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A Morse theoretical approach for the boundary mean curvature problem on $\mathbb{B}^4$

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Abstract

In this paper we investigate existence as well as multiplicity of scalar flat metric of prescribed boundary mean curvature on the standard 4-dimensional ball. Due to the existence of critical point at infinity, the standard variational methods cannot be applied. To overcome this difficulty, we prove that in a neighborhood of critical points at infinity, a Morse lemmas at infinity reduction holds, then develop a whole Morse theory of this noncompact variational problem. In particular we establish, under generic boundary condition Morse inequalities at infinity, which give a lower bound on the number of solutions to the above problem in terms of the total contribution of the critical point at infinity to the difference of topology between the level sets of the associated Euler–Lagrange functional. As further application of this Morse theoretical approach, we prove more existence results and extend a topological invariant introduced by A. Bahri.

Keywords: Boundary mean curvature; Blow up analysis; Morse theory; Morse lemma at infinity; Critical points at infinity; Morse inequalities

Contents

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1. Introduction and main results

In this paper, we study some nonlinear elliptic equation involving the Sobolev trace critical exponent, associated to conformal deformations of Riemannian metrics on manifolds with boundary. Namely, given a Riemannian manifold with boundary (M,g) of dimension $n \geqslant 3$, with scalar curvature R_g and boundary mean curvature H_g, let $g' = v^{\frac{4}{n-2}} g$, where v is a smooth positive function, be a conformal metric. Then new curvatures $R_{g'}$ and $H_{g'}$ are related by

$$\begin{cases} -4\dfrac{n-1}{n-2}\Delta_g v + R_g v = R_{g'} v^{\frac{n+2}{n-2}}, & \text{in } M, \\ \dfrac{2}{n-2}\dfrac{\partial v}{\partial \nu} + H_g v = H_{g'} v^{\frac{n}{n-2}}, & \text{on } \partial M, \end{cases} \tag{1.1}$$

see e.g. [5]. In the above equation, ν denotes the inward unit vector perpendicular to ∂M, with respect to the metric g.

The problem we are interested in arises naturally when looking at Eq. (1.1): given a function $H : \partial M \to \mathbb{R}$, does there exist a metric g' conformal to g such that $R_{g'} \equiv 0$ and $H_{g'} \equiv H$?

From Eq. (1.1), the problem is equivalent to finding a smooth positive solution v of the equation

$$\begin{cases} -4\dfrac{n-1}{n-2}\Delta_g v + R_g v = 0, & \text{in } M, \\ \dfrac{2}{n-2}\dfrac{\partial v}{\partial \nu} + H_g v = H v^{\frac{n}{n-2}}, & \text{on } \partial M. \end{cases} \tag{1.2}$$

In this paper, as well as in its second part [1], we are interested in the case of standard balls where the noncompact group of conformal transformations of the ball, acts on the equation giving rise to Kazdan–Warner type obstructions, just as in the celebrated scalar curvature (or Nirenberg) problem (see [39]). Namely, let $\mathbb{B}^n$ be the unit ball in $\mathbb{R}^n$ with Euclidean metric g_0. Its boundary will be denoted by $\mathbb{S}^{n-1}$ and will be endowed with the standard metric still denoted by g_0. Let H be a smooth function on $\mathbb{S}^{n-1}$.

In this case our problem becomes

$$\begin{cases} \Delta_{g_0} u = 0, \quad u > 0, & \text{in } \mathbb{B}^n, \\ \dfrac{\partial u}{\partial \nu} + \dfrac{n-2}{2} u = \dfrac{n-2}{2} H u^{\frac{n}{n-2}}, & \text{on } \mathbb{S}^{n-1}. \end{cases} \tag{1.3}$$

Our aim is to give sufficient conditions on H such that problem (1.3) admits a positive solution (of class C^2). It is easy to see that a necessary condition for solving the problem is that H has to be positive somewhere and as we already mentioned, there is at least another obstruction to solving the problem, the so-called Kazdan–Warner conditions.

This problem has been studied by A. Chang, X. Xu and P. Yang, see [21], who proved a perturbative result for Eq. (1.3) (that is in the case where H is close to a constant function). In [33], Escobar and Garcia studied problem (1.3) for $n = 3$ in the nonperturbative setting, proving that blow-ups of solutions (of subcritical approximations) occur at one point and also obtained compactness and existence results. They also considered the four-dimensional case under some flatness conditions of the function H near its critical points. In [28], Djadli, Malchiodi and Ould Ahmedou considered the problem on the four-dimensional ball. They performed a refined blow-up analysis à la Schoen [46,47] and Y.Y. Li [40,41] and gave an Euler–Hopf criterium reminiscent to the one given by Bahri and Coron [13] and Chang, Gursky, Yang [20] for the prescribed scalar curvature on the three-dimensional sphere. Related problems have been studied in [4,17,25–27,35,36,45].

In this paper, we consider the prescribed boundary mean curvature problem with zero scalar curvature on $\mathbb{B}^4 = \{x \in \mathbb{R}^4;\ \|x\| < 1\}$. This problem is equivalent to the equation

$$(\mathrm{P}_H)\qquad \begin{cases} \Delta u = 0 \quad \text{and} \quad u > 0 & \text{in } \mathbb{B}^4, \\ \dfrac{\partial u}{\partial \nu} + u = H u^2 & \text{on } \partial\mathbb{B}^4, \end{cases}$$

where $H : \partial\mathbb{B}^4 \longrightarrow \mathbb{R}^+$ is a given C^2 function and ν is the inward normal vector on $\partial\mathbb{B}^4$, with respect to the standard metric g_0.

Our choice of the dimension four is motivated by the special features of the noncompactness in this dimension. In fact, one looking to the possible formations of blow up points, it comes out that the strong interaction of the *bubbles* in dimensions three forces all blow up points to be single while in dimension greater or equal five such any interaction of two bubbles is negligible with respect to the self interaction, while in dimension four there is a phenomenon of balance [28]. Similar phenomena occur for the scalar curvature on spheres [10,16].

Moreover, the main analytic difficulties of this problem are due to the presence of critical exponent on the right-hand side of our equation. Indeed, due to the fact that the embedding $H^1(M) \hookrightarrow L^{\frac{2(n-1)}{n-2}}(\partial M)$ is not compact, the Euler–Lagrange functional J associated to our problem fails to satisfy *the Palais–Smale condition*. That is there exist noncompact sequences along which the functional is bounded and its gradient goes to zero. Therefore, it is not possible to apply the standard variational methods to prove existence of solutions. Moreover, we notice that the above problem is a natural analogue to the well-known *scalar curvature problems on closed manifolds*, to find a positive smooth solution to the following equation:

$$(\mathrm{SC})\qquad -c_n \Delta_g u + R_g u = K u^{(n+2)/(n-2)}, \qquad u > 0 \quad \text{in } M,$$

and to which much works have been devoted (see [3,6–11,15,20–24,34,37,38,40,41,48,50] and the references therein).

When H is a constant function, the problem is a variant of the so-called *the Yamabe problem on manifolds with boundary* which also has been studied through many works. See [18,30–32, 35,36], and the references therein.

