,j})_{1 \leqslant i, j \leqslant p} \in \mathscr{M}_p(\mathbb{R}) \tag{13}$$

with

$$m_{ii} = H_\Omega(x_i, x_i); \qquad m_{ij} = -G_\Omega(x_i, x_j), \qquad i \neq j \tag{14}$$

and define

$$\rho_\Omega(x) = \rho_\Omega(x_1, ..., x_p) \tag{15}$$

as the smallest eigenvalue of $M_\Omega(x)(\rho_\Omega(x) = -\infty$ if $x_i = x_j$, for some $i \neq j)$.

We introduce the set $I_p(\Omega)$

$$I_p(\Omega) = \{x = (x_1, ..., x_p) \in \Omega^p / \rho_\Omega(x) \leqslant 0\} \tag{16}$$

and, for $\eta > 0$, we set

$$J_{b_p \pm \eta} = \{u \in H_0^1(\Omega) / J(u) \leqslant b_p \pm \eta\}.$$

The topological contribution of critical points at infinity can be described as follows.

THEOREM 1.3 ([6, 14]). *Assume that zero is a regular value of ρ. For $\eta > 0$ small enough the relative homology $H*(J_{b_p+\eta} \cap V_\Omega(p, \eta), J_{b_p-\eta} \cap V_\Omega(p, \eta))$ is isomorphic to $H*(\Omega^p \times_{\sigma_p} D^{p-1}, \Omega^p \times S^{p-2} \cup_{\sigma_p} I_p \times D^{p-1})$, where the index σ_p means that we took the quotient of the space by the action of the symmetric group σ_p.*

Critical points at infinity contribute substantially, according to Theorem 1.3, to the differences of topology in the level sets of J. It is therefore clearly essential to know the behavior of I_p and its topology in order to prove further existence and multiplicity results for problems of type (P). In a previous work [2], we studied I_p for the case $p = 2$ and Ω an annulus;

we study it here for p arbitrary. Establishing such a result in the general case is both more difficult and more interesting. It provides us with the first explicit computation of the difference of topology induced by the critical points at infinity, derived from the general formula.

We now state our results, describe their qualitative meaning, and then proceed with the proof of these results. Before stating the results, we introduce some notations.

For $\varepsilon > 0$, let $A_\varepsilon = \{x \in \mathbb{R}^n / \varepsilon < |x| < 1/\varepsilon\}$, $n \geqslant 3$, we consider the nonlinear elliptic problem

$$(P_\varepsilon)\ \begin{cases} -\Delta u = u^{(n+2)/(n-2)},\ u > 0 & \text{on } A_\varepsilon, \\ u = 0 & \text{on } \partial A_\varepsilon. \end{cases}$$

Let

$$J_\varepsilon(u) = \frac{1}{(\int_{A_\varepsilon} |u|^{2n/(n-2)})^{(n-2)/n}}, \tag{17}$$

$$J_{\varepsilon, b_p \pm \eta} = \{u \in \Sigma / J_\varepsilon(u) \leqslant b_p \pm \eta\}. \tag{18}$$

Throughout this paper, for simplicity G_ε, H_ε, ρ_ε, $V_\varepsilon(p, \eta)$, $I_p(\varepsilon)$ denote G_{A_ε}, H_{A_ε}, ρ_{A_ε}, $V_{A_\varepsilon}(p, \eta)$, $I_p(A_\varepsilon)$ defined, replacing Ω by A_ε in (11), (12), (15), (8), and (16), respectively.

Our main results follow.

Proposition 1.4. *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, zero is a regular value of ρ_ε.*

Theorem 1.5. *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, $I_p(\varepsilon)$ is homotopically equivalent to A_ε^P.*

We derive, from Theorems 1.3 and 1.5, the topological contribution of the critical points at infinity.

Corollary 1.6. *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, the relative homology*

$$H_*(J_{\varepsilon, b_p+\eta} \cap V_\varepsilon(p, \eta), J_{\varepsilon, b_p-\eta} \cap V_\varepsilon(p, \eta)) = \begin{cases} 0 & \text{if } * \geqslant 1 \\ F & \text{if } * = 0, \end{cases}$$

for $\eta > 0$ small enough (F is some abelian group).

This result proves that, for expanding annuli ($\varepsilon \to 0$), and a given p, $I_p(\varepsilon)$ will topologically "stabilize" for ε small enough, and becomes homotopically equivalent to A_ε^p. Along a deformation of the domain, from a thin to a very large annulus for example, the difference of topology at infinity changes considerably, for a given p: I_2, for example, starts from the diagonal of $A_\varepsilon \times A_\varepsilon$, for a thin annulus (see [1]), and transforms into $A_\varepsilon \times A_\varepsilon$, topologically, for a large annulus (see [2] and Theorem 1.5). Since the variational theory, when completed with the critical points at infinity, holds, this can only occur if there is an exchange between the solutions of (P) and the critical points at infinity: some of the first type transform into critical points at infinity and vice-versa. An example of such a phenomenon is given by O. Rey in [13] (see also [9]). Our results provide the behavior of I_p at the two extremes of the "spectrum": a case where I_p is topologically very "rich" (for thin annulus) and there are many critical points at infinity. A case where I_p is topologically very "poor" (large annulus) and there are probably more solutions. We hope that the intermediate cases are interpolation between these two extreme cases.

The remainder of the paper is organized as follows: In Section 2, we give some careful estimates about the Green's function and its behavior; these results are of independent interest. In Sections 3 and 4, we give the proof of our results.

2. TECHNICAL LEMMAS

In this section, we use the following notation:

$$\Omega_\varepsilon = \left\{ x \in \mathbb{R}^n / 1 < |x| < \frac{1}{\varepsilon} \right\}, \qquad \varepsilon > 0,$$

$$W_\varepsilon = \{ x \in \mathbb{R}^n / \varepsilon < |x| < 1 \}, \qquad \varepsilon > 0,$$

$$\partial_i \Omega_\varepsilon = \partial B(0, 1) \text{: interior boundary of } \Omega_\varepsilon,$$

$$\partial_e \Omega_\varepsilon = \partial B \left(0, \frac{1}{\varepsilon} \right) \text{: exterior boundary of } \Omega_\varepsilon,$$

$$B_1 = B(0, 1), \qquad {}^c B_1 = \mathbb{R}^n \backslash B(0, 1),$$

$$d_1 = d(x_1, \partial_i \Omega_\varepsilon), \qquad x_1 \in \Omega_\varepsilon.$$

LEMMA 2.1. *Let $x_1 \in \Omega_\varepsilon$ such that $|x_1| \to c \geqslant 1$ when $\varepsilon \to 0$. Then, for any $y \in \Omega_\varepsilon$, we have*

$$\frac{\partial H_{\Omega_\varepsilon}}{\partial a}(x_1, y) = \frac{\partial H_{{}^c B_1}}{\partial a}(x_1, y) + (n-2) \frac{x_1}{|x_1|^n} \varepsilon^{n-2} \left(1 - \frac{1}{|y|^{n-2}} \right) + O(\varepsilon^{n-1})$$

where $\partial / \partial a$ is the derivative with respect to the first variable.

Proof. We have

$$\begin{cases} -\Delta_y\left(\dfrac{\partial H_{\Omega_\varepsilon}}{\partial a}(x_1, y) - \dfrac{\partial H_{^cB_1}}{\partial a}(x_1, y)\right) = 0 & \text{on } \Omega_\varepsilon, \\ \dfrac{\partial H_{\Omega_\varepsilon}}{\partial a}(x_1, y) - \dfrac{\partial H_{^cB_1}}{\partial a}(x_1, y) & \\ \qquad = \begin{cases} 0 & \text{on } \partial_i \Omega_\varepsilon, \\ -(n-2)\dfrac{(x_1 - y)}{|x_1 - y|^n} - \dfrac{\partial H_{^cB_1}}{\partial a}(x_1, y) & \text{on } \partial_e \Omega_\varepsilon. \end{cases} \end{cases}$$

According to [2], we have

$$\frac{\partial H_{^cB_1}}{\partial a}(x_1, y) = -(n-2)\frac{x_1}{|x_1|^n}\varepsilon^{n-2} + O(\varepsilon^{n-1}) \qquad \text{for } y \quad \text{s.t. } |y| = \frac{1}{\varepsilon}.$$

Hence, we have

$$\begin{cases} -\Delta_y\left(\dfrac{\partial H_{\Omega_\varepsilon}}{\partial a}(x_1, y) - \dfrac{\partial H_{^cB_1}}{\partial a}(x_1, y)\right) = 0 & \text{on } \Omega_\varepsilon, \\ \dfrac{\partial H_{\Omega_\varepsilon}}{\partial a}(x_1, y) - \dfrac{\partial H_{^cB_1}}{\partial a}(x_1, y) & \\ \qquad = \begin{cases} 0 & \text{on } \partial_i \Omega_\varepsilon, \\ (n-2)\dfrac{x_1 \varepsilon^{n-2}}{|x_1|^n} + O(\varepsilon^{n-1}) & \text{on } \partial_e \Omega_\varepsilon. \end{cases} \end{cases}$$

That is,

$$\begin{aligned} &\frac{\partial H_{\Omega_\varepsilon}}{\partial a}(x_1, y) - \frac{\partial H_{^cB_1}}{\partial a}(x_1, y) \\ &\qquad = (n-2)\frac{x_1}{|x_1|^n}\varepsilon^{n-2}\left(1 - \frac{1}{|y|^{n-2}}\right) + O(\varepsilon^{n-1}). \end{aligned}$$

Lemma 2.1 is thereby proved.

LEMMA 2.2. *Let $x_1 \in \Omega_\varepsilon$. There exist $\varepsilon_* > 0$, $d_* > 0$ such that for any $\varepsilon < \varepsilon_*$ and $d < d_*$. We have*

$$\frac{\partial H_{\Omega_\varepsilon}}{\partial n_1}(x_1, x_1) = \frac{n-2}{2^{n-2} d_1^{n-1}} + o\left(\frac{1}{d_1^{n-1}}\right),$$

where n_1 *is the outward normal to* $\partial_i \Omega_\varepsilon$ *and* x_1' *and* x_1' *is the only point of* $\partial_i \Omega_\varepsilon$ *such that* $d_1 = |x_1 - x_1'|$.

