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ON THE PRESCRIBED SCALAR CURVATURE ON 3-HALF SPHERES: MULTIPLICITY RESULTS AND MORSE INEQUALITIES AT INFINITY

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Abstract. We consider the existence and multiplicity of riemannian metrics of prescribed mean curvature and zero boundary mean curvature on the three dimensional half sphere (S^3_+, g_c) endowed with its standard metric g_c. Due to Kazdan-Warner type obstructions, conditions on the function to be realized as a scalar curvature have to be given. Moreover the existence of *critical point at infinity* for the associated Euler Lagrange functional, makes the existence results harder to be proved. However it turns out that such noncompact orbits of the gradient can be treated as usual critical point once a *Morse Lemma at infinity* is performed. In particular their topological contribution to the level sets of the functional can be computed. In this paper we prove that, under generic conditions on K, this *topology at infinity* is a lower bound for the number of metrics in the conformal class of g_c having prescribed scalar curvature and zero boundary mean curvature.

1. **Introduction.** Let (M^n, g) be n-dimensional riemannian manifold with boundary, $n \geq 3$, and let $\tilde{g} = u^{4/(n-2)} g$, be a conformal metric to g, where u is a smooth positive function, then the scalar curvatures R_g, $R_{\tilde{g}}$ and the mean curvatures h_g, $h_{\tilde{g}}$, with respect to g and $\tilde{g}$ respectively, are related by the following equations:

$$(P_{K,H}) \quad \begin{cases} -c_n \Delta_g u + R_g u & = & R_{\tilde{g}} u^{\frac{n+2}{n-2}} & \text{in} & \overset{\circ}{M} \\ \frac{2}{n-2} \frac{\partial u}{\partial \nu} + h_g u & = & h_{\tilde{g}} u^{\frac{n}{n-2}} & \text{on} & \partial M, \end{cases} \tag{1}$$

see e.g. [5]. In the above equation, ν denotes the outward unit normal to ∂M, with respect to the metric g.

In view of the above equations the following natural question arises:
Given two functions $K : M \to \mathbb{R}$ and $H : \partial M \to \mathbb{R}$, does there exist a metric $\tilde{g}$ conformally equivalent to g such that $R_{\tilde{g}} = K$ and $h_{\tilde{g}} = H$?

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The answer to this equation is equivalent to finding a smooth positive solution u of the following equation

$$(P_{K,H}) \quad \begin{cases} -c_n\Delta_g u + R_g u &= K u^{\frac{n+2}{n-2}} \quad \text{in} \quad \overset{\circ}{M} \\ \frac{2}{n-2}\frac{\partial u}{\partial \nu} + h_g u &= H u^{\frac{n}{n-2}} \quad \text{on} \quad \partial M. \end{cases}$$

We observe that the above problem is a natural generalization of the well known *Scalar Curvature Problems on Closed manifolds*: to find a positive smooth solution to the following equation:

$$(\mathbf{SC}) \quad - c_n\,\Delta_g u + R_g\,u = K u^{(n+2)/(n-2)},\ u > 0 \text{ in } M$$

and to which much works have been devoted (see [1], [3], [4], [6], [8], [9], [10], [12], [14], [15], [19], [20], [22], [23], [24], [25], [32], [40], [45], [46] and the references therein).
When K and H are constants, the problem is called *The Yamabe Problem on Manifolds with boundary*. It has also been studied through the works [18], [29], [30], [31], [33], [34], [35], [36] and the references therein.
This problem was first studied by P. Cherrier [26] in 1984, who proved regularity up to the boundary of weak H^1 solutions. For further works on this equation and related ones please see [2], [16], [17], [27], [29], [31], [35], [36], [37], [39] and the references therein. The main analytic difficulties of our problem are due to the presence of critical exponents on the right hand side of our equation. Indeed due to the fact that the embeddings $H^1(M) \rightarrow L^{2n/(n-2)}(M)$ and $H^1(M) \rightarrow L^{2(n-1)/(n-2)}(\partial M)$ are not compact, the Euler-Lagrange functional J associated to our problem fails to satisfy *the Palais Smale condition*. That is there exist noncompact sequences along which the functional is bounded and its gradient goes to zero. Therefore it is not possible to apply the standard variational methods to prove existence of solution, although the functional in the positive case has a *mountain pass* structure.
From another part, in the family of problems $(P_{K,H})$ we single out two extreme cases. Namely the one where we prescribe the scalar curvature under minimal boundary conditions which amounts to solving the following equations:

$$(P_K) \quad \begin{cases} -c_n\Delta_g u + R_g\, u &= K u^{\frac{n+2}{n-2}} \quad \text{in} \quad \overset{\circ}{M} \\ \frac{2}{n-2}\frac{\partial u}{\partial \nu} + h_g u &= 0 \quad \text{on} \quad \partial M. \end{cases}$$

The second one is when we prescribe the mean curvature of scalar flat metric, which correspond to solve the following equation:

$$(P_H) \quad \begin{cases} -c_n\Delta_g u + R_g u &= 0 \quad \text{in} \quad \overset{\circ}{M} \\ \frac{2}{n-2}\frac{\partial u}{\partial \nu} + h_g u &= H u^{\frac{n}{n-2}} \quad \text{on} \quad \partial M. \end{cases}$$

Although these are particular cases they summarize somehow all the analytic difficulties of the family of problems $(P_{K,H})$, in the sense that all intermediate cases are interpolations between these two extreme ones. From another part these two problems have different analytic features as far as the lack of compactness and existence results are concerned. While we can prove corresponding statements of all the existence results for the prescribed scalar curvature problem on closed manifolds for the problem (P_H), this is no longer true in general for the problem (P_K) and the problems $(P_{K,H})$ which behave like (P_K). Indeed in this case we have new solutions created by the boundary effect which have no equivalent on a closed manifold from one part and from another part the boundary effect makes the blow up picture more complicated because of the existence under generic conditions on

K of *bubbles* having concentration points on the interior of the manifolds as well as on the boundary. Such a situation which makes the blow up analysis much harder cannot occur for the problem (P_H). Such a consideration conducts us to focus in the article on the problem (P_K). From a variational viewpoint the main difficulty in dealing with the above problem lies in the lack of compactness of the flow lines of the gradient flow and such a noncompactness is usually due to the invariance under the action of a noncompact group. From another part we have proved that such a noncompactness does occur only if the manifold is conformally equivalent to the standard half sphere, at least in the three dimensional case.

1.1. **The Case of the half sphere.** In this case we are reduced to look for positive solutions of the following problem:

$$(P_K) \qquad \begin{cases} -\Delta_g u + \frac{n(n-2)}{4} u &= K u^{\frac{n+2}{n-2}} \quad \text{in} \quad S^n_+ \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on} \quad \partial S^n_+, \end{cases} \tag{2}$$

where g is the standard metric of $S^n_+ = \{x \in \mathbb{R}^{n+1} \, / \, |x| = 1, \, x_{n+1} > 0\}$.
For the two-dimensional case, there are analogous equations on the 2-dimensional half sphere S^2_+ involving exponential nonlinearities.
Problem (2) has not always a solution, indeed there are *Kazdan Warner type obstructions* just as it is the case for the scalar curvature on spheres.
Although problem (2) shares many common features with the scalar curvature problem on spheres, it has its own features as far as the lack of compactness is concerned. Such a difference can be made clear if we look at the behavior of blowing up subcritical approximations to both problems. Indeed concerning the problem of the prescribed scalar curvature on spheres, we recall that on S^2 and S^3 there is at most one blow up point, a fact that has been proved by A. Chang and Paul Yang [21], Bahri-Coron [12] and Chang-Gursky-Yang [19] and in [45], R. Schoen and D. Zhang developped a Morse theory for the scalar curvature problem on S^3. For S^4 there are multiple blow up points but the points of concentration of the bubbles should satisfy a balance condition as it has been shown by Ben Ayed-Chen-Chtioui-Hammami [14] and YY Li [41]. In [40], Yanyan Li proved that there is at most one blow up point on S^n, $n \geq 3$, under the optimal β-flatness condition on K near its critical points which is $\beta > n-2$. Finally on any sphere S^n, $n \geq 5$ and any Morse function K satisfying that $\Delta K \neq 0$ at its critical points there holds that for any p-tuple of points $(a_1, \cdots, a_p)$ which are critical points of K satisfying that $\Delta K(a_i) < 0$ there is a sequence of blowing up solutions concentrating at these points [10]. .
Regarding now our problem (2) and, for generic function K there are actually multiple blow up points on S^3_+, and there are blowing up approximate solutions concentrating at any configuration of points $(x_1, \cdots, x_k)$ of critical points of K satisfying $\Delta K(x_i) < 0$ for an interior point and $(\partial K(x_j))/(\partial \nu) > 0$ at any boundary point. Therefore the blow up picture is completely different and needs a more careful analysis. Moreover the presence of the boundary creates a new type of solutions which have no counterpart on S^n.

In collaboration with Djadli and Malchiodi [27] the second author gave sufficient condition on K (and more generally on the pair (K, H) for the problem $(P_{K,H})$) to insure existence of solutions. See also Lei Zhang [49] for related existence and compactness results. Our aim here is to give multiplicity results which include already the existence results mentioned above but our approach here is drastically different.

1.2. **Multiplicity Results for metrics of prescribed scalar curvature on S^3_+.** To state our existence results we need to set the following notations:
Let G be the Green's function of L_g on S^3_+ under Neumann boundary condition and K be a C^2 positive Morse function on $\overline{S^3_+}$. we say that K satisfies the condition

$$(C1) \qquad \text{if } a \in \partial S^3_+, \nabla_T K(a) = 0, \qquad \text{then } \frac{\partial K}{\partial \nu}(a) \neq 0.$$

Moreover we single out the following subset of potential concentration points:

$$\mathcal{F}_\infty := \left\{ q \in \partial S^3_+ ; \nabla_T K(q) = 0, \frac{\partial}{\partial \nu} K(q) > 0 \right\}.$$

Now to $(q^1, \cdots, q^N) \subseteq \mathcal{F}_\infty$, we associate a Matrix $M = (M_{ij})$ defined by:

$$\begin{cases} M_{jj} = \frac{(\partial K/\partial \nu)(q^j)}{K(q^j)^{3/2}}, & j \in \{1, \dots, N\} \\ M_{lj} = -4\sqrt{2} \frac{G_{q^l}(q^j)}{K(q^l)^{1/4} K(q^j)^{1/4}}, & l, j \in \{1, \dots, N\}, \quad l \neq j. \end{cases} \tag{3}$$

Here $G_{q^l}(.)$ denotes the Green's function of the conformal Laplacian on S^3_+ with pole at q^l. See please its expression in the formula (41). Let $\rho = \rho(q^1, \dots, q^N)$ denote the least eigenvalue of M.
Before stating our first result, we will say that a solution w of (P_K) is nondegenerate if 0 is not an eigenvalue of the operator $-\Delta_g + n(n-2)/2 - Kw^{4/(n-2)}$.

Theorem 1.1. *Let l_1 be the cardinal of $\mathcal{F}_\infty$ and $m(q^i) := Morse(K_1, q^i)$ where K_1 is the restriction of K to ∂S^3_+.*
We assume that K satisfies condition (**C1**) *and that for any $(q^1, ..., q^l) \subseteq \mathcal{F}_\infty$, the matrix $M(q^1, ..., q^l)$ is nondegenerate. We also assume that all the solutions of the problem (P_K) are nondegenerate. Then the number of solutions is lower-bounded by*

$$\left| 1 - \sum_{s=1}^{l_1} \sum_{\tau_s = (i_1, \dots, i_s)/\rho(\tau_s) > 0} (-1)^{3s-1-\sum_{j=1}^{s} m(q^{i_j})} \right|.$$

Theorem 1.1 can be seen as *Morse type inequality result* in the sense that we give here a lower bound for the number of solutions in terms of the *topology at infinity* that is the total contribution of noncompact orbits of the gradient flow associated to the Euler Lagrange functional (its *critical point at infinity*). Recall that *Morse inequalities* give a lower bound on the number of critical points of a *Morse function* in terms of the *Betti numbers* of the underlying manifold. In our case the space of variation is contractible and hence has no topology. However due to the non-compactness of the problem there are *critical points at infinity* whose topological contribution to the difference of topology between the level sets of the functional can be computed thanks to a *Morse Lemma at Infinity* which provides new coordinates in which the gradient flow takes a quite simple *normal form*.
In the following we give a brief description on the main ingredients behind the proof of theorem 1.1.
Our argument uses a careful analysis of the lack of compactness of the Euler Lagrange functional J associated to problem (2). Namely we study the noncompact orbits of the gradient flow of J the so called *critical points at infinity* following the terminology of A. Bahri [9]. These are noncompact orbits of J along which J is bounded and its gradient goes to zero. These *critical point at infinity* can be treated as usual critical point once a Morse Lemma at infinity is performed from which we can derive just as in the classical Morse Theory the difference of topology induced by

these noncompact orbits and compute their Morse index. Such a Morse Lemma at infinity which is a cornerstone in our analysis is obtained through the construction of a suitable pseudogradient for which the Palais Smale condition is satisfied along the decreasing flow lines, as long as these flow lines do not enter the neighborhood of a finite numbers of critical points of K_1 such that the related matrix M (see (3)) is positive definite. Moreover along the flow lines of such a pseudogradient there can be only finitely many blow up points. Furthermore if some blow up points are close and the interactions between them is large, then the flow lines starting from there will enter the zone with at least one less blow up point.
Similar Morse Lemma has been established for the prescribed scalar curvature problem on the spheres S^3, S^4 under the hypothesis that the problem has no solution by A. Bahri and J. M. Coron [12], see also [14]. Since our aim is to prove multiplicity rather than only the existence we have to perform our Morse Lemma without such an assumption, a situation which creates a new difficulty namely to deal with the possibility of existence of *a critical point at infinity of new type* consisting of a sum of bubbles plus a solution of (P_K). By performing a *Morse Lemma* and by constructing a pseudogradient near an ε-neighborhood of such a potential critical point at infinity, we rule out such a possibility on S^3_+. Moreover in our case the presence of the boundary makes the analysis more involved. Indeed it turns out that the interaction of the bubbles and the boundary creates a phenomenon of new type which is not present in the sphere's case.
Finally we notice that similar statement to Theorem 1.1 for the prescribed curvature problem on the 3-dimensional sphere has been obtained by R. Schoen and D. Zhang [45], where due to the fact that in their case only single blow up occurs, the formula is much simpler. Their proof which is drastically different from ours involves a refined analysis for blowing up subcritical approximations.
Now we give another multiplicity result under more general assumptions than the ones of Theorem 1.1. To state it we need to introduce new conditions and notations: Assume that $K_1 = K_{|\partial S^3_+}$ has only nondegenerate critical points $y_0, y_1, ..., y_s$, and set the following notations:

$$\begin{aligned}
\mathcal{C}_+ =& \{y/\ \nabla K_1(y) = 0 \text{ and } \frac{\partial K}{\partial \nu}(y) > 0\},\\
\mathcal{C}_0^+ =& \{y/\ \nabla K_1(y) = 0,\ \frac{\partial K}{\partial \nu}(y) = 0 \text{ and } \Delta K(y) < 0\},\\
{\mathcal{F}_\infty}^N =& \{(q_1, ..., q_N) \in (\mathcal{C}_+)^N / q_i \neq q_j \text{ for } i \neq j\}.
\end{aligned}$$

To each $(q_1, ..., q_N) \in {\mathcal{F}_\infty}^N$, we associate as we did before an $N \times N$ symmetric matrix $M = M(q_1, ..., q_N)$ defined by (3). We assume that
(A_1) For any $N \leq \text{card}(\mathcal{C}_+)$ and any $(q_1, ..., q_N) \in {\mathcal{F}_\infty}^N$, the associated matrix $M(q_1, ..., q_N)$ is nondegenerate.
Furthermore, we assume that
(A_2) For each critical point y of K_1 such that $(\partial K/\partial \nu)(y) = 0$, we have $\Delta K(y) \neq 0$ and near y, $\partial K/\partial \nu$ and $(-\Delta K)$ have the same sign.