Unlike [28], we revisit problem (P_H) through topological and dynamical methods, in particular through a Morse theoretical approach. Our aim is to give existence as well as multiplicity results. We notice that it is standard to prove, see e.g. [2], that if $H \in C^{1,\alpha}(S^3)$ for some $\alpha \in (0,1)$, then there exists a solution $v \in C^{2,\alpha}$ of (P_H). We denote by Ξ the operator which associates to H the solution v of (P_H), and we extend the definition of Ξ to the case of weak solutions of (P_H).

Throughout this paper, we denote by $\mathcal{H}$ the subclass of positive functions $H \in C^2(S^3)$ which have only nondegenerate critical points $y_0, y_1, \ldots, y_\ell$ satisfying that

$$H(y_0) \geqslant H(y_1) \geqslant \cdots \geqslant H(y_\ell) \quad \text{and} \quad \Delta H(y_i) \neq 0, \quad \text{for } i = 0, 1, \ldots, \ell.$$

We denote

$$\mathcal{F}_\infty := \{ q \in S^3;\ \nabla_{\mathrm{T}} H(q) = 0,\ \Delta_{S^3} H(q) < 0 \},$$

where $\nabla_{\mathrm{T}} H$ denotes the tangential gradient of H.

To every $(y_{i_1}, \ldots, y_{i_N}) \subseteq \mathcal{F}_\infty$, we associate the matrix $M = (M_{ij})$ defined by

$$\begin{cases} M_{jj} = -C \dfrac{\Delta H(y_{i_j})}{H(y_{i_j})^3}, & j \in \{1, \ldots, N\}, \\ M_{lj} = -C' \dfrac{G_{y_{i_l}}(y_{i_j})}{H(y_{i_l}) H(y_{i_j})}, & l, j \in \{1, \ldots, N\},\ l \neq j, \end{cases} \tag{1.4}$$

where

$$C = \frac{\bar{c}^3}{18} \int_{\mathbb{R}^3} \frac{|x|^2\, dx}{(1+|x|^2)^3} \quad \text{and} \quad C' = \bar{c}^3 \int_{\mathbb{R}^3} \frac{dx}{(1+|x|^2)^2}.$$

Here $G_q(\cdot)$ denotes the Green's function for the operator Ξ with pole q.

Let $\rho = \rho(y_{i_1}, \ldots, y_{i_N})$ denotes the least eigenvalue of M and set

$$\mathcal{H}^+ := \{ H \in \mathcal{H} \text{ such that for any } (q^1, \ldots, q^l) \subseteq \mathcal{F}_\infty,\ M(q^1, \ldots, q^l) \text{ is nondegenerate} \}.$$

Before stating our first result, recall that a solution of (P_H) is said to be nondegenerate, if the associated linearized operator does not have zero as an eigenvalue.

Theorem 1.1. *Assume that the function $H \in \mathcal{H}^+$, for every $\varepsilon < \varepsilon_0$ there exists $H_\varepsilon \in \mathcal{H}^+$ such that*:

1. $\|H - H_\varepsilon\| < \varepsilon$; *$H$ and H_ε have the same critical points with the same Morse indices.*
2. *The problem $(\mathrm{P}_{H_\varepsilon})$ has only nondegenerate solutions.*
3. *To a solution of the problem $(\mathrm{P}_{H_\varepsilon})$ corresponds a solution to the problem (P_H).*
4. *Set l_1 to be the cardinal of ${\mathcal{F}_\infty}^+ := \{q \in \mathcal{F}_\infty;\ \rho > 0\}$, we set $L_0 := \sup_{1 \leqslant s \leqslant l_1} (4s - 1 - \sum_{j=1}^{l_1} k_{i_j})$. Then it holds*:

$$\left| 1 - \sum_{s=1}^{l_1} \sum_{\rho(y_{i_1}, \ldots, y_{i_s}) > 0} (-1)^{s-1+\sum_{j=1}^{s} k_{i_j}} \right| \leqslant N_{L_0+1},$$

where N_{L_0+1} *is the number of solution of* $(\mathrm{P}_{H_\varepsilon})$ *of Morse index* $\leqslant L_0+1$ *and* $k_{i_j}=3-\operatorname{ind}(H,y_{i_j})$. *Here* $\operatorname{ind}(H,y_{i_j})$ *denotes the Morse index of* H *at* y_{i_j}.

Remark 1.2. We emphasize that the solutions of problems (P_H) and $(\mathrm{P}_{H_\varepsilon})$ may be not necessarily in one-to-one correspondence. Indeed some nondegenerate solutions of problem $(\mathrm{P}_{H_\varepsilon})$ may through the homotopy that we built there come together to give birth of one degenerate solution of the problem (P_H). A situation that we cannot rule out does happen even in the finite-dimensional case. The reason is that, although the "topology at infinity" does not change during our deformation process, one cannot give a precise lower bound on the number of critical points of the associated Euler–Lagrange functional since a degenerate critical point may have contribution to the homology of its corresponding critical groups in the dimensions between the (strict) Morse index and the (generalized) one, that is the Morse index plus the nullity.

However we assume that all the solutions of the problem (P_H) are nondegenerate one and recovers the same lower bound as the one given in Theorem 1.1. See please the following corollary.

Corollary 1.3. *Assume that the function* $H\in\mathcal{H}^+$ *and that all the solutions of the problem* (P_H), *having their Morse indices* $\leqslant L_0$, *where* L_0 *is defined in Theorem* 1.1, *are nondegenerate, then it holds*:

$$\left|1-\sum_{s=1}^{l_1}\sum_{\rho(y_{i_1},\dots,y_{i_s})>0}(-1)^{s-1+\sum_{j=1}^{s}k_{i_j}}\right|\leqslant N_{\ell_0+1},$$

where N_{ℓ_0+1} *is the number of solutions of* (P_H) *of Morse index* $\leqslant L_0+1$, *and* $k_{i_j}=3-\operatorname{ind}(H,y_{i_j})$. *Here* $\operatorname{ind}(H,y_{i_j})$ *denotes the Morse index of* H *at* y_{i_j}.

Theorem 1.1 can be seen as Marino–Prodi type resolution as well as a *Morse type inequality result* in the sense that we give here a lower bound for the number of solutions in terms of the *topology at infinity* that is the total contribution of noncompact orbits of the gradient flow associated to the Euler–Lagrange functional (its *critical point at infinity*). Recall that *Morse inequalities* give a lower bound on the number of critical points of a *Morse function* in terms of the *Betti numbers* of the underlying manifold. In our case, the space of variation is contractible and hence has no topology. However, due to the noncompactness of the problem, there are *critical points at infinity* whose topological contribution to the difference of topology between the level sets of the functional can be computed thanks to a *Morse lemma at infinity* which provides new coordinates in which the gradient flow takes a quite simple *normal form*.

In the following, we give a brief description of the main ingredients behind the proof of Theorem 1.1.

A first main ingredient in the proof of Theorem 1.1 is a Marino–Prodi type approximation of our problem by a new one whose critical points are all nondegenerate. Such an approximation is performed using a perturbation of the function H. Building on the refined blow up analysis à la Schoen of Djadli, Malchiodi and Ould Ahmedou, see [28], we prove a priori estimates for solutions of the perturbed problem when the function H lies in some classes of C^2-functions on $\partial\mathbb{B}^4$. Using these a priori estimates and a continuity method argument, we prove that each solution of the perturbed problem gives rise through the above continuity argument to a solution of the unperturbed problem.