Proof. On the one hand, we have, from Lemma 2.1,

$$\frac{\partial H_{\Omega_\varepsilon}}{\partial x_1}(x_1, x_1)$$
$$= 2\frac{\partial H_{{}^cB_1}}{\partial a}(x_1, x_1) + 2(n-2)\frac{x_1}{|x_1|^n}\varepsilon^{n-2}\left(1 - \frac{1}{|x_1|^{n-2}}\right) + O(\varepsilon^{n-1}).$$

On the other hand, we have

$$H_{{}^cB_1}(x_1, y) = |\tilde{x}_1|^{n-2}\,|\tilde{y}|^{n-2}\,H_{B_1}(\tilde{x}_1, \tilde{y}),$$

where

$$\tilde{x}_1 = \frac{x_1}{|x_1|^2} \qquad \text{and} \qquad \tilde{y} = \frac{y}{|y|^2}.$$

Hence

$$\frac{\partial H_{{}^cB_1}}{\partial \tilde{x}_1}(x_1, y) = (n-2)\,\tilde{x}_1\,|\tilde{x}_1|^{n-4}\,|\tilde{y}|^{n-2}\,H_{B_1}(\tilde{x}_1, \tilde{y})$$
$$+ |\tilde{x}_1|^{n-2}\,|\tilde{y}|^{n-2}\frac{\partial H_{B_1}}{\partial \tilde{x}_1}(\tilde{x}_1, \tilde{y}).$$

According to [12] and [11], we have

$$H_{B_1}(\tilde{x}_1, \tilde{y}) = \frac{1}{|\tilde{x}_1 - \tilde{y} - 2\tilde{d}_1 n_1|^{n-2}} + o\left(\frac{1}{\tilde{d}_1^{\,n-2}}\right) \qquad \text{with} \quad \tilde{d}_1 = \frac{d_1}{|x_1|}$$

and

$$\frac{\partial H_{B_1}}{\partial \tilde{x}_1}(\tilde{x}_1, \tilde{y}) = -(n-2)\frac{1}{|\tilde{x}_1 - \tilde{y} - 2\tilde{d}_1 n_1|^n}$$
$$\times [\tilde{x}_1 - \tilde{y} - 2n_1 \cdot (\tilde{x}_1 - \tilde{y})\, n_1 + 2\tilde{d}_1 n_1]$$
$$+ o\left(\frac{1}{\tilde{d}_1^{\,n-1}}\right) \qquad \text{as} \quad d_1 \to 0.$$

We derive

$$\begin{aligned}\frac{\partial H_{{}^cB_1}}{\partial x_1}(x_1, y) = &-\frac{(n-2)\,x_1}{|x_1|^n\,|y|^{n-2}\,|\tilde{x}_1-\tilde{y}-2\tilde{d}_1 n_1|^{n-2}}\\ &-\frac{(n-2)}{|x_1|^n\,|y|^{n-2}\,|\tilde{x}_1-\tilde{y}-2\tilde{d}_1 n_1|^{n}}\\ &\times[\tilde{x}_1-\tilde{y}-2n_1\cdot(\tilde{x}_1-\tilde{y})\cdot n_1+2\tilde{d}_1 n_1]\\ &+\frac{2(n-2)\,x_1}{|x_1|^{n+2}\,|y|^{n-2}\,|\tilde{x}_1-\tilde{y}-2\tilde{d}_1 n_1|^{n}}\\ &\times[\tilde{x}_1-\tilde{y}-2n_1\cdot(\tilde{x}_1-\tilde{y})\cdot n_1+2\tilde{d}_1 n_1]\,x_1+o\left(\frac{1}{d^{n-1}1}\right)\end{aligned}$$

and hence we have

$$\frac{\partial H_{{}^cB_1}}{\partial x_1}(x_1, x_1) = (n-2)\,\frac{n_1}{|x_1|^{n-1}\,2^{n-1}d_1^{n-1}}+o\left(\frac{1}{|x_1|^{n-1}\,d_1^{n-1}}\right).$$

Thus

$$\begin{aligned}\frac{\partial H_{\Omega_\varepsilon}}{\partial n_1}(x_1, x_1) = &\frac{n-2}{2^{n-2}d_1^{n-1}}+o\left(\frac{1}{d_1^{n-1}}\right)\\ &-2(n-2)\,\frac{\varepsilon^{n-2}}{|x_1|^{n-1}}\left(1-\frac{1}{|x_1|^{n-2}}\right)+O(\varepsilon^{n-1}).\end{aligned}$$

Our lemma follows.

Lemma 2.3. *Let $x_1, x_2 \in \Omega_\varepsilon$ such that $|x_2| \to c > 1$, when $\varepsilon \to 0$. There exist $\varepsilon_* > 0$, $d_* > 0$ such that*

$$\varepsilon < \varepsilon_*, \qquad d_1 \leqslant d_* \Rightarrow \frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1, x_2) < 0,$$

where n_1 is the outward normal to $\partial_i \Omega_\varepsilon$ at x_1' and x_1' is the only point of $\partial_i \Omega_\varepsilon$ such that

$$d_1 = |x_1 - x_1'|.$$

Proof. On the one hand, we have, from Lemma 2.1,

$$\frac{\partial G_{\Omega_\varepsilon}}{\partial a}(x_1, x_2) = \frac{\partial G_{^cB_1}}{\partial a}(x_1, x_2) - (n-2)\frac{x_1}{|x_1|^n}\varepsilon^{n-2}\left(1 - \frac{1}{|x_2|^{n-2}}\right)$$
$$+ O(\varepsilon^{n-1}).$$

On the other hand, we have

$$G_{^cB_1}(x_1, x_2) = \frac{1}{|x_1|^{n-2}\,|x_2|^{n-2}} G_{B_1}(\tilde{x}_1, \tilde{x}_2),$$

where

$$\tilde{x}_1 = \frac{x_1}{|x_1|^2} \qquad \text{and} \qquad \tilde{x}_2 = \frac{x_2}{|x_2|^2}.$$

Thus

$$\frac{\partial G_{^cB_1}}{\partial a}(x_1, x_2) = -(n-2)\frac{x_1}{|x_1|^n\,|x_2|^{n-2}} G_{B_1}(\tilde{x}_1, \tilde{x}_2)$$
$$+ \frac{1}{|x_1|^n\,|x_2|^{n-2}} \frac{\partial G_{B_1}}{\partial a}(\tilde{x}_1, \tilde{x}_2)$$
$$- \frac{2x_1}{|x_1|^{n+1}\,|x_2|^{n-2}} \frac{\partial G_{B_1}}{\partial \tilde{n}_1}(\tilde{x}_1, \tilde{x}_2)$$

with $\tilde{n}_1 = x_1/|x_1| = -n_1$. We derive

$$\frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1', x_2) = \frac{1}{|x_2|^{n-2}} \frac{\partial G_{B_1}}{\partial \tilde{n}_1}(x_1', \tilde{x}_2) + (n-2)\,\varepsilon^{n-2}\left(1 - \frac{1}{|x_2|^{n-2}}\right)$$
$$+ O(\varepsilon^{n-1}).$$

Furthermore, a direct computation shows that

$$\frac{\partial G_{B_1}}{\partial \tilde{n}_1}(x_1', \tilde{x}_2) = -(n-2)\frac{1 - |\tilde{x}_2|^2}{|\tilde{x}_2 - x_1'|^n}.$$

Thus

$$\frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1', x_2) = -(n-2)\frac{1-|\tilde{x}_2|^2}{|x_2|^{n-2}\,|\tilde{x}_1 - x_1'|^n} + (n-2)\,\varepsilon^{n-2}\left(1-\frac{1}{|x_2|^{n-2}}\right) + O(\varepsilon^{n-1})$$
$$\leqslant -M \qquad (M>0)$$

for ε small enough. Observe that

$$\frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1, x_2) = \frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1', x_2) + O\left(d_1 \sup_{x\in[x_1', x_1]}\left|\frac{\partial^2 G_{\Omega_\varepsilon}}{\partial a\,\partial n_1}(x, x_2)\right|\right).$$

Pointing out the harmonic property of G_{Ω_ε} in a certain subdomain D (see Fig. 1) and using the Poisson formula, we have

$$\frac{\partial^2 G_{\Omega_\varepsilon}}{\partial a\,\partial n_1}(x, x_2) = -\int_{\partial D}\frac{\partial^3 G_D}{\partial a\,\partial n_1\,\partial\gamma_z}(x, z)\,G_{\Omega_\varepsilon}(z, x_2)\,d\sigma(z).$$

Here and below, σ is the area measure on a $(n-1)$-dimensional surface and γ_z is the outward normal to ∂D at z.

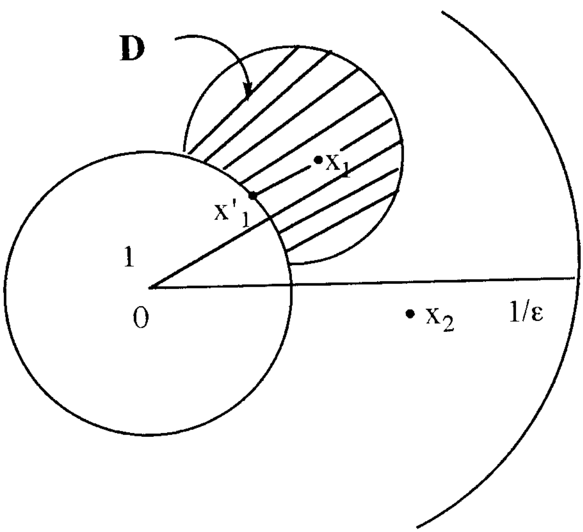

FIGURE 1

We derive

$$\left|\frac{\partial^2 G_{\Omega_\varepsilon}}{\partial a\,\partial n_1}(x, x_2)\right| \leqslant c \qquad \text{for any} \quad x \in [x_1', x_1].$$

Thus

$$\frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1, x_2) \leqslant -M + O(d_1)$$

For ε small enough.

Lemma 2.3 is thereby proven.