Now, we are able to state our second multiplicity result.

Theorem 1.2. *Under the assumptions* (A_1) *and* (A_2)*, if all the solutions of* (P_K) *are nondegenerate, then the number of solutions is lower-bounded by*

$$\left|1 - \sum_{y \in \mathcal{C}_0^+} (-1)^{m(y)} - \sum_{N=1}^{\#\mathcal{C}_+} \sum_{\tau_N = (y_{j_1}, \dots, y_{j_N}) \in \mathcal{F}_\infty{}^N / \rho(\tau_N) > 0} (-1)^{3N-1-\sum_{k=1}^N m(y_{j_k})}\right|.$$

Here, $m(y)$ *denotes the Morse index of* K_1 *at* y*.*

Actually theorem 1.2 shows that instead of assuming that all critical points of K_1 are nondegenerate in the sense that $(\partial K)/(\partial \nu) \neq 0$ it is possible to allow some flatness. Namely let us assume that at any critical point q of K_1 with $(\partial K(q))/(\partial \nu) = 0$ there exists some real number $\beta := \beta(q) \in [2, 3)$ such that in some geodesic normal coordinate system centered at q (recall that the boundary is totally geodesic) it holds:

$$(A_f) \qquad \begin{cases} K(y) = K(0) \ + \ b_1|y_1|^\beta + b_2|y_2|^\beta + b_3|y_3|^\beta \ + \ R(y), \\ b_i \neq 0 \ \forall\, i, \quad \sum_{j=1}^3 b_j \neq 0 \text{ and if } \sum_{j=1}^3 b_j < 0 \text{ then } b_3 < 0, \end{cases}$$

where $\displaystyle\sum_{s=0}^{[\beta]} |\nabla^s R(y)||y|^{-\beta-s} = o(1)$ as y tends to zero. Here ∇^s denotes all possible derivatives of order s and $[\beta]$ the integer part of β. We introduce now a new subset of critical point of K_1 namely:

$$\mathcal{C}_f^+ = \{q/\ \nabla K(q) = 0, \quad \text{and } \sum_{j=1}^3 b_j(q) < 0\}. \tag{4}$$

Now, we are ready to state a generalization of theorem 1.1.

Theorem 1.3. *Under the assumptions* (A_1) *and* (A_f)*, if all the solutions of* (P_K) *are nondegenerate, then the number of solutions is lower-bounded by*

$$\left|1 - \sum_{q \in \mathcal{C}_f^+} (-1)^{3-\iota(q)} - \sum_{N=1}^{\#\mathcal{C}_+} \sum_{\tau_N = (y_{j_1}, \dots, y_{j_N}) \in \mathcal{F}_\infty{}^N / \rho(\tau_N) > 0} (-1)^{3N-1-\sum_{k=1}^N m(y_{j_k})}\right|$$

Here, $\iota(y) := \#\{b_j(q); q \in \mathcal{C}_f, b_j(q) < 0\}$*.*

Now we introduce a new assumption: we say that K satisfies $(\mathbf{A_3})$ if :
(A_3) For each critical point y of K_1, we have $(\partial K/\partial \nu)(y) \leq 0$.
We derive then the following multiplicity result:

Corollary 1. *Under the assumptions* (A_2) *and* (A_3)*, if all the solutions of* (P_K) *are nondegenerate, then the number of solutions is lower-bounded by*

$$\left|1 - \sum_{y \in \mathcal{C}_0^+} (-1)^{m(y)}\right|,$$

where $m(y)$ *denotes the Morse index of* K_1 *at* y*.*

2. **Variational Structure and Preliminaries.** In this section we recall the functional setting and the variational problem and its main features. Problem (P_K) has a variational structure. The functional is

$$J(u) = \frac{||u||^2}{\left(\int_{S^n_+} K|u|^{2n/(n-2)}\right)^{(n-2)/n}}$$

defined on $H^1(S^n_+,\mathbb{R})\setminus\{0\}$ equipped with the norm

$$||u||^2=\int_{S^n_+}|\nabla u|^2+\frac{n(n-2)}{4}\int_{S^n_+}u^2.$$

We denote by Σ the unit sphere of $H^1(S^n_+,\mathbb{R})$ and we set $\Sigma^+=\{u\in\Sigma:\ u\geq 0\}$. The Palais-Smale condition fails to be satisfied for J on Σ^+. In order to characterize the sequences failing the Palais-Smale condition, we need to introduce some notations.
For $a\in\overline{S^n_+}$ and $\lambda>0$, let

$$\delta_{(a,\lambda)}(x)=c_0\frac{\lambda^{(n-2)/2}}{(\lambda^2+1+(1-\lambda^2)\cos d(a,x))^{(n-2)/2}}$$

where d is the geodesic distance on (S^n_+,g) and c_0 is chosen so that

$$-\Delta\delta_{(a,\lambda)}+\frac{n(n-2)}{4}\delta_{(a,\lambda)}=\delta_{(a,\lambda)}^{\frac{n+2}{n-2}}\quad \text{in } S^n_+.$$

For $a\in\overline{S^n_+}$, we define $\varphi_{(a,\lambda)}$ to be the function defined on S^n_+ by

$$-\Delta\varphi_{(a,\lambda)}+\frac{n(n-2)}{4}\varphi_{(a,\lambda)}=\delta_{(a,\lambda)}^{\frac{n+2}{n-2}}\text{ in } S^n_+,\quad \frac{\partial\varphi_{(a,\lambda)}}{\partial\nu}=0\text{ on }\partial S^n_+.$$

It is easy to remark that $\varphi_{(a,\lambda)}=\delta_{(a,\lambda)}$ if the point $a\in\partial S^n_+$.
For $\varepsilon>0$ and $p\in\mathbb{N}^*$, let us define

$$\begin{aligned}V(p,\varepsilon)=&\{u\in\Sigma:\ \exists\, a_1,...,a_p\in\overline{S^n_+},\exists\,\lambda_1,...,\lambda_p>0,\exists\,\alpha_1,...,\alpha_p>0\\ &\text{s.t. } ||u-\sum_{i=1}^p\alpha_i\varphi_i||<\varepsilon\,,\ \Big|\frac{\alpha_i^{4/(n-2)}K(a_i)}{\alpha_j^{4/(n-2)}K(a_j)}-1\Big|<\varepsilon,\\ &\lambda_i>\varepsilon^{-1},\varepsilon_{ij}<\varepsilon\text{ and }\lambda_id_i<\varepsilon\text{ or }\lambda_id_i>\varepsilon^{-1}\}\end{aligned}\tag{5}$$

where $\varphi_i=\varphi_{(a_i,\lambda_i)}$, $\varepsilon_{ij}=(\lambda_i/\lambda_j+\lambda_j/\lambda_i+\lambda_i\lambda_j(1-\cos d(a_i,a_j))/2)^{(2-n)/2}$ and $d_i=d(a_i,\partial S^n_+)$.
For w a solution of (P_K) we also define $V(p,\varepsilon,w)$ as

$$\{u\in\Sigma/\exists\,\alpha_0>0\text{ s. t. } u-\alpha_0w\in V(p,\varepsilon)\text{ and }|\alpha_0^4J(u)^3-1|<\varepsilon\}.\tag{6}$$

The failure of the Palais-Smale condition can be described following the ideas introduced in [12], [42], [48]. Such a description is by now standard and reads in our case as follows. Let ∇J be the gradient of J.

Proposition 1. *Let $(u_j)\in\Sigma^+$ be a sequence such that $\nabla J(u_j)$ tends to zero and $J(u_j)$ is bounded. Then, there exist an integer $p\in\mathbb{N}^*$, a sequence $\varepsilon_j>0$, ε_j tends to zero, and an extracted of u_j's, again denoted u_j, such that $u_j\in V(p,\varepsilon_j,w)$ where w is zero or a solution of (P_K).*

If a function u belongs to $V(p,\varepsilon)$, we assume, for the sake of simplicity, that $\lambda_id_i<\varepsilon$ for $i\leq q$ and $\lambda_id_i>\varepsilon^{-1}$ for $i>q$. We consider the following minimization problem for $u\in V(p,\varepsilon)$ with ε small

$$\min_{\alpha_i>0,\,\lambda_i>0,\,a_i\in\partial S^n_+,\,b_i\in S^n_+}\Big\|u-\sum_{i=1}^q\alpha_i\delta_{(a_i,\lambda_i)}-\sum_{i=q+1}^p\alpha_i\varphi_{(b_i,\lambda_i)}\Big\|.\tag{7}$$

We then have the following proposition which defines a parametrization of the set $V(p,\varepsilon)$. It follows from corresponding statements in [9], [12], [44].

Proposition 2. *For any $p \in \mathbb{N}^*$, there is $\varepsilon_p > 0$ such that if $\varepsilon < \varepsilon_p$ and $u \in V(p,\varepsilon)$, the minimization problem (7) has a unique solution (up to permutation). In particular, we can write $u \in V(p,\varepsilon)$ as follows*

$$u = \sum_{i=1}^{q} \bar{\alpha}_i \delta_{(\bar{a}_i,\bar{\lambda}_i)} + \sum_{i=q+1}^{p} \bar{\alpha}_i \varphi_{(\bar{a}_i,\bar{\lambda}_i)} + v,$$

where $(\bar{\alpha}_1,...,\bar{\alpha}_p,\bar{a}_1,...,\bar{a}_p,\bar{\lambda}_1,...,\bar{\lambda}_p)$ is the solution of (7) and $v \in H^1(S^n_+)$ such that

$$(V_0) \quad ||v|| \leq \varepsilon, \quad (v,\psi) = 0 \text{ for } \psi \in \bigcup_{i\leq q,\, j>q} \left\{\delta_i, \frac{\partial \delta_i}{\partial \lambda_i}, \frac{\partial \delta_i}{\partial a_i}, \varphi_j, \frac{\partial \varphi_j}{\partial \lambda_j}, \frac{\partial \varphi_j}{\partial a_j}\right\}.$$

In the following we will say that $v \in (V_0)$ if v satisfies (V_0). We have also the following proposition whose proof is similar, up to minor modification to corresponding statements in [9] (see also [44])

Proposition 3. *There exists a C^1 map which, to each $(\alpha_1,...,\alpha_p,a_1,...,a_p,\lambda_1,...,\lambda_p)$ such that $\sum_{i=1}^{q}\alpha_i\delta_i + \sum_{i=q+1}^{p}\alpha_i\varphi_i \in V(p,\varepsilon)$ with small ε, associates $\overline{v} = \overline{v}_{(\alpha_i,a_i,\lambda_i)}$ satisfying*

$$J\left(\sum_{i=1}^{q}\alpha_i\delta_i + \sum_{i=q+1}^{p}\alpha_i\varphi_i + \overline{v}\right) = \min_{v\in(V_0)} J\left(\sum_{i=1}^{q}\alpha_i\delta_i + \sum_{i=q+1}^{p}\alpha_i\varphi_i + v\right).$$

Moreover, there exists $c > 0$ such that the following holds

$$||\overline{v}|| \leq c\left(\sum_{i\leq p}(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}) + \sum_{i>q}\frac{1}{(\lambda_i d_i)^2} + \sum_{k\neq r}\varepsilon_{kr}(log(\varepsilon_{kr}^{-1}))^{1/3}\right).$$

Let w be a nondegenerate solution of (P_K). The following proposition defines a parameterization of the set $V(p,\varepsilon,w)$. Its proof follows closely a similar statement in [10].

Proposition 4. *There is $\varepsilon_0 > 0$ such that if $\varepsilon \leq \varepsilon_0$ and $u \in V(p,\varepsilon,w)$, then the problem*

$$\min\{\big|\big|u - \sum_{i=1}^{q}\alpha_i\,\delta_{(a_i,\lambda_i)} - \sum_{i=q+1}^{p}\alpha_i\,\varphi_{(a_i,\lambda_i)} - \alpha_0(w+h)\big|\big|,\, \alpha_i > 0,$$
$$\lambda_i > 0,\, a_i \in \partial S^n_+\,(for\, i\leq q),\, a_i \in S^n_+\,(for\, i>q),\ h \in T_w(W_u(w))\}$$

has a unique solution $(\overline{\alpha},\overline{\lambda},\overline{a},\overline{h})$. Thus, we write u as follows:

$$u = \sum_{i=1}^{q}\overline{\alpha}_i\delta_{(\overline{a}_i,\overline{\lambda}_i)} + \sum_{i=q+1}^{p}\overline{\alpha}_i\varphi_{(\overline{a}_i,\overline{\lambda}_i)} + \overline{\alpha}_0(w+\overline{h}) + v,$$

where v belongs to $H^1(S^n_+) \cap T_w(W_s(w))$ and it satisfies (V_0), $T_w(W_u(w))$ and $T_w(W_s(w))$ are the tangent spaces at w to the unstable and stable manifolds of w.