Besides the reduction to the nondegenerate case, the proof uses a careful analysis of the lack of compactness of the Euler–Lagrange functional J associated to problem (P_H). Namely, we study the noncompact orbits of the gradient flow of J the so called *critical points at infinity* following the terminology of A. Bahri [10]. There are the noncompact orbits of J along which J is bounded and its gradient goes to zero. These *critical points at infinity* can be treated as usual critical points once a Morse lemma at infinity is performed from which we can derive just as in the classical Morse theory the difference of topology induced by these noncompact orbits and compute their Morse index. Such a Morse lemma at infinity which is a cornerstone in our analysis is obtained through the construction of a suitable pseudogradient for which the *Palais–Smale* condition is satisfied along the decreasing flow lines, as long as these flow lines do not enter the neighborhood of a finite numbers of critical points of H such that the related matrix M (see (1.4)) is positive definite. Moreover, along the flow lines of such a pseudogradient there can be only finitely many blow up points. Furthermore, if some blow up points are close and the interactions between them is large, then the flow lines starting from there will enter the zone with at least one less blow up points.

Similar Morse lemma has been established for the prescribed scalar curvature problem on the spheres S^3, S^4 under the hypothesis that the problem has no solution by A. Bahri and J.M. Coron [13] and Ben Ayed, Chen, Chtioui and Hammami [16]. Since our aim is to prove multiplicity rather than only existence, we have to perform our Morse lemma without such an assumption, a situation which creates a new difficulty namely to deal with the possibility of existence of *a new type of critical points at infinity* consisting of a sum of bubbles plus a solution of the problem. Performing now a *Morse lemma at infinity* near an ε-neighborhood of such a potential critical points at infinity, we rule out such a possibility for the problem (P_H) on the four-dimensional ball $\mathbb{B}^4$.

Finally, we notice that related statement to Corollary 1.3 has been proved by Escobar and Garcia [33] on the 3-dimensional ball, where due to the fact that in their case only single blow up points occur, their formula is much more simple. Their proof which is drastically different from ours involves a refined analysis for blowing up subcritical approximations. As another corollary of Theorem 1.1, we recover an existence result, proved in [28] using topological degree argument and blow up analysis.

Corollary 1.4. *Assume that the function $H \in \mathcal{H}^+$, if*

$$\sum_{s=1}^{l_1} \sum_{\rho(y_{i_1},\dots,y_{i_s})>0} (-1)^{s-1+\sum_{j=1}^{s} k_{i_j}} \neq 1$$

then problem (P_H) has at least one solution.

As above, $k_{i_j} = 3 - \mathrm{ind}(H, y_{i_j})$ where $\mathrm{ind}(H, y_{i_j})$ denotes the Morse index of H at y_{i_j}.

Next, we give another kind of existence results. Unlike Theorem 1.1 and Corollaries 1.3 and 1.4, where global information was used, these results use only local information. Namely, under suitable conditions on the function H we prove that some local difference of topology can only be explained by the presence of critical points.

Theorem 1.5. *We assume that $H \in \mathcal{H}$. If the following conditions hold,*

(H$_1$) $y_1 \in \mathcal{F}_\infty$:
(H$_2$) $\rho(y_0, y_1) < 0$,
(H$_3$) $H(y)^{\frac{-3}{2}} > H(y_0)^{\frac{-3}{2}} + H(y_1)^{\frac{-3}{2}}\ \forall y \in \mathcal{F}_\infty \setminus \{y_0, y_1\}$,

then problem (P$_H$) *has a solution of Morse index* k_1 *or* $k_1 + 1$, *where* k_1 *is the coindex of* y_1 *as critical point of* H.

Now, to prove further existence results for the problem (P$_H$) we extend to the variational framework of boundary critical nonlinearities a topological invariant denoted by μ, developed by A. Bahri in his studies of some Yamabe type problems, see [11].

To state our results, we need to introduce some assumptions and notations. Let Z be a pseudogradient of H of Morse–Smale type (that is the intersection of the stable and the unstable manifolds of the critical points of H are transverse). Let

$$X = \overline{W_s(y_1)} = W_s(y_1) \cup W_s(y_0),$$

where $W_s(y_i)$ is the stable manifold of y_i for the pseudogradient Z. Notice that X is a manifold without boundary, we denote by k_1 its dimension.

Let $C_{y_0}(X)$ the following set:

$$C_{y_0}(X) = \bigl\{\alpha\delta_{y_0} + (1-\alpha)\delta_x \bigm| \alpha \in [0,1],\ x \in X\bigr\},$$

where δ_x is the Dirac measure at x. For λ large enough, we introduce a map

$$\begin{aligned} f_\lambda : C_{y_0}(X) &\to \Sigma^+ \\ \alpha\delta_{y_0} + (1-\alpha)\delta_x &\to \frac{\alpha\tilde{\delta}_{(y_0,\lambda)} + (1-\alpha)\tilde{\delta}_{(x,\lambda)}}{\|\alpha\tilde{\delta}_{(y_0,\lambda)} + (1-\alpha)\tilde{\delta}_{(x,\lambda)}\|}. \end{aligned}$$

For λ large enough, we define the intersection number (modulo 2) of $f_\lambda(C_{y_0}(X))$ with $W_s(y_0, y_1)_\infty$,

$$\mu(y_0) = f_\lambda\bigl(C_{y_0}(X)\bigr) \cdot W_s(y_0, y_1)_\infty,$$

where $W_s(y_0, y_1)_\infty$ is the stable manifold of the critical point at infinity $(y_0, y_1)_\infty$ (see Corollary 6.3 below) for a decreasing pseudogradient V for the Euler–Lagrange functional associated to (P$_H$) which is transverse to $f_\lambda(C_{y_0}(X))$. This intersection number is well defined, see [44]. We then have the following result.

Theorem 1.6. *Let* $H \in \mathcal{H}$. *If the following conditions holds*:

(H$_4$) $\rho(y_0, y_1) > 0$,
(H$_5$) $\mu(y_0) = 0$,

then problem (P$_H$) *has a solution of Morse index* k_1 *or* $k_1 + 1$, *where* k_1 *is the dimension of the Morse complex at infinity* X.

The remainder of the paper is organized as follows. Section 2 is devoted to a Marino–Prodi type reduction, where we reduce our problem to one which has only nondegenerate critical points. In Section 3, we set up the variational structure and we recall some well-known facts. In Section 4, we perform an accurate expansion of J and its gradient near potential critical points at infinity, in Section 5 we construct a suitable pseudogradient and in Section 6 we prove a Morse lemma at infinity near such points. In Section 7, we derive a Morse inequality at infinity and prove Theorem 1.1. Lastly in Section 8, we prove the remaining existence results.

2. Marino–Prodi type resolution

The celebrated Marino–Prodi theorem asserts that in the ε-neighborhood of any C^2-function defined on a C^2-Finsler manifold M such that the $D^2 f(p)$ at any critical point $p \in M$ of f is a Fredholm operator, there is a C^2 function having only nondegenerate critical points (see [43]).