LEMMA 2.4. *Let $x_1, x_2 \in \Omega_\varepsilon$ such that $|x_2| \to +\infty$, when $\varepsilon \to 0$. There exist $\varepsilon_* > 0$, $d* > 0$ such that*

$$\varepsilon < \varepsilon_*, \qquad d_1 \leqslant d_* \Rightarrow \frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1, x_2) < 0,$$

where n_1 is the outward normal to $\partial_i \Omega_\varepsilon$ at x_1' and x_1' is the only point of $\partial_i \Omega_\varepsilon$ such that

$$d_1 = |x_1 - x_1'|.$$

Proof. ε will be taken small enough in the following: We choose $R > 0$ such that the annular region $A = B(0, R) \backslash B(0, 1) \subset \Omega_\varepsilon$ and $x_1 \in A$. Let $D_\varepsilon = \Omega_\varepsilon \backslash A$. We then have

$$\Delta_y \left(\frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1, y) \right) = 0 \qquad \text{on } D_\varepsilon.$$

By Lemma 2.3, we derive

$$\frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1, y) \leqslant -M \qquad \text{for} \quad y \in \partial_i D_\varepsilon = \partial B(0, R).$$

On the other hand, we have

$$\frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1, y) = 0 \qquad \text{for} \quad y \in \partial_e D_\varepsilon = \partial B\left(0, \frac{1}{\varepsilon}\right).$$

The maximum principle implies that

$$\frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1, y) \leqslant 0 \qquad \text{for} \quad y \in D_\varepsilon.$$

Our lemma follows.

LEMMA 2.5. *Let $x_1, x_2 \in \Omega_\varepsilon$ such that $d_1 \leqslant d_2$, $|x_1 - x_2| \nrightarrow 0$, $|x_1| \to 1$ and $|x_2| \to 1$ when $\varepsilon \to 0$. Then, there exist $\varepsilon_* > 0$, $d_* > 0$ such that*

$$\varepsilon < \varepsilon_*, \qquad d_1 \leqslant d_* \Rightarrow \frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1, x_2) < 0,$$

where n_1 is the outward normal to $\partial_i \Omega_\varepsilon$ at x'_1 and x'_1 is the only point of $\partial_i \Omega_\varepsilon$ such that

$$d_1 = |x_1 - x'_1|.$$

Proof. As $|x_1 - x_2| \nrightarrow 0$, we can choose $R > 0$ and $y \in \Omega_\varepsilon$ such that

$$B(y, R) \subset \Omega_\varepsilon, \qquad \partial_i \Omega_\varepsilon \cap \partial B(y, R) = \{x'_1\} \qquad \text{and} \qquad x_2 \notin B\left(y, \frac{3R}{2}\right).$$

Let D_1 be a subdomain as in Fig. 2, for any $x \in \partial B(y, R/4)$, we have

$$\begin{aligned} G_{\Omega_\varepsilon}(x, x_2) &\geqslant G_{D_1}(x, x_2) \\ &= G_{D_1}(x, x'_2) + \nabla_2 G_{D_1}(x, x'_2) \cdot (x_2 - x'_2) + O(d_2^2) \\ &= -d_2 \frac{\partial G_{D_1}}{\partial n_2}(x, x'_2) + O(d_2^2), \end{aligned}$$

where x'_2 is the only point of $\partial_i \Omega_\varepsilon$ such that $d_2 = |x_2 - x'_2|$ and n_2 is the outward normal to $\partial_i \Omega_\varepsilon$ at x'_2. We then derive

$$G_{\Omega_\varepsilon}(x, x_2) \geqslant c d_2 \qquad \text{on } \partial B\left(y, \frac{R}{4}\right),$$

where $c = c(\partial B(y, R/4))$ is independent of ε.

We introduce an auxillary function v by defining

$$v(x) = e^{-\alpha r^2} - e^{-\alpha R^2},$$

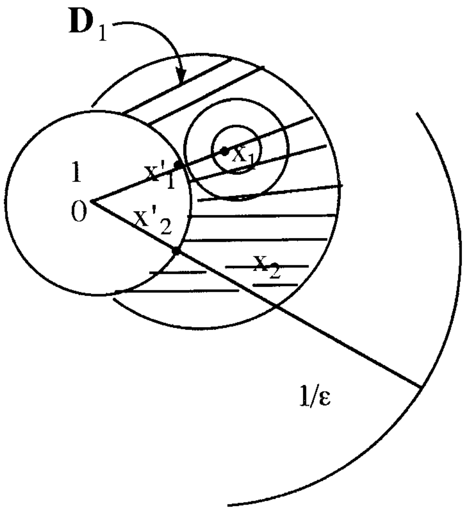

FIGURE 2

where $r = |x - y|$ and α is a positive constant sufficiently large so that $\Delta v \geq 0$ throughout the annular region $A = B(y, R) \backslash B(y, R/4)$. We then have

$$G_{\Omega_\varepsilon}(x, x_2) \geq c d_2 v(x) \qquad \text{on} \quad \partial B\left(y, \frac{R}{4}\right).$$

On the other hand, for $x \in \partial B(y, R)$, we have

$$G_{\Omega_\varepsilon}(x, x_2) - c d_2 v(x) = G_{\Omega_\varepsilon}(x, x_2) \geq 0.$$

Let

$$w_\varepsilon(x) = c d_2 v(x) - G_{\Omega_\varepsilon}(x, x_2).$$

We then have

$$\Delta w_\varepsilon \geq 0 \quad \text{on } A \qquad \text{and} \qquad w_\varepsilon \leq 0 \quad \text{on } \partial A.$$

The weak maximum principle implies that $w_\varepsilon(x) \leq 0$ throughout A. Thus

$$\frac{\partial w_\varepsilon}{\partial n_1}(x'_1) \geq 0$$

that is,

$$\frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1', x_2) \leqslant c d_2 v'(R) = -c_1 d_2, \qquad c_1 > 0.$$

On the other hand, we have

$$\frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1, x_2) = \frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1', x_2) + O\left(d_1 \operatorname*{Sup}_{x \in [x_1', x_1]} \left| \frac{\partial^2 G_{\Omega_\varepsilon}}{\partial a \, \partial n_1}(x, x_2) \right| \right)$$

and

$$\begin{aligned} \frac{\partial^2 G_{\Omega_\varepsilon}}{\partial a \, \partial n_1}(x, x_2) &= \frac{\partial^2 G_{\Omega_\varepsilon}}{\partial a \, \partial n_1}(x, x_2') + O\left(d_2 \operatorname*{Sup}_{y \in [x_2, x_2']} \left| \frac{\partial^3 G_{\Omega_\varepsilon}}{\partial b \, \partial a \, \partial n_1}(x, y) \right| \right) \\ &= O\left(d_2 \operatorname*{Sup}_{y \in [x_2, x_2']} \left| \frac{\partial^3 G_{\Omega_\varepsilon}}{\partial b \, \partial a \, \partial n_1}(x, y) \right| \right). \end{aligned}$$

Observing that G_{Ω_ε} is harmonic in some subdomain D_2 (see Fig. 3) and by the Poisson formula, we have

$$\frac{\partial^3 G_{\Omega_\varepsilon}}{\partial b \, \partial a \, \partial n_1}(x, y) = -\int_{\partial D_2} \frac{\partial^3 G_{D_2}}{\partial a \, \partial n_1 \, \partial \gamma_z}(x, z) \frac{\partial G_{\Omega_\varepsilon}}{\partial b}(z, y) \, d\sigma(z).$$

Using the convergence, in the C^1-topology on every compact set of $\mathbb{R}^n$, of G_{Ω_ε} to G_{cB_1}, we derive that

$$\left| \frac{\partial G_{\Omega_\varepsilon}}{\partial b}(z, y) \right| \leqslant C.$$

Thus

$$\frac{\partial^3 G_{\Omega_\varepsilon}}{\partial b \, \partial a \, \partial n_1}(x, y) = O(1).$$

Then

$$\frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1, x_2) \leqslant -c d_2 + O(d_1 d_2).$$

Our lemma follows.

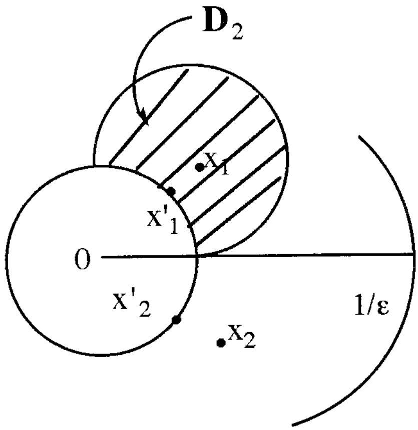

FIGURE 3

Combining Lemmas 2.3, 2.4, and 2.5, we obtain the following lemma.

LEMMA 2.6. *Let $x_1, x_2 \in \Omega_\varepsilon$ such that $d_1 \leqslant d_2$, $|x_1 - x_2| \nrightarrow 0$ when $\varepsilon \to 0$. Then, there exist $\varepsilon_* > 0$, $d_* > 0$ such that*

$$\varepsilon < \varepsilon_*, \qquad d_1 \leqslant d_* \Rightarrow \frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1, x_2) < 0,$$

where n_1 is the outward normal to $\partial_i \Omega_\varepsilon$ at x_1' and x_1' is the only point of $\partial_i \Omega_\varepsilon$ such that

$$d_1 = |x_1 - x_1'|.$$

LEMMA 2.7. *Let $x_1, x_2 \in \Omega_\varepsilon$ such that $d_1 \leqslant d_2 = d(x_2, \partial_i \Omega_\varepsilon)$, $|x_1 - x_2| \to 0$, $\rho_{\Omega_\varepsilon}(x_1, x_2) \geqslant 0$ and $|x_1 - x_2|/d_2$ remains bounded when $\varepsilon \to 0$. Then, there exist $\varepsilon_* > 0$, $d_* > 0$ such that*

$$\varepsilon < \varepsilon_*, \qquad d_1 \leqslant d_* \Rightarrow \frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1, x_2) < 0,$$

where n_1 is the outward normal to $\partial_i \Omega_\varepsilon$ at x_1' and x_1' is the only point of $\partial_i \Omega_\varepsilon$ such that

$$d_1 = |x_1 - x_1'|.$$

Proof. Two cases may occur.

(1) $d_1/d_2 \to 0$;

(2) d_2/d_1 is bounded.

Let us consider the first case.

We introduce the map

$$T\colon \Omega_\varepsilon \to T(\Omega_\varepsilon) = \tilde{\Omega}_\varepsilon$$

$$x \mapsto \tilde{x} = \frac{x - x_2}{d_2}.$$

Therefore $\tilde{\Omega}_\varepsilon$ "converges" in the C^1-topology on every compact set to a half-space π (see [1]).