Following A. Bahri we introduce the following definitions and notations

Definition 2.1. *A critical point at infinity* of J on Σ^+ is a limit of a flow line $u(s)$ of the equation:

$$\begin{cases} \frac{\partial u}{\partial s} = -\nabla J(u) \\ u(0) = u_0 \end{cases}$$

such that $u(s)$ remains in $V(p, \varepsilon(s), w)$ for $s \geq s_0$.
Here w is either zero or a solution of (P_K) and $\varepsilon(s)$ is some function tending to zero when $s \to \infty$. Using Proposition 4, $u(s)$ can be written as:

$$u(s) \ = \sum_{i=1}^{p} \alpha_i(s)\, \varphi_{(a_i(s), \lambda_i(s))} + \alpha_0(s)(w + h(s)) \ + v(s).$$

Denoting $a_i := \lim_{s\to\infty} a_i(s)$ and $\alpha_i = \lim_{s\to\infty} \alpha_i(s)$, we denote by

$$(a_1, \cdots, a_p, w)_\infty \text{ or } \sum_{i=1}^{p} \alpha_i\, \varphi_{(a_i,\infty)} + \alpha_0 w$$

such a critical point at infinity. If $w \neq 0$ it is called of *w-type*.

For such a *critical point at infinity* stable and unstable manifolds are associated and they are easily determined once a Morse type reduction is performed, see please [10].

3. Ruling out the existence of critical point at Infinity in $V(p, \varepsilon, w)$ for $w \neq 0$.

In this section, for $u \in V(p, \varepsilon, w)$, we will write $u = \sum_{i=1}^{p} \alpha_i \varphi_{(a_i, \lambda_i)} + \alpha_0(w+h) + v$ and for simplicity, we will assume that for $i \leq q$, we have $a_i \in \partial S^3_+$ and therefore $\varphi_i = \delta_i$ and for $i > q$, we have $a_i \in S^3_+$ and this implies by the definition of $V(p, \varepsilon, w)$ that $\lambda_i d(a_i, \partial S^3_+) \to \infty$.

Proposition 5. *For $\varepsilon > 0$ small enough and $u = \sum_{i=1}^{p} \alpha_i \varphi_{(a_i, \lambda_i)} + \alpha_0(w + h) + v \in V(p, \varepsilon, w)$, we have the following expansion*

$$\begin{aligned} J(u) = \frac{(S_3/2)\sum_{i=1}^{p} \mu_i \alpha_i^2 + \alpha_0^2 ||w||^2}{((S_3/2)\sum_{i=1}^{p} \mu_i \alpha_i^6 K(a_i) + \alpha_0^6 ||w||^2)^{\frac{1}{3}}} & \left[1 - 2c_2 \alpha_0 \sum_{i=1}^{p} \mu_i \alpha_i \frac{w(a_i)}{\lambda_i^{1/2}} \right. \\ & - c_2 \sum_{i \neq j} \mu_{(i,j)} \alpha_i \alpha_j \varepsilon_{ij} - 2c_2 \sum_{i,j > q} \alpha_i \alpha_j \frac{H(a_i, a_j)}{\sqrt{\lambda_i \lambda_j}} + f_2(h) + \alpha_0^2 Q_2(h, h) \\ & \left. + f_1(v) + Q_1(v, v) + o\left(\sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{1}{\lambda_i^{\frac{1}{2}}} + \sum_{i > q} \frac{1}{\lambda_i d_i} + ||v||^2 + ||h||^2 \right) \right] \end{aligned}$$

where $\mu_i = 1$ *for* $i \leq q$, $\mu_i = 2$ *for* $i > q$, $\mu_{(i,j)} = 1$ *if* $i, j \leq q$, *and* $\mu_{(i,j)} = 2$ *if not and*

$$Q_1(v,v) = \frac{1}{\gamma_1}||v||^2 - \frac{5}{\beta_1}\int_{S^3_+} K\sum_{i=1}^{p}(\alpha_i\varphi_i)^4 v^2 - \frac{5}{\beta_1}\alpha_0^4\int_{S^3_+} Kw^4v^2,$$

$$Q_2(h,h) = \frac{1}{\gamma_1}||h||^2 - \frac{5}{\beta_1}\alpha_0^4\int_{S^3_+} Kw^4h^2, \qquad c_2 = \frac{1}{2}c_0^6\int_{\mathbb{R}^3}\frac{dx}{(1+|x|^2)^{\frac{5}{2}}}$$

$$f_1(v) = -\frac{2}{\beta_1}\int_{S^3_+} K(\sum_{i=1}^{p}\alpha_i\varphi_i)^5 v, \qquad S_3 = c_0^6\int_{\mathbb{R}^3}\frac{dx}{(1+|x|^2)^3}$$

$$f_2(h) = 2\frac{\alpha_0}{\gamma_1}\sum_i \alpha_i(\varphi_i, h) - \frac{2\alpha_0}{\beta_1}\int_{S^3_+} K(\sum_i \alpha_i\varphi_i + \alpha_0 w)^5 h,$$

$$\beta_1 = \frac{S_3}{2}(\sum_{i=1}^{p}\mu_i\alpha_i^6 K(a_i)) + \alpha_0^6||w||^2, \quad \gamma_1\frac{S_3}{2}(\sum_{i=1}^{p}\mu_i\alpha_i^2) + \alpha_0^2||w||^2.$$

Proof. Before starting the proof, we mention that it will be convenient to perform some stereographic projection in order to reduce our problem to $\mathbb{R}^3_+$. Let $D^{1,2}(\mathbb{R}^3_+)$ denote the completion of $C_c^\infty(\overline{\mathbb{R}^3_+})$ with respect to Dirichlet norm. The stereographic projection π_a through a point $a \in \partial S^3_+$ induces an isometry $i : H^1(S^3_+) \to D^{1,2}(\mathbb{R}^3_+)$ according to the following formula

$$(iv)(x) = \left(\frac{2}{1+|x|^2}\right)^{1/2} v(\pi_a^{-1}(x)), \qquad v \in H^1(S^3_+),\ x \in \mathbb{R}^3_+.$$

In particular, the following holds true, for every $v \in H^1(S^3_+)$

$$\int_{S^3_+}(|\nabla v|^2 + \frac{3}{4}v^2) = \int_{\mathbb{R}^3_+}|\nabla(iv)|^2 \quad \text{and} \quad \int_{S^3_+}|v|^6 = \int_{\mathbb{R}^3_+}|iv|^6.$$

In the sequel, we will identify the function K and its composition with the stereographic projection π_a. We will also identify a point b of S^3_+ and its image by π_a. These facts will be assumed as understood in the sequel. We need to estimate $N(u) = ||u||^2$ and $D^3 = \int_{S^3_+} K(x)u^6$.
Expanding $N(u)$, it is equal to

$$\sum_{i=1}^{p}\alpha_i^2||\varphi_i||^2 + 2\alpha_i\alpha_0(\varphi_i, w+h) + \alpha_0^2(||h||^2 + ||w||^2) + ||v||^2 + \sum_{i\neq j}\alpha_i\alpha_j(\varphi_i,\varphi_j)$$

Observe that

$$||\varphi_i||^2 = \frac{S_3}{2}\ (\text{if } i \leq q)\,;\ ||\varphi_i||^2 = S_3 + 2c_2\frac{H(a_i,a_i)}{\lambda_i} + o(\frac{1}{\lambda_i d_i})\ (\text{if } i > q) \tag{8}$$

$$(\varphi_i,\varphi_j) = \begin{cases} 2c_2\varepsilon_{ij} + 2c_2\frac{H(a_i,a_j)}{\sqrt{\lambda_i\lambda_j}} + o(\varepsilon_{ij} + \sum\frac{1}{\lambda_k d_k}) & \text{if } i,j > q \\ \mu_{(i,j)}c_2\varepsilon_{ij}(1+o(1)) & \text{if not,} \end{cases} \tag{9}$$

where $\mu_{(i,j)} = 1$ if $i, j \leq q$ and $\mu_{(i,j)} = 2$ if i or $j > q$, and

$$(\varphi_i, w) = \int_{S^3_+}\delta_i^5 w = \mu_i c_2\frac{w(a_i)}{\sqrt{\lambda_i}} + o(\frac{1}{\sqrt{\lambda_i}}), \tag{10}$$

where $\mu_i = 1$ if $i \leq q$ and $\mu_i = 2$ if $i > q$. Thus

$$\begin{aligned} N =&\gamma_1 + 2\alpha_0 \sum_{i=1}^{p} c_2\mu_i\alpha_i \frac{w(a_i)}{\sqrt{\lambda_i}} + \alpha_i(\varphi_i, h) + c_2 \sum_{i\neq j} \alpha_i\alpha_j\mu_{(i,j)}\varepsilon_{ij} + \alpha_0^2||h||^2 \\ &+ 2c_2 \sum_{i,j>q} \alpha_i\alpha_j \frac{H(a_i,a_j)}{\sqrt{\lambda_i\lambda_j}} + ||v||^2 + o\Big(\sum_{i=1}^{p} \frac{1}{\sqrt{\lambda_i}} + \frac{1}{\lambda_i d_i} + \sum_{i\neq j} \varepsilon_{ij}\Big) \end{aligned} \tag{11}$$

For the denominator, we write

$$\begin{aligned} D^3 =& \int K(\sum_{i=1}^{p} \alpha_i\varphi_i)^6 + \alpha_0^6 \int Kw^6 + 6\alpha_0 \int K(\sum_{i=1}^{p} \alpha_i\varphi_i)^5 w \\ &+ 6\alpha_0^5 \int K(\sum_{i=1}^{p} \alpha_i\varphi_i)w^5 + 6\int_{S^3_+} K(\sum_{i=1}^{p} \alpha_i\varphi_i + \alpha_0 w)^5(\alpha_0 h + v) \\ &+ 15\int_{S^3_+} K(\sum_{i=1}^{p} \alpha_i\varphi_i + \alpha_0 w)^4(\alpha_0^2h^2 + v^2 + 2\alpha_0 hv) \\ &+ O(\sum \int w^4\varphi_i^2 + w^2\varphi_i^4) + O(||v||^3 + \|h\|^3). \end{aligned} \tag{12}$$

Observe that

$$\begin{aligned} \int_{S^3_+} K(\sum_{i=1}^{p} \alpha_i\varphi_i)^6 =& \sum_{i=1}^{p} \alpha_i^6 K(a_i)\frac{S_3}{2}\mu_i + 6c_2 \sum_{i\neq j} \alpha_i^5\alpha_j K(a_i)\mu_{(i,j)}\varepsilon_{ij} \\ &+ 12c_2 \sum_{i,j>q} \alpha_i^5\alpha_j K(a_i)\frac{H(a_i,a_j)}{\sqrt{\lambda_i\lambda_j}} + O(\frac{1}{\lambda_i}) + o(\varepsilon_{ij}) + o(\sum_{i>q} \frac{1}{\lambda_i d_i}). \end{aligned} \tag{13}$$

$$\int_{S^3_+} Kw^6 = ||w||^2; \quad \int_{S^3_+} Kw^5\varphi_i = c_2\mu_i \frac{w(a_i)}{\lambda_i^{1/2}} + o(\frac{1}{\sqrt{\lambda_i}}), \tag{14}$$

$$\int_{S^3_+} K(\sum \alpha_i\varphi_i)^5 w = c_2 \sum \mu_i\alpha_i^5 K(a_i)\frac{w(a_i)}{\lambda_i^{1/2}} + o(\frac{1}{\sqrt{\lambda_i}}), \tag{15}$$

$$\int_{S^3_+} \varphi_i^4 w^2 + \varphi_i^2 w^4 = O(\frac{1}{\lambda_i}), \tag{16}$$

$$\begin{aligned} \int K(\sum \alpha_i\varphi_i + \alpha_0 w)^4 vh &= O\left(\int (\sum \alpha_i\varphi_i)^4|v||h| + (\sum \alpha_i\varphi_i)^3 w|v||h|\right) \\ &= O\left(\|v\|^3 + \|h\|^3 + 1/\lambda_i^{3/2}\right), \end{aligned} \tag{17}$$

where we have used that $h \in T_w(W_u(w))$ and $v \in T_w(W_s(w))$ and the fact that h belongs to a finite dimensional space which implies that $\|h\|_\infty \leq c\|h\|$.
Concerning the linear form of v, since $v \in T_w(W_s(w))$, it can be written as

$$\begin{aligned} \int K(\sum_{i=1}^{p} \alpha_i\varphi_i + \alpha_0 w)^5 v &= \int K(\sum_{i=1}^{p} \alpha_i\varphi_i)^5 v + O\Big(\sum_{i=1}^{p} \int (\varphi_i^4 w + \varphi_i w^4)|v|\Big) \\ &= f_1(v) + O\left(\frac{||v||}{\sqrt{\lambda_i}}\right). \end{aligned} \tag{18}$$

Finally, we have

$$\int K(\sum_{i=1}^{p}\alpha_i\varphi_i+\alpha_0 w)^4h^2=\alpha_0^4\int Kw^4h^2+o(||h||^2) \tag{19}$$

$$\int K(\sum_{i=1}^{p}\alpha_i\varphi_i+\alpha_0 w)^4v^2=\sum_{i=1}^{p}\int K(\alpha_i\varphi_i)^4v^2+\alpha_0^4\int Kw^4v^2+o(||v||^2). \tag{20}$$

Combining (11),...,(20), the result follows. □

Lemma 3.1. *We have*
(a) $Q_1(v,v)$ is a quadratic form positive definite in
$$E_v=\{v\in H^1(S^n_+)/v\in T_w(W_s(w)) \text{ and } v \text{ satisfies } (V_0)\}.$$
(b) $Q_2(h,h)$ is a quadratic form negative definite in $T_w(W_u(w))$.