In the same spirit, we will perturb our initial equation (P_H) into a new equation whose solutions are all nondegenerate. Recall that a solution is said to be nondegenerate when the linearized operator does not admit zero eigenvalues. The main feature of our perturbation is that through a continuity method argument and some a priori estimates, we associate to any solution of the perturbed equation a solution of our original problem.

Theorem 2.1. *Assume that $H \in \mathcal{H}^+$ and that the problem (P_H) has only isolated solutions. Then there exist $\varepsilon > 0$ and a function $h \in C^2(S^3)$ such that for $H_\varepsilon := H + \varepsilon h \in \mathcal{H}^+$ and the problem*

$$(\mathrm{P}_{H_\varepsilon}) \quad \begin{cases} \Delta u = 0 & \text{in } \mathbb{B}^4, \\ \dfrac{\partial u}{\partial \nu} + u = H_\varepsilon u^2 & \text{on } \partial \mathbb{B}^4, \end{cases} \tag{2.1}$$

there hold:

1. *The perturbed equation* (2.1) *has only nondegenerate solutions.*
2. *To a solution of the problem* $(\mathrm{P}_{H_\varepsilon})$ *corresponds a solution to the problem* (P_H).

A main ingredient in the proof of Theorem 2.1 is the following a priori estimate for the family of problems (2.1) when the parameter ε is chosen sufficiently small. Namely we have

Lemma 2.2. *For any $H \in \mathcal{H}^+$ there exists $\delta := \delta(H)$ and $C := C(H) < \infty$ such that for all $\tilde{H} \in C^2(S^3)$ with $\|\tilde{H} - H\|_{C^2} < \delta$ and for every u solution of the problem*

$$(\mathrm{P}_h) \quad \begin{cases} \Delta u = 0 & \text{in } \mathbb{B}^4, \\ \dfrac{\partial u}{\partial \nu} + u = \tilde{H} u^2 & \text{on } \partial \mathbb{B}^4, \end{cases}$$

we have that

$$C^{-1} < u < C \quad \text{on } \overline{\mathbb{B}^4}, \qquad \|u\|_{C^3} < C.$$

Proof. Arguing by contradiction, we suppose that there exists a sequence of $H_\varepsilon \to H$ in $C^2(\overline{\mathbb{B}^4})$ such that $\max_{\overline{\mathbb{B}^4}} u_\varepsilon \to \infty$ for some solution of (2.1). It follows then from [28, Proposition 5.1] that u_ε has only isolated simple blow up $(p_1, \dots, p_N)$ and that all the blow up points are located on the boundary $\partial\mathbb{B}^4 := \mathbb{S}^3$.

We claim that $N \geqslant 2$. Indeed, for $N = 1$ let $p_\varepsilon \to p$ the blow up point, it follows then from [28, Proposition 5.1] that

$$u_\varepsilon(p_\varepsilon)u_\varepsilon \to aG(p_\varepsilon, .) \quad \text{in } C^2\big(\overline{\mathbb{B}^4} \setminus \{p\}\big),$$

where $a > 0$ and $G(p_\varepsilon, .)$ is the Green's function of the operator Ξ.

By making a suitable stereographic projection to transform $\mathbb{B}^4$ to $\mathbb{R}^4_+$ and p_ε to 0, u_ε is transformed to v_ε which satisfies the equation

$$\begin{cases} \Delta v_\varepsilon = 0 & \text{in } \mathbb{R}^4_+, \\ -\dfrac{\partial v_\varepsilon}{\partial y_4} = H_\varepsilon v_\varepsilon^2 & \text{on } \partial\mathbb{R}^4_+, \end{cases}$$

Applying Pohozaev identity (see [28, Proposition 6.1]), we obtain

$$\int\limits_{\mathbb{R}^3} \sum_{i=1}^{3} y_i \cdot \frac{\partial H_\varepsilon}{\partial y_i} v_\varepsilon^3 = 0.$$

Using [28, Lemmas 2.8, 2.9] we derive that

$$\int\limits_{\mathbb{R}^3} \sum_{i=1}^{3} y_i \cdot \frac{\partial H_\varepsilon}{\partial y_i} v_\varepsilon^3 = \Delta H_\varepsilon(0) \int\limits_{\mathbb{R}^3} |x|^2 v_\varepsilon^3 + o\big(\big(v_\varepsilon(0)\big)^{-2}\big).$$

It follows now from the above estimates and [28, Lemma 2.8] that

$$0 = \Delta H_\varepsilon(p_\varepsilon)\big(u_\varepsilon(p_\varepsilon)\big)^{-2} + o\big(\big(u_\varepsilon(p_\varepsilon)\big)^{-2}\big),$$

which contradicts the fact that $H_\varepsilon \in \mathcal{H}^+$ and our claim follows.

Now coming back to the proof of Lemma 2.2 and using again [28, Proposition 5.1] we derive that, when $N \geqslant 2$ it holds:

$$\sum_{l=1}^{N} M_{lj}\lambda_l = 0,$$

where $\lambda_l > 0$ for $1 \leqslant l \leqslant N$, which contradicts the fact that $H_\varepsilon \in \mathcal{H}^+$. Therefore our lemma follows. □

Proof of Theorem 2.1. We first observe that for $\varepsilon \in (0, 1)$ the function $H_\varepsilon := H + \varepsilon h$, where $h \in C^2(S^3)$ with $|\nabla h| < |\nabla H|$ has the same critical point as the function H itself. We choose $\varepsilon_1 \in (0, 1)$ such that for every $0 \leqslant \varepsilon \leqslant \varepsilon_1$, there holds

$$H_\varepsilon := H + \varepsilon h \in \mathcal{H}^+ \quad \text{and} \quad \|H - H_\varepsilon\|_{C^2(S^3)} < \delta(H),$$

where $\delta(H)$ is the one defined in Lemma 2.2.

Now it follows from standard elliptic theory, the solution operator corresponding to the equation (P_H) is a Fredholm operator. It follows then from Sard–Smale theorem, see e.g. [51], that for generic function h and for all $\varepsilon \in (0, \varepsilon_1)$ the problem $(\mathrm{P}_{H_\varepsilon})$, defined exactly as in the case of problem (P_H) but with H_ε instead of H, has only nondegenerate solutions. It follows then that for every $\varepsilon \in (0, \varepsilon_1)$ there exists a function H_ε satisfying properties 1 and 2 of Theorem 2.1.

Now let u_{ε_0} be a solution of $(\mathrm{P}_{H_{\varepsilon_0}})$ and let

$$I := \bigl\{t \in [0, \varepsilon_0] \text{ such that } (\mathrm{P}_{H_t}) \text{ has a solution } u_t\bigr\}.$$

Observe that I is not empty since $\varepsilon_0 \in I$ and that it follows from Schauder theory that it is open. Moreover, it follows from the a priori estimates of Lemma 2.2, that it is closed. Hence $I = [0, \varepsilon_0]$ and our original problem (P_H) has a solution. The proof of our theorem follows. □

3. Variational structure and preliminary results

Our problem (P_H) enjoys a variational structure. Indeed, solutions of (P_H) correspond to positive critical points of the functional:

$$J(u) = \frac{1}{(\int_{\partial\mathbb{B}^4} H u^3 \, d\sigma_{g_0})^{\frac{2}{3}}},$$

defined on

$$\Sigma = \biggl\{u \in H^1\bigl(\mathbb{B}^4\bigr),\ \|u\|^2 = \int\limits_{\mathbb{B}^4} |\nabla u|^2 \, dv_{g_0} + \int\limits_{\partial\mathbb{B}^4} u^2 \, d\sigma_{g_0} = 1\biggr\}.$$

However, it is delicate from a variational viewpoint because the functional J does not satisfy the *Palais–Smale* condition (P.S. for short). This means that there exist a sequences along which J is bounded, its gradient goes to zero and which do not converge. The analysis of sequences failing P.S. condition can be analyzed along the ideas introduced in [13,49]. In order to describe such a characterization in our case, we need to introduce some notations.