Since $G_{\tilde{\Omega}_\varepsilon}(\cdot, \tilde{x}_2)$ converges in the C^1-topology on every compact set that does not contain $\tilde{x}_2$ to $G_\pi(\cdot, \tilde{x}_2)$, our claim follows.

Lastly, we study the second case. Observe that since d_1 and d_2 are of the same order and $\rho_{\Omega_\varepsilon}(x_1, x_2) \geqslant 0$ we have that $d_1/|x_1 - x_2|$ remains bounded. According to Lemma 2.1, we have

$$\begin{aligned}
\frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1, x_2) &= -(n-2)\frac{(x_1 - x_2)\cdot n_1}{|x_1 - x_2|^n} + (n-2)\frac{\varepsilon^{n-2}}{|x_1|^{n-1}}\left(1 - \frac{1}{|x_2|^{n-2}}\right)\\
&\quad + O(\varepsilon^{n-1}) - \frac{\partial H_{{}^cB_1}}{\partial n_1}(x_1, x_2)\\
&= -(n-2)\frac{(x_1 - x_2)\cdot n_1}{|x_1 - x_2|^n} + O(d_2\varepsilon^{n-2}) + O(\varepsilon^{n-1})\\
&\quad - \frac{\partial H_{{}^cB_1}}{\partial n_1}(x_1, x_2)\\
&= -(n-2)\frac{(x_1 - x_2)\cdot n_1}{|x_1 - x_2|^n} + O(d_2\varepsilon^{n-2}) + O(\varepsilon^{n-1})\\
&\quad - (n-2)\cdot\frac{1}{|x_1|^{n-1}}\cdot\frac{1}{|x_2|^{n-2}} H_{B_1}(\tilde{x}_1, \tilde{x}_2)\\
&\quad + \frac{1}{|x_1|^n}\cdot\frac{1}{|x_2|^{n-2}}\frac{\partial H_{B_1}}{\partial \tilde{n}_1}(\tilde{x}_1, \tilde{x}_2)\\
&\quad - \frac{2}{|x_1|^{n-1}\,|x_2|^{n-2}}\frac{\partial H_{B_1}}{\partial \tilde{n}_1}(\tilde{x}_1, \tilde{x}_2),
\end{aligned}$$

where $\tilde{x}_1 = x_1/|x_1|^2$, $\tilde{x}_2 = x_2/|x_2|^2$, and $\tilde{n}_1$ is the outward normal to $\partial B(0,1)$ at x_1'. According to [11] we have

$$\begin{aligned}\frac{\partial H_{B_1}}{\partial \tilde{n}_1}(\tilde{x}_1, \tilde{x}_2) = &-\frac{(n-2)}{|\tilde{x}_1 - \tilde{x}_2 + 2d_1\tilde{n}_1|^n} \\ &\times [\tilde{x}_1 - \tilde{x}_2 - 2\tilde{n}_1(\tilde{x}_1 - \tilde{x}_2)\,\tilde{n}_1 - 2\tilde{d}_1\tilde{n}_1]\cdot\tilde{n}_1 + o\left(\frac{1}{\tilde{d}_1^{n-1}}\right) \\ &= \frac{n-2}{|\tilde{x}_1 - \tilde{x}_2 + 2\tilde{d}_1\tilde{n}_1|^n}((\tilde{x}_1 - \tilde{x}_2)\cdot\tilde{n}_1 + 2\tilde{d}_1) + o\left(\frac{1}{\tilde{d}_1^{n-1}}\right),\end{aligned}$$

where $\tilde{d}_1 = d_1/|x_1|$. From the identities

$$\begin{aligned}(x_1 - x_2)\cdot n_1 &= d_2 - d_1 + o(d_1) \\ |x_1 - x_2 + 2d_1 n_1|^2 &= |x_1 - x_2|^2 + 4d_1^2 + 4d_1(d_1 - d_1) + o(d_1^2) \\ &= |x_1 - x_2|^2 + 4d_1 d_2 + o(d_1^2) \\ |\tilde{x}_1 - \tilde{x}_2 + 2\tilde{d}_1\tilde{n}_1|^2 &= |\tilde{x}_1 - \tilde{x}_2|^2 + 4\tilde{d}_1\tilde{d}_2 + o(\tilde{d}_1^2),\end{aligned}$$

we deduce that

$$\begin{aligned}\frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1, x_2) = &-\frac{(n-2)}{|x_1|^{n-1}\,|x_2|^{n-2}} H_{B_1}(\tilde{x}_1, \tilde{x}_2) - (n-2)\frac{d_2 - d_1}{|x_2 - x_1|^n} \\ &- \frac{(n-2)(\tilde{d}_1 + \tilde{d}_2)}{(|\tilde{x}_1 - \tilde{x}_2|^2 + 4\tilde{d}_1\tilde{d}_2)^{n/2}} + o\left(\frac{1}{d_1^{n-1}}\right) \\ &< 0\end{aligned}$$

for d_1 and ε small enough.

Our lemma follows.

LEMMA 2.8. *Let $x_1, x_2 \in \Omega_\varepsilon$ such that $d_1 \leqslant d_2$, $|x_1 - x_2| \to 0$ and $|x_1 - x_2|/d_2 \to +\infty$ when $\varepsilon \to 0$. Then, there exist $\varepsilon_* > 0$, $d_* > 0$ such that*

$$\varepsilon < \varepsilon_*, \qquad d_1 \leqslant d_* \Rightarrow \frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1, x_2) < 0,$$

where n_1 is the outward normal to $\partial_i \Omega_\varepsilon$ at x_1' and x_1' is the only point of $\partial_i \Omega_\varepsilon$ such that

$$d_1 = |x_1 - x_1'|.$$

Proof. We perform the following blow-up.

$$T\colon \Omega_\varepsilon \to T(\Omega_\varepsilon) = \tilde{\Omega}_\varepsilon$$
$$x \mapsto \tilde{x} = \frac{x - x_2'}{|x_1 - x_2|},$$

where x_2' is the only point of $\partial_i \Omega_\varepsilon$ such that $d_2 = d(x_2, \partial_i \Omega_\varepsilon) = |x_2 - x_2'|$. Observe that

$$|\tilde{x}_1 - \tilde{x}_2| = 1, \quad \tilde{d}_1 = d(\tilde{x}_1, \partial\tilde{\Omega}_\varepsilon) = \frac{d_1}{|x_1 - x_2|} \to 0, \quad \tilde{d}_2 = d(\tilde{x}_2, \partial\tilde{\Omega}_\varepsilon) \to 0.$$

By arguments similar to the ones used in the proof of Lemma 2.5, we obtain

$$\frac{\partial G_{\tilde{\Omega}_\varepsilon}}{\partial n_1}(x_1^*, \tilde{x}_2) \leqslant -c\tilde{d}_2,$$

where x_1^* is the only point of $\partial\tilde{\Omega}_\varepsilon$ such that $|\tilde{x}_1 - x_1^*| = \tilde{d}_1$ and $c > 0$.

On the other hand, we have

$$\frac{\partial G_{\tilde{\Omega}_\varepsilon}}{\partial n_1}(\tilde{x}_1, \tilde{x}_2) = \frac{\partial G_{\tilde{\Omega}_\varepsilon}}{\partial n_1}(x_1^*, \tilde{x}_2) + O\left(\tilde{d}_1 \operatorname*{Sup}_{x \in [\tilde{x}_1, x_1^*]} \left|\frac{\partial^2 G_{\tilde{\Omega}_\varepsilon}}{\partial a\, \partial n_1}(x, \tilde{x}_2)\right|\right)$$

and

$$\frac{\partial^2 G_{\tilde{\Omega}_\varepsilon}}{\partial a\, \partial n_1}(x, \tilde{x}_2) = \frac{\partial^2 G_{\tilde{\Omega}_\varepsilon}}{\partial a\, \partial n_1}(x, x_2^*) + O\left(\tilde{d}_2 \operatorname*{Sup}_{y \in [\tilde{x}_2, x_2^*]} \left|\frac{\partial^3 G_{\tilde{\Omega}_\varepsilon}}{\partial b\, \partial a\, \partial n_1}(x, y)\right|\right)$$
$$= O\left(\tilde{d}_2 \operatorname*{Sup}_{y \in [\tilde{x}_2, x_2^*]} \left|\frac{\partial^3 G_{\tilde{\Omega}_\varepsilon}}{\partial b\, \partial a\, \partial n_1}(x, y)\right|\right),$$

where x_2^* is the only point of $\partial\tilde{\Omega}_\varepsilon$ such that $|\tilde{x}_2 - x_2^*| = \tilde{d}_2$.

Using the harmonic property of $G_{\tilde{\Omega}_\varepsilon}$ in a certain subdomain D_4 (see Fig. 4), we derive that

$$\frac{\partial^3 G_{\tilde{\Omega}_\varepsilon}}{\partial b\, \partial a\, \partial n_1}(x, y) = -\int_{\partial D_4} \frac{\partial^3 G_{D_4}}{\partial a\, \partial n_1\, \partial \gamma_z}(x, z)\, \frac{\partial G_{\tilde{\Omega}_\varepsilon}}{\partial b}(z, y)\, d\sigma(z),$$

where γ_z is the outward normal to ∂D_4 at z. As ε tends to zero, we have

$$\left|\frac{\partial G_{\tilde{\Omega}_\varepsilon}}{\partial b}(z, y)\right| < c.$$

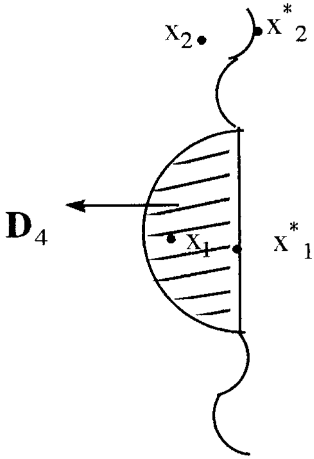

FIGURE 4

Thus

$$\frac{\partial G_{\tilde{\Omega}_\varepsilon}}{\partial n_1}(\tilde{x}_1, \tilde{x}_2) \leqslant -c\tilde{d}_2 + O(\tilde{d}_1 \tilde{d}_2).$$

Our lemma follows.