Proof. Claim (a) can be proved following the proof of Lemma 6 of [10]. Regarding Claim (b), it follows from the fact that $h\in T_w(W_u(w))$ and $\alpha_0^4\gamma_1=\beta_1+o(1)$. □

Corollary 2. *Let $u=\sum_{i=1}^p\alpha_i\varphi_{(a_i,\lambda_i)}+\alpha_0(w+h)+v\in V(p,\varepsilon,w)$. There is an optimal $(\overline{v},\overline{h})$ and a change of variables $v-\overline{v}\to V$ and $h-\overline{h}\to H$ such that J reads as*

$$J(u)=J\left(\sum_{i=1}^{p}\alpha_i\varphi_{(a_i,\lambda_i)}+\alpha_0 w+\overline{h}+\overline{v}\right)+||V||^2-||H||^2.$$

Furthermore we have the following estimates

$$||\overline{h}||\leq\sum_i\frac{c}{\lambda_i^{\frac{1}{2}}} \quad\text{and}\quad ||\overline{v}||\leq\sum_i\frac{c}{\lambda_i}+\sum_{i>q}\frac{c}{(\lambda_i d_i)^2}+c\sum\varepsilon_{kr}(Log\varepsilon_{kr}^{-1})^{\frac{1}{3}}.$$

Hence

$$\begin{aligned}J(u)=&\frac{(S_3/2)\sum_{i=1}^p\mu_i\alpha_i^2+\alpha_0^2||w||^2}{((S_3/2)\sum_{i=1}^p\mu_i\alpha_i^6K(a_i)+\alpha_0^6||w||^2)^{\frac{1}{3}}}\Bigg[1-2c_2\alpha_0\sum_{i=1}^{p}\alpha_i\mu_i\frac{w(a_i)}{\lambda_i^{1/2}}\\&-c_2\sum_{i\neq j}\alpha_i\alpha_j\mu_{(i,j)}\varepsilon_{ij}-2c_2\sum_{i,j>q}\alpha_i\alpha_j\frac{H(a_i,a_j)}{\sqrt{\lambda_i\lambda_j}}\\&+o\left(\sum_{i\neq j}\varepsilon_{ij}+\sum_{i>q}\frac{1}{\lambda_i d_i}+\sum_{i=1}^{p}\frac{1}{\lambda_i^{1/2}}\right)\Bigg]+||V||^2-||H||^2.\end{aligned}$$

Proof. The expansion of J with respect to h (respectively to v) is very close, up to a multiplicative constant, to $Q_2(h,h)+f_2(h)$ (respectively $Q_1(v,v)+f_1(v)$). Since Q_2 is negative definite (respectively Q_1 is positive definite), there is a unique maximum $\overline{h}$ in the space of h (respectively a unique minimum $\overline{v}$ in the space of v). Furthermore, it is easy to derive $||\overline{h}||\leq c||f_2||$ and $\|\overline{v}\|\leq c\|f_1\|$. The estimate of $\overline{v}$ follows from Proposition 3. For the estimate of $\overline{h}$, we use the fact that for each $h\in T_w(W_u(w))$ which is a finite dimensional space, we have $\|h\|_\infty\leq c\|h\|$. Therefore, we derive that $\|f_2\|=O(\sum\lambda_i^{-1/2})$. Then our result follows. □

Corollary 3. *Let K be a C^2 positive function and let w be a nondegenerate critical point of J in Σ^+ with a finite Morse index. Then, for each $p\in\mathbb{N}^*$, there is no critical points or critical points at infinity in the set $V(p,\varepsilon,w)$, that means we can construct a pseudogradient of J so that the Palais-Smale condition is satisfied along the decreasing flow lines.*

The proof of this corollary is immediately from the above corollary and the fact that $w > 0$ in $\overline{S^3_+}$.

4. **A Morse Lemma near critical points at infinity.** In this section we perform a Morse type reduction in a neighborhood of critical points at infinity following the original ideas introduced by A. Bahri and J. M. Coron [12] and then refined developed and adapted to variational problems on manifolds with boundary through the works of A. Bahri and his students see [10], [14], [15], [16], [17] and the references therein .

Such a *normal form* for the gradient flow is then used to characterize the critical points at infinity of J and to compute the difference of topology they induce between the level sets of the functional.

Performing such a reduction relies on the construction of a global pseudogradient flow for the functional J. Such a pseudogradient is constructed so that along its flow lines there can be only finitely many isolated blow up points. Outside of $(\cup_p V(p,\varepsilon/2))$, we use $-\nabla J$ which satisfies the Palais-Smale condition in this set. In $V(p,\varepsilon)$, we construct a vector field W. Such a flow is defined by combining two basic facts. On the one hand, the first one comes from the Morse Lemma at infinity which moves points and concentrations as follows: points move according to ∇K, concentrations move so as to decrease the functional J. On the other hand, there is another pseudogradient when the α_i's are not in their maximum values. We need to convex-combine both flows to keep the pseudogradient property, to avoid the creation of new asymptotes.

We will prove that the first pseudogradient and the second pseudogradient will combine to increase the total interaction $\sum_{i\neq j}\varepsilon_{ij}$ when the points are very close. Now we proceed with our construction.

In the sequel we denote by $V_b(p,\varepsilon)$ the subset of $V(p,\varepsilon)$ such that all the concentration points are on the boundary.

Proposition 6. *Assume K satisfies $(C1)$. For any $p \geq 1$, there exists a pseudogradient W so that the following holds:*
There is a constant $c > 0$ independent of $u = \sum_{i=1}^p \alpha_i\delta_i \in V_b(p,\varepsilon)$ so that
(i) $(-\nabla J(u), W)$

$$\geq c\sum_{i=1}^{p}\Big(\frac{1}{\lambda_i}|\frac{\partial K(a_i)}{\partial\nu}| + \frac{|\nabla_T K(a_i)|}{\lambda_i} + |1 - J(u)^3\alpha_i^4 K(a_i)| + \sum_{k\neq r}\varepsilon_{kr}\Big)$$

(ii) $(-\nabla J(u+\overline{v}), W + \dfrac{\partial\overline{v}}{\partial(\alpha_i,a_i,\lambda_i)}(W))$

$$\geq c\sum_{i=1}^{p}\Big(\frac{1}{\lambda_i}|\frac{\partial K(a_i)}{\partial\nu}| + \frac{|\nabla_T K(a_i)|}{\lambda_i} + |1 - J(u)^3\alpha_i^4 K(a_i)| + \sum_{k\neq r}\varepsilon_{kr}\Big)$$

(iii) $|W|$ is bounded. Furthermore, the only cases where the maximum of the λ_i's is not bounded is when the concentration points $(a_1,\cdots,a_p)$ satisfy: each point a_i is close to a critical point y_{j_i} of K_1 with $j_i \neq j_k$ for $i \neq k$, $(\partial K/\partial\nu)(y_{j_i}) > 0$ and $\rho(y_{i_1},...,y_{i_p}) > 0$, where $\rho(y_{i_1},...,y_{i_p})$ denotes the least eigenvalue of $M(y_{i_1},...,y_{i_p})$.

Before given the proof of this Proposition, we derive from it a precise characterization of the critical points at infinity and compute their Morse Index. This is the content of the following corollary.

Corollary 4. *Assume that K satisfies $(C1)$, then the only critical points at infinity of J correspond to $\sum_{i=1}^p K(y)^{-1/4}\delta_{(y_{j_i},\infty)}$ where $p \geq 1$ and the y_{j_i}'s satisfy $(\partial K/\partial \nu)(y_{j_i}) > 0$ and $\rho(y_{j_1}, ..., y_{j_p}) > 0$. Furthermore such a critical point at infinity has a Morse index equal to $(3p - 1 - \sum_{j=1}^p index(K_1, y_{i_j}))$, where $index(K_1, y_{i_j})$ is the Morse index of K_1 at y_{i_j}.*

Proof. ¿From Proposition 6, we know that the only region where the λ_i's are unbounded is when each point a_i is close to a critical point y_{j_i}, with $j_i \neq j_k$ for $i \neq k$ and $\rho(y_{i_1}, ..., y_{i_p}) > 0$. In this region, arguing as in [10] and [14], we can find a change of variable

$$(a_1, ..., a_p, \lambda_1, ..., \lambda_p) \to (\tilde{a}_1, ..., \tilde{a}_p, \tilde{\lambda}_1, ..., \tilde{\lambda}_p) := (\tilde{a}, \tilde{\lambda})$$

such that

$$\begin{aligned} J\Big(\sum_{i=1}^p \alpha_i\delta_i + \overline{v}\Big) &= \psi(\alpha, \tilde{a}, \tilde{\lambda}) \\ &:= \frac{S_3^{2/3}\sum_{i=1}^p \alpha_i^2}{(4\sum_{i=1}^p \alpha_i^6 K(a_i))^{1/3}}\left(1 + (c-\eta)\Big(\sum_{i=1}^p \frac{1}{K(y_{j_i})}\Big)^{-1} {}^t\Lambda M(\tau_p)\Lambda\right) \end{aligned} \tag{21}$$

where $\alpha = (\alpha_1, ..., \alpha_p)$, c is a positive constant, η is a small positive constant, ${}^t\Lambda = (\tilde{\lambda}_1^{-1/2}, ..., \tilde{\lambda}_p^{-1/2})$, $\tau_p = (y_{j_1}, ..., y_{j_p})$ and $S_3 = \int_{\mathbb{R}^3}\delta_{(o,1)}^6$.
This yields a split of variables $\tilde{a}$ and $\tilde{\lambda}$, thus it is easy to see that if $\tilde{a}$ is equal to $(y_{j_1}, ..., y_{j_p})$, only $\tilde{\lambda}$ can move. Since $\rho(y_{i_1}, ..., y_{i_p}) > 0$, in order to decrease the functional J, we have to increase $\tilde{\lambda}$, and we obtain a critical point at infinity only in this case.
It remains to compute the Morse index of such a critical point at infinity. In order to compute such a Morse index, we observe that $M(\tau_p)$ is definite positive and the function ψ possesses, with respect to the variables α_i's, an absolute degenerate maximum with one dimensional nullity space. Then the Morse index of such a critical point at infinity is equal to $(p - 1 - \sum_{i=1}^p(2 - \text{index}(K_1, y_{j_i})))$. Thus our result follows. □

Proof of Proposition 6. The construction of our pseudogradient with the required properties will be constructed by distinguishing different regions in $V_b(p, \varepsilon)$. In each of these regions an appropriate pseudogradient will be constructed. Then our W will be a convex combination of all cases.
Claim (ii) relies on Claim (i) and the estimate of $||\nabla J(u+\overline{v})||^2$, $||\overline{v}||^2$ and $J"(u)\overline{v}W$ (see Lemma B.4 of [14]). We prove that they are very small with respect to the lower bound of (i). Hence, once Claim (i) is proved, Claim (ii) follows.
Without loss of generality, we can assume that $\lambda_1 \leq ... \leq \lambda_p$. For M and C_1 large positive constants and η a small positive constant such that ηM is large, we define

$$\begin{aligned} \mathcal{D} =& \{1\} \cup \{i/\lambda_k \leq M\lambda_{k-1}\ \forall\, k \text{ s.t. } 2 \leq k \leq i\}, \\ \mathcal{E} =& \{i/d(a_i, y) > \frac{\eta}{2}\ \forall y \text{ s.t. } \nabla_T K(y) = 0\}, \\ \mathcal{C}_k =& \{i/d(a_i, y_k) < \eta\}, \text{ where } y_k \text{ is a critical point of } K_1, \\ L =& \left\{i/|1 - J(u)^3\alpha_i^4 K(a_i)| \geq C_1\left(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right)\right\}. \end{aligned}$$

1st case. There exist $i, j \in \mathcal{D}$ such that $d(a_i, a_j) < 4\eta$.
Let i_0 be the least index in $\mathcal{D}$ such that there exists $j_0 < i_0$ satisfying $d(a_{i_0}, a_{j_0}) <$

4η. It follows then that we have $\lambda_{j_0} \le \lambda_{i_0}$ and for any $i, j < i_0$, we have $d(a_i, a_j) \ge 4\eta$. Now we define

$$Y_1 = -\sum_{k=i_0}^{p} 2^k \alpha_k \lambda_k \frac{\partial \delta_k}{\partial \lambda_k} \text{ and } Y_\alpha = -\sum_{i\in L} \text{sign}\left(1 - J(u)^3 \alpha_i^4 K(a_i)\right) \delta_i. \tag{22}$$

Using Proposition 10 of the appendix and the fact that C_1 is a large constant, we derive that

$$(-\nabla J(u), Y_\alpha) \ge c \sum_{i\in L} |1 - J(u)^3 \alpha_i^4 K(a_i)| + O\Big(\sum_{k\ne i} \varepsilon_{ki}\Big). \tag{23}$$

Furthermore some computations show that

$$-\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = \frac{\varepsilon_{ij}}{2}\left(1 - 2\frac{\lambda_j}{\lambda_i}\varepsilon_{ij}^2\right) ; -2^i \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} - 2^j \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \ge \varepsilon_{ij}(1 + o(1)). \tag{24}$$

Thus, using again Proposition 10, we derive that:

$$(-\nabla J(u), Y_1) \ge c \sum_{k\ge i_0} \left(\sum_{r\ne k} \varepsilon_{kr} + O\left(\frac{1}{\lambda_k}\right)\right) + o\left(\sum_{r\ne l} \varepsilon_{rl}\right). \tag{25}$$

Now notice that $1/\lambda_{i_0} = o(\varepsilon_{i_0 j_0})$, indeed from the fact that: $\lambda_{j_0} \le \lambda_{i_0}$, we obtain that:

$$\frac{1}{\lambda_{i_0}^2 \varepsilon_{i_0 j_0}^2} = \frac{1}{\lambda_{i_0}\lambda_{j_0}} + \frac{\lambda_{j_0}}{\lambda_{i_0}^3} + \frac{\lambda_{j_0}}{2\lambda_{i_0}}(1 - \cos d(a_{i_0}, a_{j_0})) \le o(1) + 4\eta^2 = o(1). \tag{26}$$