We will use the notation x for the variables belonging to the unit ball $\mathbb{B}^n, n \geqslant 3$, or to the half space $\mathbb{R}^n_+$ defined by $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n,\ x_n > 0\}$. We will also use the notation $x = (x', x_n)$ for $x \in \mathbb{R}^n_+$.

It will be convenient to perform some stereographic projection in order to reduce the above problem to $\mathbb{R}^n_+$. Let $D^{1,2}(\mathbb{R}^n_+)$ denotes the completion of $C^\infty_{\mathrm{c}}(\overline{\mathbb{R}^n_+})$, with respect to the Dirichlet norm. The stereographic projection π_q through an appropriate point $q \in \mathbb{S}^{n-1}$ induces an isometry $i : H^1(\mathbb{B}^n) \longrightarrow D^{1,2}(\mathbb{R}^n_+)$ according to the following formula:

$$iu(x) = \left(\frac{2}{|x'|^2 + (x_n + 1)^2}\right)^{\frac{n-2}{2}} u\left(\frac{2x'}{|x'|^2 + (x_n + 1)^2}, \frac{|x'|^2 + x_n - 1}{|x'|^2 + (x_n + 1)^2}\right),$$

where $x' = (x_1, \ldots, x_{n-1})$. In particular, we can check that the following relations hold true for every $u \in H^1(\mathbb{B}^n)$,

$$\int_{\mathbb{B}^n} |\nabla u|^2 + \frac{n-2}{2} \int_{\partial \mathbb{B}^n} u^2 = \int_{\mathbb{R}^n_+} |\nabla i u|^2 \quad \text{and} \quad \int_{\partial \mathbb{B}^n} |u|^{2\frac{(n-1)}{n-2}} = \int_{\partial \mathbb{R}^n_+} |iu|^{2\frac{(n-1)}{n-2}}.$$

In the sequel, we will identify the function H and its composition with the stereographic projection π_q. We will also identify a point x of $\mathbb{B}^n$ and its image by π_q. These facts will be assumed as understood in the sequel.

For $a \in \partial \mathbb{R}^n_+$ and $\lambda > 0$, define the function:

$$\delta_{a,\lambda}(x) = \bar{c} \frac{\lambda^{\frac{n-2}{2}}}{((1+\lambda x_n)^2 + \lambda^2 |x' - a'|^2)^{\frac{n-2}{2}}},$$

where $x \in \mathbb{R}^n_+$, and $\bar{c}$ is chosen such that $\delta_{a,\lambda}$ satisfies the following equation,

$$\begin{cases} \Delta u = 0 \quad \text{and} \quad u > 0 & \text{in } \mathbb{R}^n_+, \\ -\dfrac{\partial u}{\partial x_n} = u^{\frac{n}{n-2}} & \text{on } \partial \mathbb{R}^n_+. \end{cases}$$

Set

$$\tilde{\delta}_{a,\lambda} = i^{-1}(\delta_{a,\lambda}).$$

Let for $\varepsilon > 0$, $p \in \mathbb{N}^*$ and w either a solution of (P_H) or zero,

$$V(p,\varepsilon,w) = \begin{cases} u \in \Sigma \quad \text{s.t. } \exists a_1, \ldots, a_p \in S^{n-1},\ \exists \alpha_0, \alpha_1, \ldots, \alpha_p > 0, \\ \exists \lambda_1, \ldots, \lambda_p > \varepsilon^{-1} \quad \text{with } \left\| u - \alpha_0 w - \sum_{i=1}^p \alpha_i \tilde{\delta}_{a_i,\lambda_i} \right\| < \varepsilon, \quad \varepsilon_{ij} < \varepsilon\ \forall i \neq j, \\ \left| \dfrac{\alpha_i^{\frac{2}{n-2}} H(a_i)}{\alpha_j^{\frac{2}{n-2}} H(a_j)} - 1 \right| < \varepsilon\ \forall i, j = 1, \ldots, p, \quad \text{and} \\ \left| \alpha_0 J(u)^{\frac{n-1}{2}} - 1 \right| < \varepsilon, \end{cases}$$

where

$$\varepsilon_{ij} = \left(\frac{1}{\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2} \right)^{\frac{n-2}{2}}.$$

If u is a function in $V(p,\varepsilon,w)$, one can find an optimal representation, following the ideas introduced in [11] and [12]. Namely we have

Lemma 3.1. *(See [11,12].) For any $p \in \mathbb{N}^*$, there is $\varepsilon_p > 0$ such that if $\varepsilon \leqslant \varepsilon_p$ and $u \in V(p,\varepsilon,w)$, then the minimization problem*

$$\min \left\{ \left\| u - \sum_{i=1}^p \alpha_i \tilde{\delta}_{(a_i,\lambda_i)} - \alpha_0 (w + h) \right\|,\ \alpha_i > 0,\ \lambda_i > 0,\ a_i \in S^{n-1},\ h \in T_w\big(W_{\mathrm{u}}(w)\big) \right\}$$

has a unique solution $(\bar{\alpha}, \bar{\lambda}, \bar{a}, \bar{h})$. *Thus, we can write* u *as follows*:

$$u = \sum_{i=1}^{p} \alpha_i \tilde{\delta}_{(a_i,\lambda_i)} + \alpha_0 (w + h) + v,$$

where v *belongs to* $H^1(\mathbb{B}^n) \cap T_w(W_{\mathrm{s}}(w))$ *and satisfies* (V_0), *where* $T_w(W_{\mathrm{u}}(w))$ *and* $T_w(W_{\mathrm{s}}(w))$ *are the tangent spaces at* w *of the unstable and the stable manifolds of* w, *and* (V_0) *are the following conditions*:

$$(\mathrm{V}_0): \quad \begin{cases} \langle v, \varphi_i \rangle = 0 & \text{for } i = 1, \ldots, p, \text{ and } \varphi_i = \tilde{\delta}_i, \partial \tilde{\delta}_i / \partial \lambda_i, \partial \tilde{\delta}_i / \partial a_i, \\ \langle v, w \rangle = 0, \\ \langle v, h \rangle = 0 & \text{for all } h \in T_w\big(W_{\mathrm{u}}(w)\big), \end{cases}$$

where, $\tilde{\delta}_i = \tilde{\delta}_{a_i,\lambda_i}$ *and* $\langle .,. \rangle$ *denote the scalar product defined on* $H^1(\mathbb{B}^n)$ *by*

$$\langle u, v \rangle = \int_{\mathbb{B}^n} \nabla u \nabla v \, dv_{g_0} + \frac{n-2}{2} \int_{\mathbb{S}^{n-1}} uv \, d\sigma_{g_0}.$$

Notice that Lemma 3.1 is also true if we take $w = 0$, therefore $h = 0$ and u in $V(p, \varepsilon)$.