Combining Lemmas 2.7 and 2.8, we obtain

LEMMA 2.9. *Let $x_1, x_2 \in \Omega_\varepsilon$ such that $d_1 \leqslant d_2$, $|x_1 - x_2| \to 0$ when $\varepsilon \to 0$. Then, there exist $\varepsilon_* > 0$, $d_* > 0$ such that*

$$\varepsilon < \varepsilon_*, \qquad d_1 \leqslant d_* \Rightarrow \frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1, x_2) < 0,$$

where n_1 is the outward normal to $\partial_i \Omega_\varepsilon$ at x'_1 and x'_1 is the only point of $\partial_i \Omega_\varepsilon$ such that

$$d_1 = |x_1 - x'_1|.$$

Next, we are interested in the behavior of $\rho_{\Omega_\varepsilon}$, for $\rho_{\Omega_\varepsilon} = 0$, near the boundary of Ω^p_ε for ε small enough. Namely, we want to prove that for $x = (x_1, ..., x_p) \in \Omega^p_\varepsilon$ close enough to the boundary of Ω^p_ε, $\rho_{\Omega_\varepsilon}(x) = 0$, then the gradient of $\rho_{\Omega_\varepsilon}$ is pointing outward Ω^p_ε for ε small enough. We start with the following lemma.

LEMMA 2.10. *There exist* $\varepsilon_* > 0$ *and* $d_* > 0$ *such that*

$$\forall \varepsilon < \varepsilon_*, \qquad \forall x = (x_1, ..., x_p) \in \Omega_\varepsilon^p$$

such that if $\rho_{\Omega_\varepsilon}(x_1, ..., x_p) = 0$, $d_k = d(x_k, \partial_i \Omega_\varepsilon) \leqslant \mathrm{Inf}_{1 \leqslant j \leqslant p}\, d(x_j, \partial_i \Omega_\varepsilon)$, $d_k < d_*$ *then*

$$\frac{\partial \rho_{\Omega_\varepsilon}}{\partial n_k}(x_1, ..., x_p) > 0,$$

where n_k *is the outward normal to* $\partial_i \Omega_\varepsilon$ *at* x'_k, *and* x'_k *is the point of* $\partial_i \Omega_\varepsilon$ *such that* $d(x_k, \partial_i \Omega_\varepsilon) = |x_k - x'_k|$.

Proof. We may assume, without loss of generality, that

$$d_1 = d(x_1, \partial_i \Omega_\varepsilon) = \inf_{1 \leqslant j \leqslant p} d(x_j, \partial_i \Omega_\varepsilon).$$

We have

$$\rho_{\Omega_\varepsilon}(x) = r(x)\, M_{\Omega_\varepsilon}(x)\, {}^t r(x) = \inf_{\|r\| = 1} r\, M_\varepsilon {}^t r,$$

where all the components of $r(x)$ are strictly positive (see [6]). Thus, in order to prove our Lemma, it is sufficient to prove that all the components of $(\partial M_\varepsilon / \partial n_1)(x)$ are positive, and $(\partial M_\varepsilon / \partial n_1)(x) \neq 0$ for d_1 and ε small enough.

The components m_{ij} of $(\partial M_\varepsilon / \partial n_1)(x)$ are given by

$$m_{11} = \frac{\partial H_{\Omega_\varepsilon}}{\partial n_1}(x_1, x_1), \qquad m_{ii} = 0 \quad \text{if} \quad i \neq 1$$

$$m_{1j} = m_{j1} = -\frac{\partial G_{\Omega_\varepsilon}}{\partial n_1}(x_1, x_j), \qquad m_{ij} = 0 \quad \text{if} \quad i \neq 1 \quad \text{and} \quad j \neq 1$$

Combining Lemmas 2.2, 2.6, and 2.9, we obtain our lemma.

LEMMA 2.11. *Let* $y \in W_\varepsilon$ *such that* $|y| \to c > 0$ *when* $\varepsilon \to 0$. *Then, for any* $x \in W_\varepsilon$, *we have*

$$\frac{\partial H_{W_\varepsilon}}{\partial b}(x, y) = \frac{\partial H_{B_1}}{\partial b}(x, y) + (n-2)\frac{y}{|y|^n}\varepsilon^{n-2}\left(1 - \frac{1}{|x|^{n-2}}\right) + O\left(\frac{\varepsilon^{n-1}}{|x|^{n-2}}\right),$$

where $\partial/\partial b$ *is the derivative with respect to the second variable.*

Proof. We have

$$\begin{cases} -\Delta_x\left(\dfrac{\partial H_{W_\varepsilon}}{\partial b}(x,y)-\dfrac{\partial H_{B_1}}{\partial b}(x,y)\right)=0 & \text{on } W_\varepsilon, \\ \dfrac{\partial H_{W_\varepsilon}}{\partial b}(x,y)-\dfrac{\partial H_{B_1}}{\partial b}(x,y) & \\ \quad =\begin{cases} -(n-2)\,\dfrac{y-x}{|y-x|^n}-\dfrac{\partial H_{B_1}}{\partial b}(x,y) & \text{on } \partial_i W_\varepsilon, \\ 0 & \text{on } \partial_e W_\varepsilon. \end{cases} \end{cases}$$

As $|x|=\varepsilon\to 0$, we have

$$\begin{aligned} H_{B_1}(x,y)&=H_{B_1}(0,y)+O(|x|)\\ &=1+O(|x|) \end{aligned}$$

and

$$\frac{\partial H_{B_1}}{\partial b}(x,y)=O(|x|).$$

Hence we have

$$\frac{\partial H_{W_\varepsilon}}{\partial b}(x,y)-\frac{\partial H_{B_1}}{\partial b}(x,y)=-(n-2)\,\frac{y}{|y|^n}+O(\varepsilon) \qquad \text{on } \partial_i W_\varepsilon.$$

Thus

$$\begin{aligned} &\frac{\partial H_{W_\varepsilon}}{\partial b}(x,y)-\frac{\partial H_{B_1}}{\partial b}(x,y)\\ &\quad=(n-2)\,\frac{y}{|y|^n}\,\varepsilon^{n-2}\left(1-\frac{1}{|x|^{n-2}}\right)+O\left(\frac{\varepsilon^{n-1}}{|x|^{n-2}}\right). \end{aligned}$$

Lemma 2.11 is thereby proved.

LEMMA 2.12. *Let $y\in W_\varepsilon$ and $d_y=d(y,\partial_e W_\varepsilon)=d(y,\partial B(0,1))$. There exist $\varepsilon_*>0$ and $d_*>0$ such that for any $\varepsilon<\varepsilon_*$ and $d_y<d_*$ we have*

$$\frac{\partial H_{W_\varepsilon}}{\partial n_y}(y,y)=\frac{n-2}{2^{n-2}d_y^{n-1}}+o\left(\frac{1}{d_y^{n-1}}\right),$$

where n_y is the outward normal to $\partial_e B_\varepsilon$ at y' and y' is the only point of $\partial_e W_\varepsilon$ such that $d_y = |y - y'|$.

Proof. From Lemma 2.11, we have

$$\frac{\partial H_{W_\varepsilon}}{\partial y}(y, y) = 2\,\frac{\partial H_{B_1}}{\partial b}(y, y) + 2(n-2)\,\frac{y}{|y|^n}\,\varepsilon^{n-2}\left(1 - \frac{1}{|y|^{n-2}}\right) + O\left(\frac{\varepsilon^{n-1}}{|y|^{n-2}}\right).$$

According to [11], we have

$$\frac{\partial H_{B_1}}{\partial b}(x, y) = -\frac{(n-2)}{|y - x + 2d_y n_y|^n}\,(y - x - 2n_y(y-x)\,n_y - 2d_y n_y) + o\left(\frac{1}{d_y^{n-1}}\right).$$

Thus

$$\frac{\partial H_{B_1}}{\partial b}(y, y) = \frac{(n-2)\,n_y}{2^{n-1} d_y^{n-1}} + o\left(\frac{1}{d_y^{n-1}}\right)$$

and hence we have

$$\frac{\partial H_{W_\varepsilon}}{\partial n_y}(y, y) = \frac{(n-2)}{2^{n-1} d_y^{n-1}} + o\left(\frac{1}{d_y^{n-1}}\right) + 2(n-2)\,\frac{\varepsilon^{n-2}}{|y|^{n-1}}\left(1 - \frac{1}{|y|^{n-2}}\right).$$

Our lemma follows.

LEMMA 2.13. *Let $x, y \in W_\varepsilon$ such that $|x - y| \nrightarrow 0$ and $|x| \to c > 0$ when $\varepsilon \to 0$. There exist $\varepsilon_* > 0$ and $d_* > 0$ such that:*

$$\varepsilon < \varepsilon_*, \qquad d(y, \partial_e W_\varepsilon) < d_* \Rightarrow \frac{\partial G_{W_\varepsilon}}{\partial n_y}(x, y) < 0,$$

where n_y is the outward normal to $\partial_e B_\varepsilon$ at y' and y' is the only point of $\partial_e W_\varepsilon$ such that $d_y = |y - y'|$.

Proof. Lemma 2.13 follows by arguments similar to the ones used in the proof of Lemma 2.6.

LEMMA 2.14. *Let $x, y \in W_\varepsilon$ such that*

(i) $d_y = d(y, \partial_e, W_\varepsilon) \leqslant d_x = d(x, \partial_e W_\varepsilon)$;

(ii) $|x - y| \to 0$ *when* $\varepsilon \to 0$.

Then, there exist $\varepsilon_* > 0$ and $d_* > 0$ such that

$$\varepsilon < \varepsilon_*, \qquad d_y < d_* \Rightarrow \frac{\partial G_{W_\varepsilon}}{\partial n_y}(x, y) < 0,$$

where n_y is the outward normal to $\partial_e W_\varepsilon$ at y' and y' is the only point of $\partial_e W_\varepsilon$ such that $d_y = |y - y'|$.

Proof. The proof of Lemma 2.14 is similar to the proof of Lemma 2.9.

From Lemmas 2.12, 2.13, and 2.14, we derive the following lemma.

LEMMA 2.15. *There exist $\varepsilon_* > 0$ and $d_* > 0$ such that*

$$\forall \varepsilon < \varepsilon_*, \qquad \forall x = (x_1, ..., x_p) \in W_\varepsilon^p,$$

such that if $\rho_{W_\varepsilon}(x_1, ..., x_p) = 0$, $d(x_k, \partial_e W_\varepsilon) \leqslant \mathrm{Inf}_{1 \leqslant j \leqslant p} d(x_j, \partial_e W_\varepsilon)$, $d_k < d_*$, $|x_j| \not\to 0$, $\forall j$ then

$$\frac{\partial \rho_{W_\varepsilon}}{\partial n_k}(x_1, ..., x_p) > 0,$$

where n_k is the outward normal to $\partial_e W_\varepsilon$ at x'_k, and x'_k is the only point of $\partial_e W_\varepsilon$ such that $d_k = |x_k - x'_k|$.