Thus, since for $k \ge i_0$ we have $\lambda_k \ge \lambda_{i_0}$, (25) becomes

$$(-\nabla J(u), Y_1) \ge c \sum_{k\ge i_0, r\ne k} \varepsilon_{kr} + \frac{c}{\lambda_{i_0}} + o\Big(\sum_{l\ne r} \varepsilon_{lr}\Big). \tag{27}$$

Using the fact that $i_0 \in \mathcal{D}$, then we can make $1/\lambda_1$ appear in the lower bound of (27) and therefore all $1/\lambda_i$. Furthermore, for $i, j < i_0$, we have $d(a_i, a_j) \ge 4\eta$, then from $1/\lambda_i$ and $1/\lambda_j$ we can make ε_{ij} appear in the lower bound. Thus, we derive that

$$(-\nabla J(u), Y_1) \ge c \sum_{i=1}^{p} \left(\frac{1}{\lambda_i}|\frac{\partial K(a_i)}{\partial \nu}| + \frac{|\nabla_T K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{k\ne r} \varepsilon_{kr}\right). \tag{28}$$

Now, define $W_1 = M_1 Y_1 + Y_\alpha$, where M_1 is a large positive constant, we obtain the desired inequality in (i) with W_1 instead of W.
2nd case. For each $i, j \in \mathcal{D}$ we have $d(a_i, a_j) \ge 4\eta$ and $\mathcal{D} \cap \mathcal{E} \ne \emptyset$.
In this case, we define

$$Y_2 = \sum_{i\in \mathcal{D}\cap\mathcal{E}} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial a_i} \frac{\nabla_T K(a_i)}{|\nabla_T K(a_i)|} - \gamma_1 \sum_{i\notin \mathcal{D}} 2^i \alpha_i \lambda_i \frac{\partial \delta_i}{\partial \lambda_i},$$

where γ_1 is a large positive constant. Using Proposition 10 of the appendix, (24) and the fact that, for $i \in \mathcal{E}$ we have $|\nabla_T K(a_i)| \geq c\eta$, we obtain

$$
\begin{aligned}
(-\nabla J(u), Y_2) \geq & c \sum_{i\in\mathcal{D}\cap\mathcal{E}} \left(\frac{c\eta}{\lambda_i} + O\left(\sum_{j\in\mathcal{D}} \frac{1}{\lambda_i} |\frac{\partial \varepsilon_{ij}}{\partial a_i}| + \sum_{j\notin\mathcal{D}} \varepsilon_{ij} \right) \right) \\
& + \gamma_1 c \sum_{i\notin\mathcal{D}, j\neq i} \varepsilon_{ij} + O\left(\sum_{k\notin\mathcal{D}} \frac{\gamma_1}{\lambda_k} \right) + o\Big(\sum_{l\neq r} \varepsilon_{lr} \Big).
\end{aligned} \tag{29}
$$

Observe that, for $k \notin \mathcal{D}$ and $i \in \mathcal{D}$, if we choose γ_1 so that $\gamma_1/(\eta M)$ is small, we derive that $\gamma_1/\lambda_k \leq \gamma_1/(M\lambda_i) = o(\eta/\lambda_i)$. Furthermore, for $i, j \in \mathcal{D}$, since $d(a_i, a_j) \geq 4\eta$, we derive that

$$
\frac{1}{\lambda_i} |\frac{\partial \varepsilon_{ij}}{\partial a_i}| \leq c\lambda_j d(a_i, a_j) \varepsilon_{ij}^3 \leq \frac{c}{\lambda_i^{3/2} \lambda_j^{1/2} \eta^2} = o\left(\frac{\eta}{\lambda_i} \right).
$$

Thus, (29) becomes

$$
(-\nabla J(u), Y_2) \geq \sum_{i\in\mathcal{D}\cap\mathcal{E}} \frac{c}{\lambda_i} + \sum_{i\notin\mathcal{D}, j\neq i} \varepsilon_{ij} + o\Big(\sum_{l\neq r} \varepsilon_{lr} \Big). \tag{30}
$$

As in the first case, we obtain (28) with Y_2 instead of Y_1 and defining $W_2 = M_1 Y_2 + Y_\alpha$, we derive the desired inequality in (i) with W_2 instead of W.

3rd case. For each $i, j \in \mathcal{D}$, we have $d(a_i, a_j) \geq 4\eta$ and $\mathcal{D} \cap N_- \neq \emptyset$, where $N_- = \{i / \exists\, y \text{ s.t. } \nabla_T K(y) = 0,\ (\partial K/\partial \nu)(y) < 0 \text{ and } d(a_i, y) < \eta\}$.
In this case, we define

$$
Y_3 = -\sum_{i\in\mathcal{D}\cap N_-} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} - \sum_{i\notin\mathcal{D}} 2^i \alpha_i \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}.
$$

Using Proposition 10 of the appendix and (24), we derive that

$$
(-\nabla J(u), Y_3) \geq c \sum_{i\in\mathcal{D}\cap N_-} \left(\frac{1}{\lambda_i} + \sum_{j\in\mathcal{D}} \varepsilon_{ij} \right) + c \sum_{k\notin\mathcal{D}} \varepsilon_{kj} + O\left(\sum_{k\notin\mathcal{D}} \frac{1}{\lambda_k} \right).
$$

As in the first case, we obtain (28) with Y_3 instead od Y_1 and we define $W_3 = M_1 Y_3 + Y_\alpha$, thus we derive the desired inequality in (i) with W_3 instead of W.

The Last case. For each $i, j \in \mathcal{D}$, we have $d(a_i, a_j) \geq 4\eta$ and $\mathcal{D} \subset N_+$, where $N_+ = \{i / \exists\, y \text{ s.t. } \nabla_T K(y) = 0,\ (\partial K/\partial \nu)(y) > 0 \text{ and } d(a_i, y) < \eta\}$.
In this case, we have, for each k, $B(y_k, \eta)$ contains at most a point of the a_i's for $i \in D$. Let us denote by $y_{i_1}, ..., y_{i_q}$ the critical points such that $a_j \in B(y_{i_j}, \eta)$, where $q = \text{card}(\mathcal{D})$. The vector field will depend on the sign of the least eigenvalue ρ of $M(y_{i_1}, ..., y_{i_q})$, where M is the matrix defined in (3). Two subcases may occur.

1st subcase. If $\rho > 0$. In this case, we define

$$
Y_6^1 = \sum_{i\in\mathcal{D}} \alpha_i \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} - \sum_{i\notin\mathcal{D}} 2^i \alpha_i \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}.
$$

Observe that, for $i, k \in \mathcal{D}$, we have $d(a_i, a_k) = d(y_{j_i}, y_{j_k}) + o(1)$. Thus, (24) implies

$$
-\lambda_i \frac{\partial \varepsilon_{ik}}{\partial \lambda_i} = \frac{\sqrt{2} + o(1)}{2[\lambda_i \lambda_k (1 - \cos d(y_{j_i}, y_{j_k}))]^{1/2}} = \frac{\sqrt{2} G(y_{j_i}, y_{j_k})}{2(\lambda_i \lambda_k)^{1/2}} + o(\varepsilon_{ik}). \tag{31}
$$

Using Proposition 10 of the appendix, (24), (31) and the fact that $J(u)^3\alpha_i^4K(a_i)=1+o(1)$, we obtain

$$\begin{aligned}(-\nabla J(u),Y_6^1)\geq&\frac{\pi\sqrt{3}}{2J(u)^{\frac12}}\sum_{i,k\in\mathcal{D}}\left(\frac{1}{\lambda_i}\frac{\frac{\partial K}{\partial\nu}(y_{j_i})}{K(y_{j_i})^{\frac32}}-\frac{4\sqrt{2}G(y_{j_i},y_{j_k})}{(K(y_{j_i})K(y_{j_k}))^{\frac14}}\frac{1}{(\lambda_i\lambda_k)^{\frac12}}\right)\\&+o\left(\sum_{i,k\in\mathcal{D}}\varepsilon_{ik}+\sum_{i\in\mathcal{D}}\frac{1}{\lambda_i}\right)+c\sum_{k\notin\mathcal{D},k\neq r}\varepsilon_{kr}+O\Big(\sum_{k\notin\mathcal{D}}\frac{1}{\lambda_k}\Big)\\\geq&\frac{\pi\sqrt{3}}{2J(u)^{\frac12}}\Lambda^tM(y_{j_1},...,y_{j_q})\Lambda+c\sum_{k\notin\mathcal{D},k\neq r}\varepsilon_{kr}\\&+o\left(\sum_{i,k\in\mathcal{D}}\varepsilon_{ik}+\sum_{i\in\mathcal{D}}\frac{1}{\lambda_i}\right)+O\Big(\sum_{k\notin\mathcal{D}}\frac{1}{\lambda_k}\Big)\end{aligned}\tag{32}$$

where $\Lambda=(\lambda_{i_1}^{-1/2},...,\lambda_{i_q}^{-1/2})^t$. Using the fact that $\rho>0$, we obtain

$$(-\nabla J(u),Y_6^1)\geq\sum_{i\in\mathcal{D}}\frac{c}{\lambda_i}+c\sum_{k\notin\mathcal{D},k\neq r}\varepsilon_{kr}.\tag{33}$$

Now, we define $W_6^1=M_1Y_6^1+Y_\alpha$ and we conclude as in the above cases.
2nd subcase. If $\rho<0$. We denote by e the eigenvector associated to ρ, we choose it to satisfy $||e||=1$ and all the components are positive. Now, as in [14], we introduce a small neighborhood of e by: for $\gamma_2>0$ small,

$$\begin{aligned}B(e,\gamma_2)=&\{x\in S^{q-1}:||x-e||\ <\gamma_2\}\\T_1(e,\gamma_2)=&\{x\in(\mathbb{R}_+)^q\setminus\{0\}:||x||^{-1}x\in B(e,\gamma_2)\}.\end{aligned}$$

We will choose γ_2 so that, for each $x\in B(e,\gamma_2)$ we have $x^tMx<\rho/2$. Two cases may occur.
- If $\Lambda\in T_1(e,\gamma_2)$. In this case, we decrease the λ_i's that is we define $Y=-\sum_{i\in\mathcal{D}}\alpha_i\lambda_i\partial\delta_i/\partial\lambda_i$.
- In the other case, we define Y by moving the vector Λ to the vector e on the sphere of radius $||\Lambda||$ (see the vector field X_3 of [14]).
Now, define $Y_6^2=Y-\sqrt{M}\sum_{i\notin\mathcal{D}}2^i\alpha_i\lambda_i\partial\delta_i/\partial\lambda_i$. Using [14] and Proposition 10 of the appendix, we obtain

$$\begin{aligned}(-\nabla J(u),Y_6^2)\geq&\sum_{i\in\mathcal{D}}\Big(\frac{c}{\lambda_i}+O(\sum_{k\notin\mathcal{D}}\varepsilon_{ki})\Big)+\sqrt{M}c\sum_{i\notin\mathcal{D}}\Big(\sum_{j\neq i}\varepsilon_{ij}+O(\frac{1}{\lambda_i})\Big)\\\geq&c\sum_{i=1}^p\frac{1}{\lambda_i}+c\sum_{k\neq r}\varepsilon_{kr}.\end{aligned}\tag{34}$$

We conclude as in the above cases by defining $W_6^2=M_1Y_6^2+Y_\alpha$.
The vector field W will be a convex combination of the above vector fields. Clearly it satisfies the claim (i). It remains to prove claim (iii). Observe that, in the cases 1, 2 and 3, the λ_i's are decreasing functions along our vector field, while in the last case, the maximum of λ_i's increases only if $\mathcal{D}^c=\emptyset$ and $\rho>0$. The proof of Proposition 6 is thereby completed. $\square$

5. **Ruling out existence of interior or mixed blow ups.** The aim of this section is to rule out the existence of critical points at infinity with interior concentration points or consisting of a mixed configurations of interiors as well as boundary points. The idea is that in a neighborhood of such possible critical points at infinity to construct a pseudogradient of J along which the concentration $\lambda_{\max} = \max(\lambda_1, \cdots, \lambda_p)$ is decreasing. We remark that we can separate the variable v from the other variables using Corollary 2 with $w = 0$. Hence we only need to construct a pseudogradient W on the variables α_i, a_i and λ_i.
We consider first the case of a configuration of purely interior blow up points.

5.1. **Ruling out interior blow ups.** In this subsection we consider an ε-neighborhood of p bubbles $\varphi_{(a_i,\lambda_i)}$, denoted as before $V(p,\varepsilon) := V(p,\varepsilon,0)$. In this case Corollary 2 implies that $J(u)$ is expanded as

$$J(u) = \frac{S_3^{2/3}\sum_{i=1}^p \alpha_i^2}{(\sum_{i=1}^p \alpha_i^6 K(a_i))^{\frac{1}{3}}}\left(1 - 2c_2\sum_{i\neq j}\varepsilon_{ij} - 2c_2\sum_{i,j}\frac{H(a_i,a_j)}{\sqrt{\lambda_i\lambda_j}}\right.$$
$$\left. + o\left(\sum_{i\neq j}\varepsilon_{ij} + \sum_{i=1}^p \frac{1}{\lambda_i d_i}\right)\right) + ||V||^2.$$

Thus, we can remark that: to decrease J, we need to decrease the λ_i's. Precisely we have the following result

Proposition 7. *For $\varepsilon > 0$ small enough and $p \geq 1$, there exists a pseudogradient W so that the following holds:*
There is a constant $c > 0$ independent of $u = \sum_{i=1}^p \alpha_i\varphi_i \in V(p,\varepsilon)$, (with $\lambda_i d_i \to \infty$ for each i) so that

(i) $$(-\nabla J(u), W) \geq c\sum\frac{1}{\lambda_i d_i} + c\sum\varepsilon_{kr}$$

(ii) $$(-\nabla J(u+\overline{v}), W + \frac{\partial\overline{v}}{\partial(\alpha_i,a_i,\lambda_i)}(W)) \geq c\sum\frac{1}{\lambda_i d_i} + c\sum\varepsilon_{kr}$$

(iii) $|W|$ is bounded. Furthermore, the λ_i's are decreasing functions.