The failure of the *Palais–Smale* condition can be characterized taking into account the uniqueness result of the corresponding problem at infinity, see e.g. Li, Zhu [42] following the ideas introduced in [19,49] as follows.

Proposition 3.2. *Let* $(u_k) \subset \Sigma^+$ *be a sequence satisfying* $J(u_k) \to c$, *a positive number and* $\partial J(u_k) \to 0$. *Then, there exist an integer* $p \geqslant 1$, *a positive sequence* $(\varepsilon_k)_k$ $(\varepsilon_k \to 0)$ *and an extracted subsequence of* (u_k), *again denoted* u_k *such that* $u_k \in V(p, \varepsilon_k, w)$, *where* w *is zero or a solution of* (P_H).

4. Expansion of the functional and its gradient near potential critical points at infinity

In this section, we will give the expansion of

$$J\left(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_{(a_i,\lambda_i)} + \alpha_0 (w + h) + v\right), \qquad \left\langle \nabla J\left(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_{(a_i,\lambda_i)}\right), \lambda_i \frac{\partial \tilde{\delta}_{(a_i,\lambda_i)}}{\partial \lambda_i} \right\rangle \quad \text{and}$$

$$\left\langle \nabla J\left(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_{(a_i,\lambda_i)}\right), \frac{1}{\lambda_i} \frac{\partial \tilde{\delta}_{(a_i,\lambda_i)}}{\partial a_i} \right\rangle,$$

where w is either a solution of (P_H) or zero. In order to simplify the notations, in the remainder we write $\tilde{\delta}_i$ instead of $\tilde{\delta}_{(a_i,\lambda_i)}$.

Proposition 4.1. *For* $\varepsilon > 0$ *small enough and* $u = \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i + v \in V(p, \varepsilon)$, *we have the following expansion*:

$$J(u)=S_4^{\frac{1}{3}}\frac{\sum_{i=1}^p\alpha_i^2}{(\sum_{i=1}^p\alpha_i^3H(a_i))^{\frac{2}{3}}}\Bigg[1-\frac{2}{3}\frac{c_1}{S_4(\sum_{i=1}^p\alpha_i^3H(a_i))}\sum_{i=1}^p\alpha_i^3\frac{\Delta H(a_i)}{\lambda_i^2}$$
$$-\frac{c_2}{\gamma}\sum_{i\neq j}\alpha_i\alpha_j\varepsilon_{ij}+f(v)+Q(v,v)+O\bigg(\sum_{k\neq r}\varepsilon_{kr}+\frac{1}{\lambda_k^2}+\|v\|^3\bigg)\Bigg],$$

where c_1 and c_2 are positive constants independent of u, with

$$Q(v,v)=\frac{1}{\gamma}\|v\|^2-\frac{2}{S_4(\sum_{i=1}^p\alpha_i^3H(a_i))}\int_{S^3}H\bigg(\sum_{i=1}^p\alpha_i\tilde{\delta}_i\bigg)v^2,$$
$$f(v)=-\frac{2}{S_4(\sum_{i=1}^p\alpha_i^3H(a_i))}\int_{S^3}H\Big(\sum\alpha_i\tilde{\delta}_i\Big)^2v,$$
$$\gamma=S_4\bigg(\sum_{i=1}^p\alpha_i^2\bigg),\qquad S_4=\bar{c}^3\int_{\mathbb{R}^3}\frac{dx}{(1+|x|^2)^3}.$$

The proof is very similar to the proof of Proposition 4.4 (see below, taking $w=0$ and therefore $h=0$), so we will omit it here.

Proposition 4.2. *For any $u=\sum_{i=1}^p\alpha_i\tilde{\delta}_i\in V(p,\varepsilon)$, we have the following expansion:*

$$\bigg\langle\nabla J(u),\lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i}\bigg\rangle=2J(u)\bigg[-c_2\sum_{j\neq i}\alpha_j\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}+J(u)^{\frac{3}{2}}\frac{2}{3}c_1\alpha_i^2\frac{\Delta H(a_i)}{\lambda_i^2}\bigg]+O\bigg(\frac{1}{\lambda_i^2}+\sum_{i\neq k}\varepsilon_{ij}\bigg).$$

Proof. We have

$$\big\langle\nabla J(u),h\big\rangle=2J(u)\bigg[\langle u,h\rangle-J(u)^{\frac{3}{2}}\int_{S^3}Hu^2h\bigg].$$

Thus,

$$\bigg\langle\nabla J(u),\lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i}\bigg\rangle=2J(u)\bigg[\bigg\langle\sum_{j=1}^p\alpha_j\tilde{\delta}_j,\lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i}\bigg\rangle-J(u)^{\frac{3}{2}}\int_{S^3}H\bigg(\sum_{j=1}^p\alpha_j\tilde{\delta}_j\bigg)^2\lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i}\bigg].$$

Observe that

$$\int_{S^3}H\bigg(\sum_{j=1}^p\alpha_j\tilde{\delta}_j\bigg)^2\lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i}=\sum_{j=1}^p\alpha_j^2\int_{S^3}H\tilde{\delta}_j^2\lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i}+2\sum_{i\neq j}\int_{S^3}H(\alpha_i\tilde{\delta}_i)\lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i}(\alpha_j\tilde{\delta}_j)$$
$$+O\bigg(\int_{S^3}\bigg(\sum_{j\neq k}\tilde{\delta}_j\bigg)\inf(\tilde{\delta}_j,\tilde{\delta}_k)^2\bigg).\tag{4.1}$$

An easy computation shows that

$$\left\langle \tilde{\delta}_i, \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} \right\rangle = 0, \tag{4.2}$$

$$\left\langle \tilde{\delta}_j, \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} \right\rangle = c_2 \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + o(\varepsilon_{ij}), \tag{4.3}$$

$$\int_{S^3} \tilde{\delta}_i \inf(\tilde{\delta}_i \tilde{\delta}_j)^2 = o(\varepsilon_{ij}), \tag{4.4}$$

$$\int_{S^3} H \tilde{\delta}_i^2 \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} = -\frac{c_1}{3} \frac{\Delta H(a_i)}{\lambda_i^2} + O\left(\frac{1}{\lambda_i^3}\right), \tag{4.5}$$

$$\int_{S^3} H \tilde{\delta}_j^2 \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} = c_2 H(a_j) \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + O\left(\varepsilon_{ij} + \frac{1}{\lambda_j^2}\right), \tag{4.6}$$

$$2 \int_{S^3} H \tilde{\delta}_i \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} \tilde{\delta}_j = c_2 H(a_i) \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + O\left(\varepsilon_{ij} + \frac{1}{\lambda_i^2}\right). \tag{4.7}$$

Using (4.1)–(4.7) and the fact that $J(u)^{\frac{3}{2}} \alpha_i H(a_i) = 1 + o(1)$ for each i, the proposition follows. □