LEMMA 2.16. *For $\lambda > 0$, we denote by h_λ the map defined by*

$$h_\lambda \colon A_\varepsilon \to h_\lambda(A_\varepsilon) = \tilde{A}_\varepsilon$$
$$x \to \tilde{x} = \lambda x.$$

Then for any $x_1, ..., x_p \in A_\varepsilon$, we have

$$\rho_{A_\varepsilon}(x_1, ..., x_p) = \lambda^{n-2} \rho_{\tilde{A}_\varepsilon}(\lambda x_1, ..., \lambda x_p).$$

Proof. The same arguments in the proof of Lemma 3.1 in [1] prove easily our Lemma.

LEMMA 2.17. *For $\varepsilon > 0$, let*

$$F_p(\varepsilon) = \{(x_1, ..., x_p) \in \Omega_\varepsilon^p / \exists i \neq j/\text{s.t. } x_i = x_j\}.$$

Then $\rho_{\Omega_\varepsilon}$ converges the C^1-topology to $\rho_{{}^cB_1}$ when $\varepsilon \to 0$ on every compact set that does not intersect V, where V is any neighborhood of $F_p(\varepsilon)$.

Proof. The proof is similar to the proof of Lemma 3.5 in [2].

LEMMA 2.18. *For $\varepsilon > 0$, let*

$$F_p(\varepsilon) = \{(x_1, ..., x_p) \in W_\varepsilon^p / \exists i \neq j \text{ s.t. } x_i = x_j\}.$$

Then ρ_{W_ε} converges in the C^1-topology to ρ_{B_1} when $\varepsilon \to 0$ on every compact set that does not intersect V, where V is any neighborhood of $F_p(\varepsilon)$.

Proof. The same arguments in the proof of Lemma 3.9 in [12] give easily our lemma.

3. PROOF OF PROPOSITION 1.4.

Let $p \in \mathbb{N}$ such that $p \geqslant 3$ (for $p = 2$ see our paper [2]).

Let $x_1, ..., x_p \in A_\varepsilon$ such that $\rho_\varepsilon(x_1, ..., x_p) = 0$. We may assume, without loss of generality, that $|x_1| \leqslant |x_2| \leqslant \cdots \leqslant |x_p|$. Two cases may occur:

1*st Case.* $\varepsilon/|x_1| \to c > 0$ when $\varepsilon \to 0$

2*nd Case.* $\varepsilon/|x_1| \to 0$ when $\varepsilon \to 0$.

1. *Study of the First Case*: $\varepsilon/|x_1| \to c > 0$.

We introduce the map:

$$A_\varepsilon \to \tilde{A}_\varepsilon$$

$$x \to \tilde{x} = \frac{x}{\varepsilon}.$$

Then $\tilde{A}_\varepsilon = \{\tilde{x} \in \mathbb{R}^n / 1 < |\tilde{x}| < 1/\varepsilon^2\}$. When $\varepsilon \to 0$, $\tilde{A}_\varepsilon$ "converges" to ${}^cB_1 = \mathbb{R}^n \backslash B(0, 1)$. Thus, denoting by $\rho_{\tilde{A}_\varepsilon}$ the function defined, replacing A_ε by $\tilde{A}_\varepsilon$ in (6), we have

$$\rho_\varepsilon(x_1, ..., x_p) = \frac{1}{\varepsilon^{n-2}} \rho_{\tilde{A}_\varepsilon}(\tilde{x}_1, ..., \tilde{x}_p) \qquad \text{(see Lemma 2.16).}$$

We distinguish two cases.

1a. $d(\tilde{x}_1, \partial_i \widetilde{A}_\varepsilon) = d(\tilde{x}_1, \partial B(0, 1)) \to 0$ when $\varepsilon \to 0$.

In this case, according to Lemma 2.10, we have

$$\frac{\partial \rho_{\widetilde{A}_\varepsilon}}{\partial \tilde{n}_1}(\tilde{x}_1, ..., \tilde{x}_p) > 0,$$

where $\tilde{n}_1$ is the outward normal to $\partial \widetilde{A}_\varepsilon$ at $\tilde{x}'_1$, and $\tilde{x}'_1$ is the only point of $\partial \widetilde{A}_\varepsilon$ such that $d(\tilde{x}_1, \partial_i \widetilde{A}_\varepsilon) = |\tilde{x}_1 - \tilde{x}'_1|$. That implies that zero is a regular value for ρ_ε.

1b. $d(\tilde{x}_1, \partial_i, \widetilde{A}_\varepsilon) \nrightarrow 0$ when $\varepsilon \to 0$

Here, two subcases may occur.

(i) There exists some constant $M > 0$ such that $\sup_{1 \leqslant i \leqslant p} |\tilde{x}_i - \tilde{x}_i| \leqslant M$ when $\varepsilon \to 0$. Therefore, we deduce from Lemma 2.17 and [2, Lemma 2.6] that $\nabla \rho_\varepsilon(x_1, ..., x_p) \neq 0$ for ε small enough.

(ii) There exists $j \in \{2, ..., p\}$ such that $|\tilde{x}_j - \tilde{x}_1| \to +\infty$ when $\varepsilon \to 0$. Two subcases may occur.

(α) $\varepsilon^2 |\tilde{x}_p| \to 1$ when $\varepsilon \to 0$

(β) $\varepsilon^2 |\tilde{x}_p| \to c < 1$ when $\varepsilon \to 0$.

Let us consider the first case (1.b.ii.α).
We introduce the map

$$\widetilde{A}_\varepsilon \to \widetilde{\widetilde{A}}_\varepsilon$$

$$\tilde{x} \to \tilde{\tilde{x}} = \frac{1}{\varepsilon^2} \frac{\tilde{x}}{|\tilde{x}|^2}.$$

Thus $\widetilde{\widetilde{A}}_\varepsilon = \widetilde{A}_\varepsilon$ and $|\tilde{\tilde{x}}_p| = 1/\varepsilon^2 |\tilde{x}_p| \to 1$ when $\varepsilon \to 0$, therefore

$$d(\tilde{\tilde{x}}_p, \partial_i \widetilde{\widetilde{A}}_\varepsilon) = d(\tilde{\tilde{x}}_p, \partial B_1) \to 0 \qquad \text{when} \quad \varepsilon \to 0.$$

Using the same arguments as in case 1.a we conclude that $\nabla \rho_\varepsilon(x_1, ..., x_p) \neq 0$ for ε small enough.

Now, we study the second subcase 1.b.ii.β. Note that $\varepsilon^2 |\tilde{x}_p| \to c < 1$ as $\varepsilon \to 0$ implies $\varepsilon^2 |\tilde{x}_j| \to c < 1$ for any $j \in \{1, ..., p\}$. We denote by $\tilde{x}_2, ..., \tilde{x}_k$ the points such that $|\tilde{x}_j - \tilde{x}_1|$ remains bounded $(2 \leqslant j \leqslant k)$ and by $\tilde{x}_{k+1}, ..., \tilde{x}_p$ the other points (that is $|\tilde{x}_j - \tilde{x}_1| \to +\infty$ as $\varepsilon \to 0$ for $j \in \{k+1, ..., p\}$). We denote by $\widetilde{G}_\varepsilon$, $\widetilde{H}_\varepsilon$ the functions defined, replacing A_ε by $\widetilde{A}_\varepsilon$ in (2), respectively, (3). We want to prove

$$\frac{\partial}{\partial t}(t^{n-2} \rho_{\widetilde{A}_\varepsilon}(t\tilde{x}_1, ..., t\tilde{x}_k, \tilde{x}_{k+1}, ..., \tilde{x}_p))_{/t=1} < 0.$$

It is sufficient to prove that all the components of

$$\frac{\partial}{\partial t}(t^{n-2}\tilde{M}_\varepsilon(t\tilde{x}_1, ..., t\tilde{x}_k, \tilde{x}_{k+1}, ..., \tilde{x}_p))$$

are negative, and $(\partial/\partial t)(t^{n-2}\tilde{M}_\varepsilon(t\tilde{x}_1, ..., t\tilde{x}_k, \tilde{x}_{k+1}, ..., \tilde{x}_p)) \neq 0$.

The components m_{ij} of $(\partial/\partial t)(t^{n-2}\tilde{M}_\varepsilon(t\tilde{x}_1, ..., t\tilde{x}_k, \tilde{x}_{k+1}, ..., \tilde{x}_p))$ are given by

$$\begin{cases} m_{ii} = \dfrac{\partial}{\partial t}(t^{n-2}\tilde{H}_\varepsilon(t\tilde{x}_i, t\tilde{x}_i)) & \text{for} \quad 1 \leqslant i \leqslant k; \\ m_{ii} > 0 \quad \text{if} \quad i > k & \\ m_{ij} = m_{ji} = -\dfrac{\partial}{\partial t}(t^{n-2}\tilde{G}_\varepsilon(t\tilde{x}_i, t\tilde{x}_j)) & \text{for} \quad 1 \leqslant i < j \leqslant k; \\ m_{ij} = m_{ji} = -\dfrac{\partial}{\partial t}(t^{n-2}\tilde{G}_\varepsilon(t\tilde{x}_i, \tilde{x}_j)) & \text{for} \quad 1 \leqslant i \leqslant k \quad \text{and} \quad j > k \\ m_{ij} = m_{ji} < 0 \quad \text{if} \quad i > k \quad \text{and} \quad j > k. & \end{cases}$$

According to [2, Lemmas 2.5 and 2.6], we have

$$\frac{\partial}{\partial t}(t^{n-2}\tilde{H}_\varepsilon(t\tilde{x}_i, t\tilde{x}_i))_{t=1} < -c \qquad \text{when} \quad \varepsilon \to 0 \ \ (c > 0) \quad \text{for } 1 \leqslant i \leqslant k$$

and

$$\frac{\partial}{\partial t}(t^{n-2}\tilde{G}_\varepsilon(t\tilde{x}_i, t\tilde{x}_j))_{t=1} > c > 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant k.$$

On the other hand, we deduce from [2, Lemma 2.7] that

$$\frac{\partial \tilde{G}_\varepsilon}{\partial a}(\tilde{x}_i, \tilde{x}_j) = (n-2)\frac{\tilde{x}_i}{|\tilde{x}_i|^n}\left[\frac{1}{|\tilde{x}_j|^{n-2}} - \varepsilon^{2(n-2)}\right] + O\left(\frac{1}{|\tilde{x}_j|^{n-1}}\right) \qquad \text{for} \quad j \in \{k+1, ..., p\} \ \text{and} \ 1 \leqslant i \leqslant k.$$

Since

$$\varepsilon^2 |\tilde{x}_j| \to c < 1 \qquad \text{when} \quad \varepsilon \to 0 \text{ for any } j \in \{k+1, ..., p\}$$

we have

$$\frac{\partial}{\partial t}(t^{n-2}\widetilde{G}_\varepsilon(t\tilde{x}_i, \tilde{x}_j))_{/t=1} > 0 \quad \text{for } j \in \{k+1, ..., p\} \text{ and } i \in \{1, ..., k\},$$

which concludes the first case.