Proof. Without loss of generality, we can assume that $\lambda_1 \leq \cdots \leq \lambda_p$. Let us define

$$W = -\sum_{i=1}^p 2^i\alpha_i\lambda_i\frac{\partial\varphi_i}{\partial\lambda_i}.$$

It follows then from Proposition 11 of the appendix that:

$$(-\nabla J(u), W) = J(u)\sum_{i\leq p}\alpha_i 2^i\left(-c_1\sum_{j\neq i}\alpha_j\lambda_i\frac{\partial\varepsilon_{ji}}{\partial\lambda_i} + c_2\sum_{j\leq p}\alpha_j\frac{H(a_j,a_i)}{(\lambda_j\lambda_i)^{1/2}}\right)$$
$$+ o\left(\sum_{k\leq p}\frac{1}{\lambda_k d_k} + \sum_{k\neq r}\varepsilon_{kr})\right). \tag{35}$$

From (24) and the fact that $H(a,.) \geq C/dist(a,\partial S^3_+)$, (35) becomes

$$(-\nabla J(u), W) \geq c\sum_{k=1}^p\frac{1}{\lambda_k d_k} + c\sum_{r\neq k}\varepsilon_{kr},$$

which implies Claim (i).
Regarding Claim (ii), it relies on the estimates of $||\nabla J(u+\overline{v})||^2$, $||\overline{v}||^2$ and $J"(u)\overline{v}W$ which are very small with respect of the lower bound of (i) (see Lemma B.4 of [14]). Hence claim (ii) follows.
Concerning Claim (iii), it follows from the definition of W and the fact that $\|\lambda_i \partial\varphi_i/\partial\lambda_i\|$ is bounded. □

As a consequence of the above Proposition, any vector field consisting in decreasing the concentrations λ_i will decrease the functional bringing $u(s) := \sum_{i=1}^p \alpha_i(s)\,\varphi_{a_i,\lambda_i}$ outside $V(p,\varepsilon_0)$ for some ε_0 very small. Hence we have:

Corollary 5. *Let $V_{in}(p,\varepsilon)$ to be the subset of $V(p,\varepsilon)$ such that $\lambda_i d_i \to \infty$ for each i. For each positive function K, in $V_{in}(p,\varepsilon)$, the functional J has no critical point or critical point at infinity.*

5.2. **Ruling out mixed blow ups.** Now we come to the case of mixte blow up consisting of a configuration where we have boundary concentrations as well as interior concentrations.

Proposition 8. *Assume that K satisfies $(C1)$. For $\varepsilon > 0$ small enough and $p \geq 2$, there exists a pseudogradient W so that the following holds:*
There is a positive constant c independent of $u = \sum_{i=1}^q \alpha_i\delta_i + \sum_{i=q+1}^p \alpha_i\varphi_i$ in $V(p,\varepsilon)$, (with $a_i \in \partial S^3_+$ for $i \leq q$ and $\lambda_i d_i \to \infty$ for each $i > q$) so that

$$(i) \quad (-\nabla J(u), W) \geq c\sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + |1 - J(u)^3\alpha_i^4 K(a_i)| + \sum_{i>q}\frac{c}{\lambda_i d_i} + c\sum_{k\neq r}\varepsilon_{kr}$$

$$(ii) \quad (-\nabla J(u+\overline{v}), W + \frac{\partial\overline{v}}{\partial(\alpha_i,a_i,\lambda_i)}(W))$$

$$\geq c\sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + |1 - J(u)^3\alpha_i^4 K(a_i)| + c\sum_{i>q}\frac{1}{\lambda_i d_i} + c\sum_{k\neq r}\varepsilon_{kr}$$

(iii) $|W|$ is bounded. Furthermore, the maximum of the λ_i's is a decreasing function.

Proof. Without loss of generality, we can assume that $\lambda_{q+1} \leq \cdots \leq \lambda_p$ and as in the proof of Proposition 7, we define

$$F_1 := -\sum_{i=q+1}^p 2^i\alpha_i\lambda_i\frac{\partial\varphi_i}{\partial\lambda_i},$$

Then we derive

$$(-\nabla J(u), F_1) \;\geq\; c\sum_{k=q+1}^p \Big(\frac{1}{\lambda_k d_k} + c\sum_{r\neq k}\varepsilon_{kr}\Big) + o\Big(\sum_{i\neq r}\varepsilon_{ir}\Big). \tag{36}$$

Now, ordering the λ_i's for $i \leq q$: $\lambda_1 \leq \cdots \leq \lambda_q$ and for a large fixed positive constant M, setting

$$D = \{1\} \cup \{2 \leq i \leq q;\ \lambda_k \leq M\lambda_{k-1} \text{ for each } k \leq i\}$$

which implies that, if $i \in D$, then λ_i and λ_1 are of the same order. Two cases may occur.
case 1: There exist $i_0 \in D$ and $j_0 > q$ such that $\lambda_{j_0} \leq M\lambda_{i_0}$.

Then from $1/(\lambda_{j_0} d_{j_0})$ which appear in the lower bound of (36) we can make appear all the variables $1/\lambda_i$, for $i \leq q$. We define

$$W_1 = F_1 + mY_\alpha - m \sum_{i \leq q} 2^i \alpha_i \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}$$

where m is a small constant. As before we deduce

$$(-\nabla J(u), W_1) \geq c \sum \varepsilon_{kr} + \sum_{i>q} \frac{c}{\lambda_i d_i} + c \sum_{i \leq p} (\frac{1}{\lambda_i} + |1 - J(u)^3 \alpha_i^4 K(a_i)|).$$

Hence Claims (i) and (ii) follow for this pseudogradient.
case 2: For each $i \in D$ and each $j > q$ we have $M\lambda_i \leq \lambda_j$.
Let $u_1 = \sum_{i \in D} \alpha_i \delta_i$. We have $u_1 \in V_b(\text{card } D, \varepsilon)$. In this set we constructed a pseudogradient, we will denote it by F_2, by Proposition 6. Thus we define $W_2 = F_1 + F_2$. This pseudogradient satisfies the claims (i) and (ii). It remains to remark that the index of the maximum of the λ_i's does not belong to D. Hence Claim (iii) follows.
Therefore along such a pseudogradient the maximum of the λ_l's is decreasing. Hence such a mixte configuration cannot be a critical point at infinity. □

6. Proofs of the Theorems.

Proof. Proof of Theorem 1.1 First of all we assume that the solutions of (P_K) are of finite number, otherwise we are done. Let $\mathcal{K}$ denote the set of critical points of J and $\mathcal{K}_\infty$ denote the set of critical points at infinity.
Let $J^\beta := \{u \in \Sigma^+ : J(u) < \beta\}$. Observe that, it is well established that if z is the only critical point of J in the set $J^{c+\varepsilon} \setminus J^{c-\varepsilon}$, assume that there are also no critical point at infinity, then

$$J^{c+\varepsilon} \simeq J^{c-\varepsilon} \cup W_u(z),$$

where $\simeq$ denotes the retraction by deformation and $W_u(z)$ denotes the unstable manifold of the critical point.
Now we want to extend such an argument to the case where there is only one critical point at infinity z_∞ in $J^{c+\varepsilon} \setminus J^{c-\varepsilon}$. Recall that a critical point at infinity under the assumptions of Theorem 1.1 is one to one correspondence to p-uple $(y_1, ..., y_p)$ of critical points of K_1 such that $\rho(y_1, ..., y_p) > 0$ (see Corollary 4). A neighborhood of (z_∞) is a subset of $V(p, \varepsilon)$ (see definition 5) such that each a_i is close to some y_{j_i}.
Now recall that in such a neighborhood the following asymptotic expansion holds (see (21) and Corollary 2)

$$J\Big(\sum_{i=1}^p \alpha_i \delta_{(a_i,\lambda_i)} + v\Big) = g(\alpha, \tilde{a})\Big(1 + c\Big(\sum_{i=1}^p \frac{1}{K(y_{j_i})}\Big)^{-1} {}^t\Lambda M(\tau_p)\Lambda\Big) + \|V\|^2$$

where ${}^t\Lambda = (\tilde{\lambda}_1^{-1/2}, ..., \tilde{\lambda}_p^{-1/2})$, $\tau_p(y_{i_1}, ..., y_{i_p})$ and

$$g(\alpha, \tilde{a}) := \frac{S_3^{2/3} \sum_{i=1}^p \alpha_i^2}{(4 \sum_{i=1}^p \alpha_i^6 K(a_i))^{1/3}}.$$

It follows from the above expansion that there is a split of variables $\alpha, \tilde{a}, \tilde{\lambda}, v$. Minimizing with respect to v, the problem is reduced to a finite dimensional one. Moreover observe that, since the matrix M is positive definite, the expansion gives us an asymptote in the λ-variable. Finally the function $g(\alpha, \tilde{a})$ (which is homogenous with respect to α-variable) possesses a degenerate absolute maximum $\overline{\alpha}$ in the α-variable and one critical point $\overline{y} := (y_{i_1}, ..., y_{i_p})$ in the a-variable. Therefore the unstable manifold of (z_∞) can be defined as

$$W_u(\overline{\alpha}, \overline{y}) \times [A, \infty)$$

where $W_u(\overline{\alpha}, \overline{y})$ is the unstable manifold of $(\overline{\alpha}, \overline{y})$ for the function g.
Therefore we have also that

$$J^{c+\varepsilon} \simeq J^{c-\varepsilon} \cup W_u(z_\infty).$$

Let C_0 be large such that $\mathcal{K} \cup \mathcal{K}_\infty \subset J^{C_0}$. Without loss of generality we assume that there exist $0 < c_1 < ... < c_N < C_0$ such that each level c_j contains only one element of $\mathcal{K} \cup \mathcal{K}_\infty$. It follows from the above that

$$J^{C_0} \simeq \cup_{z \in \mathcal{K}} W_u(z) \cup \cup_{z_\infty \in \mathcal{K}_\infty} W_u(z_\infty).$$

Now using the Euler-Poincare characteristic (denoted by χ), it holds that

$$1 = \chi(J^{C_0}) = \sum_{z \in \mathcal{K}} (-1)^{m(z)} + \sum_{z_\infty \in \mathcal{K}_\infty} (-1)^{m(z_\infty)},$$

where $m(z)$ (resp. $m(z_\infty)$) denotes the Morse index of z (resp. z_∞). Hence

$$\Big| 1 - \sum_{z_\infty \in \mathcal{K}_\infty} (-1)^{m(z_\infty)} \Big| \leq \Big| \sum_{z \in \mathcal{K}} (-1)^{m(z)} \Big| \leq \# \mathcal{K}, \tag{37}$$

where $\#\mathcal{K}$ denotes the cardinal of $\mathcal{K}$.
Finally, using Corollary 4, the proof of Theorem 1.1 is completed. □

Proof. Proof of Theorems 1.2 and 1.3 The proof of Theorems 1.2 and 1.3 goes along with the proof of Theorem 1.1. As in this proof we have to construct a pseudogradient near infinity whose stationary points are the critical points at infinity. To do so we need to prove the equivalent of Proposition 6 under the Assumptions (A_1) and (A_2) or (A_1) and (A_f). Namely we want to prove:

Proposition 9. *Assume that K satisfies (A_1) and (A_2) (resp. (A_1) and (A_f)). For $p \geq 1$, there exists a pseudogradient W so that the following holds: There is a constant $c > 0$ independent of $u = \sum_{i=1}^p \alpha_i \delta_i \in V_b(p, \varepsilon)$ so that*

(i) $(-\nabla J(u), W) \geq c \sum_{i=1}^p \Big(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^\gamma} + |1 - J(u)^3 \alpha_i^4 K(a_i)| + \sum_{k \neq r} \varepsilon_{kr} \Big)$

(ii) $(-\nabla J(u + \overline{v}), W + \frac{\partial \overline{v}}{\partial(\alpha_i, a_i, \lambda_i)}(W))$

$$\geq c \sum_{i=1}^p \Big(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^\gamma} + |1 - J(u)^3 \alpha_i^4 K(a_i)| + \sum_{k \neq r} \varepsilon_{kr} \Big)$$

where $\gamma = 2$ if K satisfies (A_2) and $\gamma = \beta$ if K satisfies (A_f).
(iii) $|W|$ is bounded. Furthermore, the only cases where the maximum of the λ_i's is not bounded are when

- *$p = 1$ and the concentration point a is near a critical point y of K_1 with $(\partial K / \partial \nu)(y) = 0$ and $-\Delta K(y) > 0$ (resp. $\sum b_j(y) < 0$).*

- *$p \geq 1$ and each concentration point a_i is close to a critical point y_{j_i} of K_1 with $j_i \neq j_k$ for $i \neq k$ and $\rho(y_{i_1}, ..., y_{i_p}) > 0$, where $\rho(y_{i_1}, ..., y_{i_p})$ denotes the least eigenvalue of $M(y_{i_1}, ..., y_{i_p})$.*

Proof. We argue in a similar way as we did in the proof of Proposition 6. As our assumptions (A_1), (A_2) and (A_f), we have more cases than what we had. In the following we consider those cases:
1st new case. For each $i, j \in \mathcal{D}$, we have $d(a_i, a_j) \geq 4\eta$, $\mathcal{D} \cap \mathcal{E} = \emptyset$, $\mathcal{D} \cap N_- = \emptyset$, $\mathcal{D} \neq \{1\}$ and $\mathcal{D} \cap N_0 \neq \emptyset$, where $N_0 = \{i / \exists\, y \text{ s.t. } \nabla K(y) = 0 \text{ and } d(a_i, y) < \eta\}$.
Observe that, since $\mathcal{D} \cap \mathcal{E} = \emptyset$ then $\mathcal{D} \subset \cup_k \mathcal{C}_k$. Furthermore, using the fact that $d(a_i, a_j) \geq 4\eta$ for each $i, j \in \mathcal{D}$, then we conclude that $d(a_i, a_j) \geq \frac{1}{2} \min\{d(y_k, y_r), k \neq r\}$. Moreover, the fact that $\mathcal{D} \neq \{1\}$ allows us to prove that for $i \in \mathcal{D} \cap N_0$, the following holds true: $\lambda_i^{-1} \partial K/\partial \nu(a_i) = o(\varepsilon_{ij})$ where $j = i - 1$ or $j = i + 1$. In fact, notice that since $\partial K/\partial \nu(y) = 0$ we get $\partial K/\partial \nu(a) \leq c\eta$ for each $a \in B(y, \eta)$. Furthermore, for $i \in \mathcal{D} \cap N_0$, either $i - 1$ or $i + 1$ belongs to $\mathcal{D}$. As in (26), it easy to see that $1/\lambda_i \leq c\sqrt{M}\varepsilon_{ij}$ where $j = i - 1$ if $i - 1 \in \mathcal{D}$, if not $j = i + 1$. Thus, if we choose η small enough ($\eta\sqrt{M}$ small), our claim follows. Moreover, for $k \notin \mathcal{D}$, we have $1/\lambda_k \leq 1/(M\lambda_i)$, where $i \in D \cap N_0$.
In this case, we define

$$Y_4 = -\sum_{i \in \mathcal{D} \cap N_0} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} - \sum_{i \notin \mathcal{D}} 2^i \alpha_i \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}.$$

Using Proposition 10 of the appendix, (24) and the above arguments, we derive that

$$(-\nabla J(u), Y_4) \geq c \sum_{i \in \mathcal{D} \cap N_0} \left(\frac{1}{\lambda_i} + \sum_{j \in \mathcal{D}} \varepsilon_{ij} \right) + c \sum_{k \notin \mathcal{D}, j \neq k} \varepsilon_{kj}. \tag{38}$$

Now, we conclude as in the second and the third cases and we define $W_4 = M_1 Y_4 + Y_\alpha$.
2nd new case. $\mathcal{D} = \{1\} \subset N_0$.