Proposition 4.3. *For any $u = \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \in V(p, \varepsilon)$, we have the following expansion*:

$$\left\langle \nabla J(u), \frac{1}{\lambda_i} \frac{\partial \tilde{\delta}_i}{\partial a_i} \right\rangle = 2J(u) \left[-c_2 \sum_{j \neq i} \frac{\alpha_j}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i} - J(u)^{\frac{3}{2}} c_3 \alpha_i^2 \frac{\nabla H(a_i)}{\lambda_i} \right] + O\left(\sum_{i \neq j} \varepsilon_{ij} + \sum_{k=1}^{p} \frac{1}{\lambda_k^2} \right).$$

Proof. We have,

$$\left\langle \nabla J(u), \frac{1}{\lambda_i} \frac{\partial \tilde{\delta}_i}{\partial a_i} \right\rangle = 2J(u) \left[\left\langle \sum_{j=1}^{p} \alpha_j \tilde{\delta}_j, \frac{1}{\lambda_i} \frac{\partial \tilde{\delta}_i}{\partial a_i} \right\rangle - J(u)^{\frac{3}{2}} \int_{S^3} H \left(\sum_{j=1}^{p} \alpha_j \tilde{\delta}_j \right)^2 \frac{1}{\lambda_i} \frac{\partial \tilde{\delta}_i}{\partial a_i} \right].$$

Observe that

$$\int_{S^3} H \left(\sum_{j=1}^{p} \alpha_j \tilde{\delta}_j \right)^2 \frac{1}{\lambda_i} \frac{\partial \tilde{\delta}_i}{\partial a_i} = \sum_{j=1}^{p} \alpha_j^2 \int_{S^3} H \tilde{\delta}_j^2 \frac{1}{\lambda_i} \frac{\partial \tilde{\delta}_i}{\partial a_i} + 2 \sum_{i \neq j} \int_{S^3} H(\alpha_i \tilde{\delta}_i) \frac{1}{\lambda_i} \frac{\partial \tilde{\delta}_i}{\partial a_i} (\alpha_j \tilde{\delta}_j) + O\left(\left(\sum_{j \neq k} \tilde{\delta}_j \right) \inf(\tilde{\delta}_j, \tilde{\delta}_k) \right), \tag{4.8}$$

$$\left\langle \tilde{\delta}_i, \frac{1}{\lambda_i}\frac{\partial \tilde{\delta}_i}{\partial a_i}\right\rangle = 0, \tag{4.9}$$

$$\left\langle \tilde{\delta}_j, \frac{1}{\lambda_i}\frac{\partial \tilde{\delta}_i}{\partial a_i}\right\rangle = \frac{c_2}{\lambda_i}\frac{\partial \varepsilon_{ij}}{\partial a_i} + o(\varepsilon_{ij}), \tag{4.10}$$

$$\int_{S^3} H\tilde{\delta}_i^2 \frac{1}{\lambda_i}\frac{\partial \tilde{\delta}_i}{\partial a_i} = c_3 \frac{\nabla H(a_i)}{\lambda_i} + O\left(\frac{1}{\lambda_i^2}\right), \tag{4.11}$$

$$\int_{S^3} H\tilde{\delta}_j^2 \frac{1}{\lambda_i}\frac{\partial \tilde{\delta}_i}{\partial a_i} = c_2 \frac{H(a_j)}{\lambda_i}\frac{\partial \varepsilon_{ij}}{\partial a_i} + O\left(\varepsilon_{ij} + \frac{1}{\lambda_j^2}\right), \tag{4.12}$$

$$2\int_{S^3} H\tilde{\delta}_i \frac{1}{\lambda_i}\frac{\partial \tilde{\delta}_i}{\partial a_i}\tilde{\delta}_j = c_2 \frac{H(a_i)}{\lambda_i}\frac{\partial \varepsilon_{ij}}{\partial a_i} + O\left(\varepsilon_{ij} + \frac{1}{\lambda_i^2}\right). \tag{4.13}$$

Combining (4.4), (4.8)–(4.13) and the fact that $J(u)^{\frac{3}{2}}\alpha_i H(a_i) = 1 + o(1)$ for each i, we easily derive our proposition. □

Next we prove a statement more general than Proposition 4.1.

Proposition 4.4. *For $\varepsilon > 0$ small enough and $u = \sum_{i=1}^p \alpha_i \tilde{\delta}_{(a_i,\lambda_i)} + \alpha_0(w + h) + v \in V(p, \varepsilon, w)$, we have the following expansion*:

$$J(u) = \frac{S_4 \sum_{i=1}^p \alpha_i^2 + \alpha_0^2\|w\|^2}{(S_4 \sum_{i=1}^p \alpha_i^3 H(a_i) + \alpha_0^3\|w\|^2)^{\frac{2}{3}}}\Bigg[1 - \frac{2c_1 \sum_{i=1}^p \alpha_i^3 \frac{\Delta H(a_i)}{\lambda_i^2}}{3[S_4(\sum_{i=1}^p \alpha_i^3 H(a_i)) + \alpha_0^3\|w\|^2]}$$
$$- \frac{c_2}{\gamma_1}\sum_{i\neq j}\alpha_i\alpha_j\varepsilon_{ij} - \frac{2c_3\alpha_0}{\gamma_1}\sum_{i=1}^p \alpha_i \frac{w(a_i)}{\lambda_i} + f_1(v) + Q_1(v,v)$$
$$+ f_2(h) + Q_2(h,h) + o\left(\sum_{k\neq r}\varepsilon_{kr} + \sum_{k=1}^p \frac{1}{\lambda_k^2} + \|v\|^2 + \|h\|^2\right)\Bigg],$$

where c_1, c_2 and c_3 are a positive constants independent of u, with

$$Q_1(v,v) = \frac{1}{\gamma_1}\|v\|^2 - \frac{2}{S_4(\sum_{i=1}^p \alpha_i^3 H(a_i)) + \alpha_0^3\|w\|^2}\int_{S^3} H\left(\sum_{i=1}^p \alpha_i\tilde{\delta}_i\right)v^2$$
$$- \frac{2\alpha_0}{S_4(\sum_{i=1}^p \alpha_i^3 H(a_i)) + \alpha_0^3\|w\|^2}\int_{S^3} Hwv^2,$$

$$Q_2(h,h) = \frac{1}{\gamma_1}\|h\|^2 - \frac{2\alpha_0}{S_4(\sum_{i=1}^p \alpha_i^3 H(a_i)) + \alpha_0^3\|w\|^2}\int_{S^3} Hwh^2,$$

$$f_1(v) = -\frac{2}{S_4(\sum_{i=1}^p \alpha_i^3 H(a_i)) + \alpha_0^3 \|w\|^2} \int_{S^3} H\left(\sum_{i=1}^p \alpha_i \tilde{\delta}_i\right)^2 v,$$

$$f_2(h) = \frac{\alpha_0}{\gamma_1} \sum_{i=1}^p \alpha_i \langle \tilde{\delta}_i, h\rangle - \frac{2\alpha_0}{S_4(\sum_{i=1}^p \alpha_i^3 H(a_i)) + \alpha_0^3 \|w\|^2} \int_{S^3} H\left(\sum_{i=1}^p \alpha_i \tilde{\delta}_i\right)^2 h,$$

$$\gamma_1 = S_4\left(\sum_{i=1}^p \alpha_i^2\right) + \alpha_0^2 \|w\|^2, \quad S_4 = \bar{c}^3 \int_{\mathbb{R}^3} \frac{dx}{(1+|x|^2)^3}.$$