2. *Study of the Second Case*: $\varepsilon/|x_1| \to 0$ *when* $\varepsilon \to 0$

We distinguish two subcases:

a. $\varepsilon|x_1| \to c > 0$ when $\varepsilon \to 0$;

b. $\varepsilon|x_1| \to 0$ when $\varepsilon \to 0$.

Study of 2.a. $\varepsilon\,|x_1| \to c > 0$ when $\varepsilon \to 0$ and $\varepsilon/|x_1| \to 0$.

In this case, we introduce the map

$$A_\varepsilon \to \widetilde{A}_\varepsilon$$
$$x \to \tilde{x} = \varepsilon x.$$

Thus $\widetilde{A}_\varepsilon = \{\tilde{x} \in \mathbb{R}^n/\varepsilon^2 < |\tilde{x}| < 1\}$ and $|\tilde{x}_1| = \varepsilon\,|x_1| \to c > 0$, therefore $\tilde{x}_j \nrightarrow 0$ for any $j \in \{1, ..., p\}$.

Two situations may occur.

(i) $d(\tilde{x}_p, \partial_e \widetilde{A}_\varepsilon) = d(\tilde{x}_p, \partial B(0,1)) \to 0$, when $\varepsilon \to 0$. In this case, from Lemma 2.15 we deduce that $(\partial\tilde{\rho}_{\widetilde{A}_\varepsilon}/\partial\tilde{n}_p)(\tilde{x}_1, ..., \tilde{x}_p) > 0$, where $\tilde{n}_p$ is the outward normal to $\partial\widetilde{A}_\varepsilon$ at $\tilde{x}'_p$ and $\tilde{x}'_p$ is the only point of $\partial\widetilde{A}_\varepsilon$ such that $d(\tilde{x}_p, \partial_e\widetilde{A}_\varepsilon) = |\tilde{x}_p - \tilde{x}'_p|$.

Thus, zero is a regular value of ρ_ε in this case.

(ii) $d(\tilde{x}_p, \partial_e, \widetilde{A}_\varepsilon) = d(\tilde{x}_p, \partial B(0,1)) \nrightarrow 0$, when $\varepsilon \to 0$. We deduce from Lemma 2.18 and the proof of Lemma 4.4 in [1] that zero is a regular value for ρ_ε for ε small enough.

Study of 2.b. $\varepsilon/|x_1| \to 0$, $\varepsilon\,|x_1| \to 0$ when $\varepsilon \to 0$

We introduce the map

$$A_\varepsilon \to \widetilde{A}_\varepsilon$$
$$x \to \tilde{x} = \frac{x}{|x_1|}.$$

Thus

$$\widetilde{A}_\varepsilon = \left\{\tilde{x} \in \mathbb{R}^n/\frac{\varepsilon}{|x_1|} < |\tilde{x}| < \frac{1}{\varepsilon\,|x_1|}\right\}$$

$\widetilde{A}_\varepsilon$ "converges" to $\mathbb{R}^n$ when $\varepsilon \to 0$.

We distinguish two subcases:

(i) $\varepsilon\,|x_p| \to c > 0$

(ii) $\varepsilon\,|x_p| \to 0$

In (i), we consider the map:

$$\tilde{A}_\varepsilon \to \tilde{\tilde{A}}_\varepsilon$$

$$\tilde{x} \to \tilde{\tilde{x}} = \frac{1}{\varepsilon\,|x_1|}\,\frac{\tilde{x}}{|\tilde{x}|^2}.$$

Then

$$\tilde{\tilde{A}}_\varepsilon = \{\tilde{\tilde{x}} \in \mathbb{R}/1 < |\tilde{\tilde{x}}| < 1/\varepsilon^2\}$$

and

$$|\tilde{\tilde{x}}_p| = \frac{1}{\varepsilon\,|x_p|} \to \frac{1}{c} > 0.$$

This is as in the first case.

Now, we study the following case.

(ii) $\varepsilon\,|x_p| \to 0$. In fact, we will prove that this case cannot occur. In this case, $\varepsilon\,|x_j| \to 0$, when $\varepsilon \to 0$ $\forall j \in \{1, ..., p\}$.

Letting $\varepsilon_1 = \varepsilon/|x_1|$ and $\varepsilon_1' = \varepsilon\,|x_1|$, we have

$$\tilde{A}_\varepsilon = \left\{\tilde{x} \in \mathbb{R}^n/\varepsilon_1 < |\tilde{x}| < \frac{1}{\varepsilon_1'}\right\}$$

and

$$\varepsilon_1'\,|\tilde{x}_j| \to 0, \qquad \text{when} \quad \varepsilon \to 0, \quad \forall j \in \{1, ..., p\}.$$

On the other hand, we state the following claim, whose proof is given at the end of the section

$$\forall j \in \{2, ..., p\} \qquad |\tilde{x}_j - \tilde{x}_1| \to +\infty, \qquad \text{when} \quad \varepsilon \to 0 \tag{19}$$

Now, we distinguish two situations:

(α) $|\tilde{x}_p|/|\tilde{x}_{p-1}| \to c > 0$;

(β) $|\tilde{x}_p|/|\tilde{x}_{p-1}| \to +\infty$.

Let us consider the first situation, that is $|\tilde{x}_p|/|\tilde{x}_{p-1}| \to c > 0$. We introduce the map

$$\tilde{A}_\varepsilon \to \tilde{\tilde{A}}_\varepsilon$$

$$\tilde{x} \to \tilde{\tilde{x}} = \frac{\tilde{x}}{|\tilde{x}|^2} \cdot |\tilde{x}_p|.$$

If $|\tilde{x}| = \varepsilon_1$, then $|\tilde{\tilde{x}}| = |\tilde{x}_p|/|\tilde{x}| = |\tilde{x}_p|/\varepsilon_1 \to +\infty$, when $\varepsilon \to 0$.

Similarly, $|\tilde{x}| = 1/\varepsilon'_1$ implies $|\tilde{\tilde{x}}| = \varepsilon'_1 |\tilde{x}_p| \to 0$, when $\varepsilon \to 0$. Therefore

$$\tilde{\tilde{A}}_\varepsilon = \left\{ \tilde{\tilde{x}} \in \mathbb{R}^n / \varepsilon'_1 |\tilde{x}_p| < |\tilde{\tilde{x}}| < \frac{|\tilde{x}_p|}{\varepsilon_1} \right\}$$

and $\tilde{\tilde{A}}_\varepsilon$ "converges" to $\mathbb{R}^n$ when $\varepsilon \to 0$.

We have

$$|\tilde{\tilde{x}}_p| = 1; \quad |\tilde{\tilde{x}}_{p-1}| = \frac{|\tilde{x}_p|}{|\tilde{x}_{p-1}|} \to c > 0; \quad 1 = |\tilde{\tilde{x}}_p| < |\tilde{\tilde{x}}_{p-1}| < \cdots < |\tilde{\tilde{x}}_1| = |\tilde{x}_p|$$

and

$$|\tilde{\tilde{x}}_1| = o\left(\frac{|\tilde{x}_p|}{\varepsilon_1}\right).$$

If $\tilde{\tilde{x}}_{p-1} \to \tilde{\tilde{x}}_p$, then $\rho_{\tilde{\tilde{A}}_\varepsilon}(\tilde{\tilde{x}}_{p-1}, \tilde{\tilde{x}}_p) < 0$ for ε small enough; this implies that $\rho_{\tilde{\tilde{A}}_\varepsilon}(\tilde{\tilde{x}}_1, ..., \tilde{\tilde{x}}_p) < 0$, a contradiction with $\rho_\varepsilon(x_1, ..., x_p) = 0$.

Now, let us recall the following result from [2, Lemma 2.11].

Lemma A [2]. *For $\varepsilon > 0$, $\varepsilon' > 0$, let $A_{\varepsilon\varepsilon'} = \{x \in \mathbb{R}^n / \varepsilon < |x| < 1/\varepsilon'\}$. Let $x_1 \in A$ such that $|x_1| = 1$. Then for ε and ε' tending to zero, $\rho_A(x_1, \cdot)$ converges in the C^1-topology on every compact set that does not contain x_1 to $\rho_{\mathbb{R}^n}(x_1, \cdot)$.*

Since $\tilde{\tilde{x}}_{p-1}$ does not converge to $\tilde{\tilde{x}}_p$, we deduce, from Lemma A, that $\rho_{\tilde{\tilde{A}}_\varepsilon}(\tilde{\tilde{x}}_{p-1}, \tilde{\tilde{x}}_p) < 0$ for ε small enough and therefore $\rho_{\tilde{\tilde{A}}_\varepsilon}(\tilde{\tilde{x}}_1, ..., \tilde{\tilde{x}}_p) < 0$, a contradiction with $\rho_\varepsilon(x_1, ..., x_p) = 0$.