In this case, a_1 is near a critical point y of K_1 which satisfies (A_2) or (A_f).
First, we deal with the case where (A_2) is satisfied. Let us define

$$Y_5 = \text{sign}(-\Delta K(y)) \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1} + \frac{\overline{c}_1}{\lambda_1} \frac{\partial \delta_1}{\partial a_1} \frac{\nabla_T K(a_1)}{|\nabla_T K(a_1)|} - \sum_{i \geq 2} 2^i \alpha_i \lambda_i \frac{\partial \delta_i}{\partial \lambda_i},$$

where $\overline{c}_1$ is a small positive constant. Using Proposition 10 of the appendix, (24) and assumption (A_2), we derive that

$$(-\nabla J(u), Y_5) \geq c \frac{|\nabla K(a_1)|}{\lambda_1} + \frac{c}{\lambda_1^2} + c \sum_{k \neq r} \varepsilon_{kr} + O(\frac{1}{\lambda_2}). \tag{39}$$

As in (26), since $2 \notin \mathcal{D}$, it is easy to see that $1/\lambda_2 = o(\varepsilon_{12})$. Now, define $W_5 = M_1 Y_5 + Y_\alpha$ and we conclude as in the above cases.
Now, we need to construct the vector field when (A_f) is satisfied. We define

$$Y_5^1 = \sum_{i=1,2} \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial (a_1)_i} \int_{\mathbb{R}^3_+} b_i \frac{|x_i + \lambda_1 (a_1)_i|^\beta}{(1 + \lambda_1 |(a_1)_i|)^{\beta - 1}} \frac{x_i}{(1 + |x|^2)^4} dx$$

$$Y_5^2 = (-\sum_{i=1}^{3} b_i) \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1} \chi(\lambda_1 |a_1|),$$

where χ is a cut off function defined by $\chi(t) = 1$ if $t \leq \xi$ and $\chi(t) = 0$ if $t \geq 2\xi$, where ξ is a small positive constant.

In this case, the vector field will be defined by

$$Y_5 := Y_5^1 + Y_5^2 - \sum_{i\geq 2} 2^i \alpha_i \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}.$$

As in (25) and using Proposition 10, we get

$$\begin{aligned}(-\nabla J(u), Y_5) \geq & c\sum_{k\neq r}\varepsilon_{kr} + O(\frac{1}{\lambda_2}) + \chi(\lambda_1|a_1|)\left(\sum_{i=1}^{3} b_i\right)^2 \frac{c'}{\lambda_1^\beta} + o(\frac{1}{\lambda_1^\beta}) \\ & + \sum_{i=1,2}\frac{c}{\lambda_1^\beta} b_i^2 \left(\int_{\mathbb{R}^3_+} \frac{|x_i+\lambda_1(a_1)_i|^\beta}{(1+\lambda_1|(a_1)_i|)^{(\beta-1)/2}}\frac{x_i}{(1+|x|^2)^4}dx\right)^2.\end{aligned} \tag{40}$$

Observe that, if $\chi(\lambda_1|a_1|) = 1$, then $\lambda_1|a_1| \leq 2\xi$ and therefore from $1/\lambda_1^\beta$ we can make appear $|\nabla K(a_1)|/\lambda_1$. In the other case, that is $\chi(\lambda_1|a_1|) \neq 1$, we have $\lambda_1|a_1| > \xi$ and therefore by Lemma 7.1, we have

$$\sum_{i=1,2}\left(\int_{\mathbb{R}^3_+} \frac{|x_i+\lambda_1(a_1)_i|^\beta}{(1+\lambda_1|(a_1)_i|)^{(\beta-1)/2}}\frac{x_i}{(1+|x|^2)^4}dx\right)^2 \geq c > 0.$$

We remark that if $\lambda_1|a_1|$ is very large, let i so that $\lambda_1|(a_1)_i| \geq c\lambda_1|a_1|$, then

$$\begin{aligned}\int_{\mathbb{R}^3_+}|x_i+\lambda_1(a_1)_i|^\beta \frac{x_i dx}{(1+|x|^2)^4} &= (\lambda_1|(a_1)_i|)^\beta \int_{\mathbb{R}^3_+} |1+\frac{x_i}{\lambda_1(a_1)_i}|^\beta \frac{x_i dx}{(1+|x|^2)^4} \\ &= c(\lambda_1|(a_1)_i|)^{\beta-1}(1+o(1)) \geq c\lambda_1^{\beta-1}|\nabla K(a_1)|.\end{aligned}$$

Hence, as in (25), (40) becomes

$$(-\nabla J(u), Y_5) \geq c\sum_{k\neq r}\varepsilon_{kr} + \frac{c}{\lambda_1^\beta} + \sum_{i\geq 2}\frac{c}{\lambda_i} + c\sum_{i\geq 1}\frac{|\nabla K(a_i)|}{\lambda_i}.$$

Now, define $W_5 = M_1 Y_5 + Y_\alpha$ and we conclude as in the above cases.
The vector field W will be a convex combination of the above vector fields just as it was the case in the proof of Proposition 6. Regarding the behavior of the concentration speeds, observe that in the first new case, the maximum of the λ_i's is a decreasing function. But in the second new case, it increases only if $p = 1$ and a_1 is near a critical point y of K_1 with $(\partial K)/(\partial\nu)(y) = 0$ and $-\Delta K(y) > 0$ (resp. $\sum b_j < 0$). The remaining is just like in the proof of Proposition 6. The proof of Proposition 9 is thereby completed. □

From Proposition 9 we deduce the critical points at infinity. This is the content of the following Corollary whose derivation from Proposition 9 follows the same lines as the one we did to derive Corollary 4 from Proposition 6.

Corollary 6. *Assume that K satisfies (A_1) and (A_2) (resp. (A_1) and (A_f)). Then, the only critical points at infinity of J correspond to:*
- $\delta_{(y,\infty)}$ where y is a critical point of K_1 satisfying $(\partial K/\partial\nu)(y) = 0$ and $\Delta K(y) < 0$ (resp. $\sum b_j(y) < 0$). Such critical point at infinity has a Morse index equal to $2 - m(y)$, where $m(y)$ denotes the morse index of K_1 at y (resp. $3 - \iota(y)$ where $\iota(y) = \#\{b_j(y) : b_j(y) < 0\}$).
- $\sum_{j=1}^{p} K(y_{i_j})^{-1/4}\delta_{(y_{i_j},\infty)}$, with $p \in \mathbb{N}^*$ and $\rho(y_{i_1},...,y_{i_p}) > 0$, where the y_i's are nondegenerate critical points of K_1. Such a critical point at infinity has a Morse index equal to $(3p - 1 - \sum_{j=1}^{p} m(y_{i_j}))$.

Now using Corollaries 4, 6 and (37), we derive Theorems 1.2 and 1.3. □

Proof of Corollary 1. Using the assumption (A_3), we derive that ${\mathcal{F}_\infty}^p$ is empty for each p. Using now Theorem 1.2, the result follows. □

7. **Appendix: Expansion of the functional and its gradient in a potential neighborhood of critical points at infinity.** This Appendix is devoted to some useful expansions of J and its gradient near a potential critical points at infinity consisting of p masses. Those propositions are extracted from [16] (with some change). In the sequel, we will write δ_i instead of $\delta_{(a_i,\lambda_i)}$.

Proposition 10. *Let $n = 3$ and $u = \sum_{i=1}^p \alpha_i\delta_i \in V_b(p,\varepsilon)$, we have the following expansions*

$$(-\nabla J(u), \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i})\frac{\nabla_T K(a_i)}{|\nabla_T K(a_i)|} \geq 2J(u)^4\alpha_i^5\left(c_4\frac{|\nabla_T K(a_i)|}{\lambda_i} - \frac{c_5}{\lambda_i^2}|D^2K(a_i)|\right) + o(\frac{1}{\lambda_i^2} + \sum_{k\neq r}\varepsilon_{kr}) + O(\sum_{j\neq i}\frac{1}{\lambda_i}|\frac{\partial\varepsilon_{ij}}{\partial a_i}|).$$

$$(\nabla J(u), \lambda_i\partial\delta_i/\partial\lambda_i) = -c_1 J(u)\sum_{j\neq i}\alpha_j\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + 2J(u)^4\alpha_i^5\left(-\frac{c_2}{\lambda_i}\frac{\partial K}{\partial\nu}(a_i) + \frac{c_3}{\lambda_i^2}\Delta K(a_i)\right) + o(\sum_{k\neq r}\varepsilon_{kr} + \frac{1}{\lambda_i^2}).$$

$$(\nabla J(u), \delta_i) = J(u)\alpha_i S_3\left(1 - J(u)^3\alpha_i^4 K(a_i)\right) + O\left(\frac{1}{\lambda_i}|\frac{\partial K}{\partial\nu}(a_i)| + \frac{1}{\lambda_i^2} + \sum_{j\neq i}\varepsilon_{ij}\right).$$

where $c_1 = \int_{\mathbb{R}^3}\frac{c_0^6}{(1+|x|^2)^{5/2}}dx = 4\sqrt{3}\pi$, $c_2 = c_0^6\int_{\mathbb{R}^3_+}\frac{x_3(|x|^2-1)}{(1+|x|^2)^4}dx = \sqrt{3}\frac{\pi}{4}$,
$c_3 = \frac{1}{3}c_0^6\int_{\mathbb{R}^3_+}\frac{|x|^2(|x|^2-1)}{(1+|x|^2)^4}dx$, $c_4 = \frac{1}{6}\int_{\mathbb{R}^3}\frac{c_0^6|x|^2}{(1+|x|^2)^4}dx$ *and* $c_5 = \int_{\mathbb{R}^3}\frac{c_0^6|x|^2}{(1+|x|^2)^3}dx$.

If a_i is near a critical point y of K_1 such that (A_f) is satisfied, then we can improve the above formula and we obtain

$$(\nabla J(u), \delta_i) = J(u)\alpha_i S_3\left(1 - J(u)^3\alpha_i^4 K(a_i)\right) + O(\frac{1}{\lambda_i^\beta} + \sum_{j\neq i}\varepsilon_{ij}),$$

$$(\nabla J(u), \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_k}) = -2J(u)^4\frac{\alpha_i^5}{\lambda_i^\beta}\int_{\mathbb{R}^3_+} b_k|x_k + \lambda_i(a_i)_k|^\beta\frac{x_k}{(1+|x|^2)^4}dx + o(\frac{1}{\lambda_i^\beta} + \sum_{k\neq r}\varepsilon_{kr}) + O(\sum_{j\neq i}\frac{1}{\lambda_i}|\frac{\partial\varepsilon_{ij}}{\partial a_i}|).$$

where $k \in \{1,2\}$ and $(a_i)_k$ is the kth component of a_i in some geodesic normal coordinates system.
Furthermore, if we assume that $\lambda_i|a_i| \leq \varepsilon$, where ε is a small positive constant, then

$$(\nabla J(u), \lambda_i\frac{\partial\delta_i}{\partial\lambda_i}) = -c_1 J(u)\sum\alpha_j\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + \frac{c'}{\lambda_i^\beta}\sum b_j + o(\sum\varepsilon_{kr} + \frac{1}{\lambda_i^\beta}).$$

Proof. Observe that

$$\int_{B(a_i,1)\cap\mathbb{R}^3_+}(K(x)-K(a_i))\delta_i^5\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}=\int\nabla K(a_i)(x-a_i)\delta_i^5\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}$$

$$+\frac{1}{2}\int D^2K(a_i)(x-a_i,x-a_i)\delta_i^5\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}+o(\frac{1}{\lambda_i^2})$$

$$=\frac{c_0^6}{2}\frac{\nabla K(a_i)}{\lambda_i}\int_{\mathbb{R}^3_+}\frac{x(1-|x|^2)}{(1+|x|^2)^4}+c_0^6\frac{\Delta K(a_i)}{12\lambda_i^2}\int_{\mathbb{R}^3_+}\frac{|x|^2(1-|x|^2)}{(1+|x|^2)^4}+o(\frac{1}{\lambda_i^2}).$$

For $j=1,2$, we denote by $(a_i)_j$ the j^{th} component of a_i.

$$\int_{B(a_i,1)\cap\mathbb{R}^3_+}(K(x)-K(a_i))\delta_i^5\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j}=\int\nabla K(a_i)(x-a_i)\delta_i^5\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j}$$

$$+\frac{1}{2}\int D^2K(a_i)(x-a_i,x-a_i)\delta_i^5\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j}+o(\frac{1}{\lambda_i^2}).$$