Finally, we study the second situation, that is $|\tilde{x}_p|/|\tilde{x}_{p-1}| \to +\infty$. At this point of the proof, we need the following estimate, whose proof is given at the end of the section.

$$G_{\tilde{A}_\varepsilon}(\tilde{x}_{p-1}, \tilde{x}_p) \sim \frac{C}{|\tilde{x}_p|^{n-2}} \qquad \text{for } \varepsilon \text{ small enough.} \tag{20}$$

On the other hand, from [2, Lemmas 2.10 and 2.12], we have the following estimates

$$H_{\tilde{A}_\varepsilon}(\tilde{x}_j, \tilde{x}_j) \sim c\,\frac{\varepsilon_1^{n-2}}{|\tilde{x}_j|^{2(n-2)}} + c_1 \varepsilon_1'^{\,n-2},$$

$$\text{when} \quad \varepsilon \to 0 \quad \forall j \in \{1, ..., p\} \tag{21}$$

$$G_{\tilde{A}_\varepsilon}(\tilde{x}_1, \tilde{x}_j) \sim \frac{c}{|\tilde{x}_j|^{(n-2)}} \qquad \forall j \in \{2, ..., p\}.$$

Thus, if $\rho_{\tilde{A}_\varepsilon}(\tilde{x}_1, \tilde{x}_{p-1}) \geqslant 0$, then $\varepsilon_1 \varepsilon_1' \, |\tilde{x}_{p-1}|^2 \geqslant c > 0$, hence $\varepsilon_1' |\tilde{x}_{p-1}|^2 \to +\infty$. Similarly, $\rho_{\tilde{A}_\varepsilon}(\tilde{x}_1, \tilde{x}_p) \geqslant 0$ implies that $\varepsilon_1 \varepsilon_1' \, |x_p|^2 \geqslant c > 0$ and that $\varepsilon_1' \, |\tilde{x}_p|^2 \to +\infty$. Using (20), (21) and $\varepsilon_1' \, |\tilde{x}_j|^2 \to +\infty$ (for $j = p-1$ or $j = p$), we derive that if

$$\rho_{\tilde{A}_\varepsilon}(\tilde{x}_{p-1}, \tilde{x}_p) \geqslant 0 \qquad \text{then} \quad (c_1 \varepsilon_1'^{\,n-2}) \cdot (c_2 \varepsilon_1'^{\,n-2}) \geqslant \frac{c}{|\tilde{x}_p|^{2(n-2)}}.$$

This implies that $\varepsilon_1' \, |\tilde{x}_p| \geqslant c$, which yields a contradiction, and the Case 2.b.ii is impossible.

We turn now to the proof of (19) and (20).

Proof of (19). We argue by contradiction. We assume that there exists $j \in \{2, ..., p\}$ such that $|\tilde{x}_j - \tilde{x}_1|$ remains bounded, when $\varepsilon \to 0$. Lemma A implies that $\rho_{\tilde{A}_\varepsilon}(\tilde{x}_1, \tilde{x}_j) < 0$ for ε small enough. Since, $\rho_{\tilde{A}_\varepsilon}(\tilde{x}_1, ..., \tilde{x}_p) \leqslant \rho_{\tilde{A}_\varepsilon}(\tilde{x}_1, \tilde{x}_j)$, $\rho_{\tilde{A}_\varepsilon}(\tilde{x}_1, ..., \tilde{x}_p)$ is strictly negative; a contradiction.

Proof of (20). On the one hand, we have

$$\begin{cases} -\Delta_y (G_{{}^cB_{\varepsilon_1}}(\tilde{x}_{p-1}, y) - G_{\tilde{A}_\varepsilon}(\tilde{x}_{p-1}, y)) = 0 & \text{on } \tilde{A}_\varepsilon \\ G_{{}^cB_{\varepsilon_1}}(\tilde{x}_{p-1}, y) - G_{\tilde{A}_\varepsilon}(\tilde{x}_{p-1}, y) & \\ \quad = \begin{cases} 0 & \text{on } \partial_i \tilde{A}_\varepsilon \\ G_{{}^cB_{\varepsilon_1}}(\tilde{x}_{p-1}, y) & \text{on } \partial_e \tilde{A}_\varepsilon, \end{cases} \end{cases}$$

where ${}^cB_{\varepsilon_1} = \mathbb{R}^n \backslash B(0, \varepsilon_1)$.

On the other hand, we have

$$G_{{}^cB_{\varepsilon_1}}(\tilde{x}_{p-1}, y) = \frac{\varepsilon_1^{n-2}}{|\tilde{x}_{p-1}|^{n-2} \, |y|^{n-2}} \, G_{B_1}\left(\varepsilon_1 \frac{\tilde{x}_{p-1}}{|\tilde{x}_{p-1}|^2}, \varepsilon_1 \frac{y}{|y|^2}\right).$$

If $|y| = 1/\varepsilon_1'$ then $|\varepsilon_1 y/|y|^2| = \varepsilon_1 \varepsilon_1' \to 0$ and $\varepsilon_1 \varepsilon_1' = o(\varepsilon_1/|\tilde{x}_{p-1}|)$ (we recall that $\varepsilon_1' \, |x_{p-1}| \to 0$), therefore

$$
\begin{aligned}
G_{B_1}&\left(\varepsilon_1 \frac{\tilde{x}_{p-1}}{|\tilde{x}_{p-1}|^2}, \varepsilon_1 \frac{y}{|y|^2}\right)\\
&= \frac{|\tilde{x}_{p-1}|^{n-2}}{\varepsilon_1^{n-2}} G_{B(0,\, |\tilde{x}_{p-1}|/\varepsilon_1)}\left(\frac{\tilde{x}_{p-1}}{|\tilde{x}_{p-1}|}, \frac{|\tilde{x}_{p-1}|}{|y|^2}\, y\right)\\
&= \frac{|\tilde{x}_{p-1}|^{n-2}}{\varepsilon_1^{n-2}} \left[G_{B(0,\, |\tilde{x}_{p-1}|/\varepsilon_1)}\left(\frac{\tilde{x}_{p-1}}{|\tilde{x}_{p-1}|}, 0\right) + O\left(\frac{|\tilde{x}_{p-1}|}{|y|}\right)\right]\\
&= \frac{|\tilde{x}_{p-1}|^{n-2}}{\varepsilon_1^{n-2}} \left[1 - \frac{\varepsilon_1^{n-2}}{|\tilde{x}_{p-1}|^{n-2}} + O(|\tilde{x}_{p-1}|\, \varepsilon_1')\right]\\
&= \frac{|\tilde{x}_{p-1}|^{n-2}}{\varepsilon_1^{n-2}} - 1 + O\left(\frac{|\tilde{x}_{p-1}|^{n-1}\, \varepsilon_1'}{\varepsilon_1^{n-2}}\right).
\end{aligned}
$$

This implies that if $y \in \partial_e \widetilde{A}_\varepsilon$,

$$
\begin{aligned}
G_{{}^cB_{\varepsilon_1}}(\tilde{x}_{p-1}, y) &= \varepsilon_1'^{n-2} - \frac{\varepsilon_1^{n-2} \varepsilon_1'^{n-2}}{|\tilde{x}_{p-1}|^{n-2}} + O(|\tilde{x}_{p-1}|\, \varepsilon_1'^{n-1})\\
&= \varepsilon_1'^{n-2} + o(\varepsilon_1'^{n-2}).
\end{aligned}
$$

This implies that

$$
G_{\widetilde{A}_\varepsilon}(\tilde{x}_{p-1}, y) = G_{{}^cB_{\varepsilon_1}}(\tilde{x}_{p-1}, y) - \varepsilon_1'^{n-2}\left(1 - \frac{\varepsilon^{n-2}}{|y|^{n-1}}\right) + o(\varepsilon_1'^{n-2}).
$$

Since

$$
\left|\frac{\varepsilon_1 \tilde{x}_p}{|\tilde{x}_p|^2}\right| = o\left(\left|\frac{\varepsilon_1 \tilde{x}_{p-1}}{|\tilde{x}_{p-1}|^2}\right|\right),
$$

we have

$$
G_{{}^cB_{\varepsilon_1}}(\tilde{x}_{p-1}, \tilde{x}_p) = \frac{1}{|\tilde{x}_p|^{n-2}} - \frac{\varepsilon_1^{n-2}}{|\tilde{x}_{p-1}|^{n-2}\, |\tilde{x}_p|^{n-2}} + O\left(\frac{|\tilde{x}_{p-1}|}{|\tilde{x}_p|^{n-1}}\right).
$$

Thus

$$
G_{{}^cB_{\varepsilon_1}}(\tilde{x}_{p-1}, \tilde{x}_p) \sim \frac{1}{|\tilde{x}_p|^{n-2}}.
$$

Therefore

$$G_{\tilde{A}_\varepsilon}(\tilde{x}_{p-1}, \tilde{x}_p) \sim \frac{C}{|\tilde{x}_p|^{n-2}};$$

(20) follows.

4. PROOF OF THEOREM 1.5.

Before giving the proof, let us recall the following result (Lemma 2.13 in [2]).

LEMMA B [2]. *Let* $\varepsilon_1 > 0$. *There exists* $\varepsilon_2 > \varepsilon_1$ *such that for any* $\varepsilon \leqslant \varepsilon_2$, $A^2_{\varepsilon_1}$ *is contained into* $I_2(\varepsilon)$.

Since, $\rho_\varepsilon(x_1, ..., x_p) \leqslant \rho_\varepsilon(x_1, x_j)$, $\forall i, j \in \{1, ..., p\}$, we derive, from Lemma B, that there exists $\varepsilon_2 > 0$ such that $A^p_{\varepsilon_1}$ is contained into $I_p(\varepsilon_2)$.

Therefore, we derive

$$I_p(\varepsilon_1) \xrightarrow{i_1} A^p_{\varepsilon_1} \xrightarrow{i_2} I_p(\varepsilon_2) \xrightarrow{i_3} A^p_{\varepsilon_2}.$$

Thus

$$H_*(I_p(\varepsilon_1)) \xrightarrow{i_{1*}} H_*(A^p_{\varepsilon_1}) \xrightarrow{i_{2*}} H_*(I_p(\varepsilon_2)) \xrightarrow{i_{3*}} H_*(A^p_{\varepsilon_2}).$$

On the other hand, Proposition 1.4 implies that there exists $\varepsilon_0 > 0$ such that $I_p(\varepsilon_1)$ is homotopically equivalent to $I_p(\varepsilon_2)$ when $\varepsilon_1 \leqslant \varepsilon_0$, therefore $i_{2*} \circ i_{1*}$ is an isomorphism, and the injection $I_p(\varepsilon_1) \to A^p_{\varepsilon_1}$ is a homotopy equivalence, since the sets $I_p(\varepsilon)$ are Euclidean neighborhood retracts. Our theorem follows.

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