Observe that

$$\left|\int_{\mathbb{R}^3_+}D^2K(a_i)(x-a_i,x-a_i)\delta_i^5\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j}\right|\leq|D^2K(a_i)|\int_{\mathbb{R}^3_+}|x-a_i|^2\delta_i^6$$

and

$$\int_{B(a_i,1)\cap\mathbb{R}^3_+}\nabla K(a_i)(x-a_i)\frac{\delta_i^5}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_j}=\int\nabla K(a_i)(x-a_i)\frac{c_0^6\lambda_i^4(x-a_i)_j}{(1+\lambda_i^2|x-a_i|^2)^4}$$

$$=c_0^6\frac{(\nabla K(a_i))_j}{\lambda_i}\int_{B(0,\lambda_i)\cap\mathbb{R}^3_+}\frac{x_j^2}{(1+|x|^2)^4}.$$

Following [16], the first part of the result is thereby completed.
Now to improve the formula when (A_f) is satisfied, we use the expansion of the function K. In fact, for the second estimate it remains to expand the following integral

$$\int_{\mathbb{R}^3_+}K(x)\delta_i^5\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(a_i)_k}=\int_{\mathbb{R}^3_+}(\sum b_j|x_j|^\beta+R(x))\frac{\lambda_i^4(x_k-(a_i)_k)}{(1+\lambda_i^2|x-a_i|^2)^4}dx$$

$$=\frac{1}{\lambda_i^\beta}\int_{\mathbb{R}^3_+}b_k|x_k+\lambda_i(a_i)_k|^\beta\frac{x_k}{(1+|x|^2)^4}dx+o(\frac{1}{\lambda_i^\beta}).$$

For the other estimate, we have

$$\int_{\mathbb{R}^3_+}K(x)\delta_i^5\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}=\int_{\mathbb{R}^3_+}(\sum b_j|x_j|^\beta+R(x))\lambda_i^3\frac{1-\lambda_i^2|x-a_i|^2}{(1+\lambda_i^2|x-a_i|^2)^4}dx$$

$$=\frac{1}{\lambda_i^\beta}\sum b_j\int_{\mathbb{R}^3_+}|x_j+\lambda_i(a_i)_j|^\beta\frac{1-|x|^2}{(1+|x|^2)^4}dx+o(\frac{1}{\lambda_i^\beta})$$

Observe that if $\lambda_i|a_i|$ is very small, it is easy to prove that

$$\int_{\mathbb{R}^3_+}|x_j+\lambda_i(a_i)_j|^\beta\frac{1-|x|^2}{(1+|x|^2)^4}dx\int_{\mathbb{R}^3_+}|x_j|^\beta\frac{1-|x|^2}{(1+|x|^2)^4}dx+o(1)<0,$$

and the second integral in the above estimate is independent of j. Hence the result follows. □

Lemma 7.1. *For each $\eta \in \mathbb{R}$ and $i = 1, 2$, we have*

$$\int_{\mathbb{R}^3_+} |x_i + \eta|^\beta \frac{x_i}{(1+|x|^2)^4} dx = 0 \qquad \textit{iff} \qquad \eta = 0.$$

Furthermore, for each $\varepsilon > 0$, there exists a positive constant $\overline{c} > 0$ such that

$$\left| \int_{\mathbb{R}^3_+} |x_i + \eta|^\beta \frac{x_i}{(1+|x|^2)^4} dx \right| \geq \overline{c}(1+|\eta|)^{\beta-1} \qquad \textit{for each} \qquad |\eta| \geq \varepsilon.$$

To give more expansions of the gradient of J near potential critical points at infinity, we need to introduce the following notation:
Let G be the Green's function of L_g on S^3_+ and H its regular part defined by

$$\begin{cases} G(x,y) = (1 - \cos(d(x,y)))^{-1} + H(x,y), \\ \Delta H = 0 \text{ in } S^3_+, \quad \partial G/\partial \nu = 0 \text{ on } \partial S^3_+ \end{cases} \tag{41}$$

The following proposition gives an accurate expansion near a potential point at infinity consisting of highly concentrated functions of the boundary as well as in the interior.

Proposition 11. [17] *For $n = 3$ and $u = \sum_{i=1}^q \alpha_i \delta_i + \sum_{i=q+1}^p \alpha_i \varphi_i \in V(p, \varepsilon)$, there exists two positive constants c_1 and c_2 such that the following holds*

$$\begin{aligned}(\nabla J(u), \lambda_j \frac{\partial \varphi_j}{\partial \lambda_j}) =& J(u) \left(-c_1 \sum_{k \neq j} \alpha_k \lambda_j \frac{\partial \varepsilon_{kj}}{\partial \lambda_j} + c_2 \sum_{k=q+1}^p \alpha_k \frac{H(a_j, a_k)}{(\lambda_j \lambda_k)^{1/2}} \right) \\ &+ o\left(\sum_{k>q} \frac{1}{\lambda_k d_k} + \sum_{k \neq r} \varepsilon_{kr} \right).\end{aligned}$$

Regarding $\varphi_{(a,\lambda)}$ we have the following lemma

Lemma 7.2. [17] *For $a \in \partial S^3_+$, we have $(\partial \delta_{(a,\lambda)})/(\partial \nu) = 0$ and therefore $\varphi_{(a,\lambda)} = \delta_{(a,\lambda)}$. For $a \notin \partial S^3_+$ with λd large, we have*

$$\varphi_{(a,\lambda)} = \delta_{(a,\lambda)} + \frac{H(a,.)}{\sqrt{\lambda}} + f_{(a,\lambda)},$$

where $f_{(a,\lambda)}$ satisfies

$$|f_{(a,\lambda)}|_{L^\infty} \leq \frac{c}{\lambda^{\frac{5}{2}} d^3}, \quad \lambda \frac{\partial f}{\partial \lambda} = O\left(\frac{1}{\lambda^{\frac{5}{2}} d^3} \right)$$

and where $d = d(a, \partial S^3_+)$.

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REFERENCES

[1] A. Ambrosetti, J. Garcia Azorero and A. Peral, *Perturbation of* $\Delta u + u^{(n+2)/(n-2)} = 0$*, the Scalar Curvature Problem in* $\mathbb{R}^n$ *and related topics*, Journal of Functional Analysis, **165** (1999), 117–149.

[2] A. Ambrosetti, Y.Y. Li and A. Malchiodi, *On the Yamabe problem and the scalar curvature problems under boundary conditions*, Math. Ann. **322** (2002), 667–699.

[3] T. Aubin, *Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire,* J. Math. Pures et Appl. **55** (1976), 269–296.

[4] T. Aubin, *Meilleures constantes de Sobolev et un théorème de Fredholm non linéaire pour la transformation conforme de la courbure scalaire*, J. Funct. Anal. **32** (1979), 148–174.

[5] T. Aubin, "Some Nonlinear Problems in Riemannian Geometry", Springer Monographs Math., Springer Verlag, Berlin 1998.

[6] T. Aubin and A. Bahri, *Méthodes de topologie algébrique pour le problème de la courbure scalaire prescrite. (French) [Methods of algebraic topology for the problem of prescribed scalar curvature]* , J. Math. Pures Appl. **76** (1997), 525–849.

[7] T. Aubin and A. Bahri, *Une hypothèse topologique pour le problème de la courbure scalaire prescrite. (French) [A topological hypothesis for the problem of prescribed scalar curvature]*, J. Math. Pures Appl. **76** (1997), 843–850.

[8] T. Aubin and E. Hebey, *Courbure scalare prescrite*, Bull. Sci. Math. **115** (1991), 125–132.

[9] A. Bahri, "Critical Points at Infinity in some Variational Problems", Pitman Res. Notes Math. Ser. **182**, Longman Sci. Tech. Harlow (1989).

[10] A. Bahri, *An invariant for Yamabe-type flows with applications to scalar curvature problems in high dimension, A celebration of J. F. Nash Jr.*, Duke Math. J. **81** (1996), 323–466.

[11] A. Bahri and J.M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of topology of the domain*, Comm. Pure Appl. Math. **41**(1988), 255–294.

[12] A. Bahri and J. M. Coron, *The scalar curvature problem on the standard three dimensional spheres*, J. Funct. Anal. **95** (1991), 106–172.

[13] A. Bahri and P. H. Rabinowitz, *Periodic solutions of* 3*-body problems*, Ann. Inst. H. Poincaré Anal. Non linéaire. **8** (1991), 561–649.

[14] M. Ben Ayed, Y. Chen, H. Chtioui and M. Hammami, *On the prescribed scalar curvature problem on 4-manifolds*, Duke Math. J. **84** (1996), 633–677.

[15] M. Ben Ayed, H. Chtioui and M. Hammami, *The scalar curvature problem on higher dimensional spheres*, Duke Math. J. **93** (1998), 379–424.

[16] M. Ben Ayed, K. El Mehdi and M. Ould Ahmedou, *Prescribing the scalar curvature under minimal boundary conditions on the half sphere*, Adv. Nonlinear Stud. **2** (2002), 93–116.

[17] M. Ben Ayed, K. El Mehdi and M. Ould Ahmedou, *The scalar curvature on the four dimensional half sphere*, Cal. Var. Partial Differential Equations **22** (2005), 465–482.

[18] S. Brendle, *A generalization of the Yamabe flow for manifolds with boundary*, Asian. J. Math. **6** (2002), 625–644.

[19] S. A. Chang, M. J. Gursky and P. Yang, *The scalar curvature equation on 2 and 3 spheres*, Calc. Var. **1** (1993), 205–229.

[20] K.C. Chang and J.Q. Liu, *On Nirenberg's problem*, Int. J. Math. (1993), 53–58.

[21] S. A. Chang and P. Yang, *Prescribing Gaussian curvature on* S^2, Acta Math. **159** (1987), 215–259.

[22] S. A. Chang and P. Yang, *A perturbation result in prescribing scalar curvature on* S^n, Duke Math. J. **64** (1991), 27–69.

[23] C.C. Chen and C.S. Lin, *Estimates of the scalar curvature via the method of moving planes I*, Comm. Pure Appl. Math. **50** (1997), 971–1017.

[24] C.C. Chen and C.S. Lin, *Estimates of the scalar curvature via the method of moving planes II*, J. Differential Geom. **49** (1998), 115–178.

[25] C.C. Chen and C.S. Lin, *Prescribing the scalar curvature on* S^n*, I. Apriori estimates*, J. Differential Geom. **57** (2001), 67–171.

[26] P. Cherrier, *Problèmes de Neumann non linéaires sur les variétés Riemaniennes*, J. Funct. Anal. **57**(1984), 154–207.

[27] Z. Djadli, A. Malchiodi and M. Ould Ahmedou, *Prescribing scalar curvature and boundary mean curvature on the three dimensional half sphere*, J. Geom. Anal., **13** (2003), 255–289.

[28] Z. Djadli, A. Malchiodi and M. Ould Ahmedou, *The Prescribing boundary mean curvature problem on* B^4, J. Differential Equations **206** (2004).

[29] J. F. Escobar, *Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary*, Ann. Math., **136** (1992), 1–50.

[30] J. F. Escobar, *The Yamabe problem on manifolds with boundary*, J. Diff. Geom., **35** (1992), 21–84.

[31] J. F. Escobar, *Conformal metrics with prescribed mean curvature on the boundary*, Cal. Var. and PDE's, **4** (1996), 559–592.

[32] J. Escobar and R. Schoen, *Conformal metrics with prescribed scalar curvature*, Inventiones Math., **86** (1986), 243–254.

[33] V. Felli and M. Ould Ahmedou, *Compactness result in conformal deformations on manifolds with boundaries*, Math. Z., **244** (2003), 373–398.

[34] V. Felli and M. Ould Ahmedou, *On a geometric equation with critical nonlinearity on the boundary*, Pacific. J. Math., **218** (2005), 75–99.

[35] Z.C. Han and Y.Y. Li, *The Yamabe problem on manifolds with boundaries: existence and compactness results*, Duke Math. J. **99** (1999), 489–542.

[36] Z.C. Han and Y.Y. Li, *The existence of conformal metrics with constant scalar curvature and constant boundary mean curvature*, Comm. Anal. Geom., **8** (2000), 809–869.

[37] E. Hebey, *The isometry concentration method in the case of a nonlinear problem with Sobolev critical exponent on compact manifolds with boundary*, Bull. Sci. Math., **116** (1992), 35–51.

[38] J. Kazdan and F. Warner, *Scalar curvature and conformal deformations of Riemannian metrics*, J. Differential Geom., **10** (1975), 113–134.

[39] Y.Y. Li, *The Nirenberg problem in a domain with boundary*, Top. Meth. Nonlin. Anal., **6** (1995), 309–329.

[40] Y.Y. Li, *Prescribing scalar curvature on S^n and related topics, Part I*, Journal of Differential Equations, **120** (1995), 319–410.

[41] Y.Y. Li, *Prescribing scalar curvature on S^n and related topics, Part II : existence and compactness*, Comm. Pure Appl. Math., **49** (1996), 437–477.

[42] P.L. Lions, *The concentration compactness principle in the calculus of variations. The limt case*, Rev. Mat. Iberoamericana, **1** (1985), I:165-201, II: 45-121.

[43] J. Milnor, "Lectures on h-Cobordism", Princeton University Press, Princeton, N.J. 1965.

[44] O. Rey, *The role of Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent*, J. Funct. Anal., **89** (1990), 1–52.

[45] R. Schoen and D. Zhang, *Prescribed scalar curvature on the n-sphere*, Cal. Var. Partial Differential Equations, **4** (1996), 1–25.

[46] M. Struwe, *A flow approach to Nirenberg problem*, to appear in Duke Math. J.

[47] M. Struwe, "Variational methods: Applications to nonlinear PDE & Hamilton systems", Springer-Verlag, Berlin 1990.

[48] M. Struwe, *A global compactness result for elliptic boundary value problems involving nonlinearities*, Math. Z., **187** (1984), 511–517.

[49] L. Zhang, *Prescribing curvatures on the three dimensional riemannian manifolds with boundaries*, Preprint 2006.

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