Proof. We need to estimate

$$N(u) = \|u\| \quad \text{and} \quad D = \int_{S^3} H u^3.$$

We have

$$N(u) = \sum_{i=1}^p \alpha_i^2 \|\tilde{\delta}_i\|^2 + \alpha_0^2\big(\|h\|^2 + \|w\|^2\big) + \|v\|^2 + \sum_{i=1}^p \alpha_i \alpha_j \langle \tilde{\delta}_i, \tilde{\delta}_j\rangle + 2\sum_{i=1}^p \alpha_i \alpha_0 \langle \tilde{\delta}_i, w+h\rangle.$$

Observe that

$$\langle h, w\rangle = 0,$$
$$\|\tilde{\delta}_i\|^2 = \|\delta_i\|^2 = S_4,$$
$$\langle \tilde{\delta}_i, \tilde{\delta}_j\rangle = c_2 \varepsilon_{ij} + o(\varepsilon_{ij}).$$

We notice also that

$$\langle \tilde{\delta}_i, w\rangle = \int_{\mathbb{R}^3} \delta_i^2 w = c_3 \frac{w(a_i)}{\lambda_i} + o\left(\frac{1}{\lambda_i}\right).$$

Thus,

$$N = \gamma_1 + c_2 \sum_{i \neq j} \alpha_i \alpha_j \varepsilon_{ij} + 2c_3 \sum_{i=1}^p \alpha_i \alpha_0 \frac{w(a_i)}{\lambda_i} + \alpha_0^2 \|h\|^2 + \|v\|^2 + \langle \tilde{\delta}_i, h\rangle$$
$$+ o\left(\sum_{i=1}^p \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij}\right). \tag{4.14}$$

For the denominator, we write

$$\begin{aligned}D = &\int_{S^3} H\biggl(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i\biggr)^3 + \alpha_0^3\int_{S^3} H(w+h)^3\\ &+ 3\alpha_0\int_{S^3} H\biggl(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i\biggr)^2(w+h) + 3\int_{S^3} H\biggl(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i + \alpha_0(w+h)\biggr)^2 v\\ &+ 3\int_{S^3} H\biggl(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i + \alpha_0(w+h)\biggr)v^2 + O\bigl(\|v\|^3\bigr)\\ &+ O\Biggl(\int\biggl(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i\biggr)\inf\biggl(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i, w+h\biggr)^2 + \sum_{i=1}^{p}\alpha_i\alpha_0\int\tilde{\delta}_i(w+h)^2\Biggr).\end{aligned}$$

We also have

$$\int_{S^3} H\biggl(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i\biggr)^3 = \sum_{i=1}^{p}\alpha_i^3\int_{S^3} H\tilde{\delta}_i^3 + 3\sum_{i\neq j}\alpha_i^2\alpha_j\int_{S^3} H\tilde{\delta}_i^2\tilde{\delta}_j + O\biggl(\sum_{i\neq j}\int_{S^3}\tilde{\delta}_i\inf(\tilde{\delta}_i\tilde{\delta}_j)^2\biggr).$$

Expansion of H around a_i and a_j give

$$\int_{S^3} H\tilde{\delta}_i^3 = H(a_i)S_4 + c_1\frac{\Delta H(a_i)}{\lambda_i^2} + O\biggl(\frac{1}{\lambda_i^3}\biggr), \tag{4.15}$$

$$\int_{S^3} H\tilde{\delta}_i^2\tilde{\delta}_j = c_2 H(a_i)\varepsilon_{ij} + O\biggl(\varepsilon_{ij} + \frac{1}{\lambda_i^2}\biggr). \tag{4.16}$$

Combining (4.4), (4.15) and (4.16) we derive that

$$\begin{aligned}\int_{S^3} H\biggl(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i\biggr)^3 = &\sum_{i=1}^{p}\alpha_i^3\biggl(H(a_i)S_4 + c_1\frac{\Delta H(a_i)}{\lambda_i^2} + O\biggl(\frac{1}{\lambda_i^3}\biggr)\biggr)\\ &+ 3\sum_{i\neq j}\alpha_i^2\alpha_j\biggl(c_2 H(a_i)\varepsilon_{ij} + O\biggl(\varepsilon_{ij} + \frac{1}{\lambda_i^2}\biggr)\biggr).\end{aligned} \tag{4.17}$$

Using the fact that h belongs to the tangent space at w, we derive that

$$\begin{aligned}\int_{S^3} H(w+h)^3 &= \int_{S^3} Hw^3 + 3\int_{S^3} Hw^2h + 3\int_{S^3} Hwh^2 + O\bigl(\|h\|^3\bigr)\\ &= \|w\|^2 + 3\int_{S^3} Hwh^2 + O\bigl(\|h\|^3\bigr).\end{aligned} \tag{4.18}$$

Also we have

$$\int_{S^3} H(w+h)\left(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i\right)^2 = c_3\sum_{i=1}^{p}\alpha_i^2\frac{H(a_i)w(a_i)}{\lambda_i} + \int_{S^3} H\left(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i\right)^2 h + o\left(\sum_{i=1}^{p}\frac{1}{\lambda_i}\right). \tag{4.19}$$

Since $v \in T_w(W_s(w))$ and $h \in T_w(W_u(w))$, the linear form on v can be written as

$$\begin{aligned}\int_{S^3} H\left(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i + \alpha_0(w+h)\right)^2 v &= \int_{S^3} H\left(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i\right)^2 v + \int_{S^3} H\big(\alpha_0(w+h)\big)^2 v \\ &\quad + O\left(\sum_{i=1}^{p}\int_{S^3}\tilde{\delta}_i|w+h||v| + \int_{S^3}\tilde{\delta}_i|w+h||v|\right) \\ &= -\frac{S_4(\sum_{i=1}^{p}\alpha_i^3 H(a_i)) + \alpha_0^3\|w\|^2}{2} f_1(v) \\ &\quad + \alpha_0^2\int_{S^3} H w^2 v + 2\int_{S^3} H w h v + O\big(\|v\|\|h\|^2\big) \\ &= -\frac{S_4(\sum_{i=1}^{p}\alpha_i^3 H(a_i)) + \alpha_0^3\|w\|^2}{2} f_1(v) \\ &\quad + O\big(\|v\|^3 + \|h\|^3\big).\end{aligned} \tag{4.20}$$

Finally, we have

$$\int_{S^3} H\left(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i + \alpha_0(w+h)\right)v^2 = \int_{S^3} H\left(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i\right)v^2 + \int_{S^3} H(\alpha_0 w)v^2 + o\big(\|v\|^2 + \|h\|^2\big). \tag{4.21}$$

Combining (4.14)–(4.21) and the fact that $\alpha_0 J(u)^{\frac{3}{2}} = 1 + o(1)$, $J(u)^{\frac{3}{2}}\alpha_i H(a_i) = 1 + o(1)$ for each i, the result follows. □

5. Construction of a pseudogradient flow

In this section, we are going to construct a pseudogradient W near infinity for the functional J using some of the ideas introduced in the proof of [11, Proposition A.2], where a related pseudogradient has been constructed for the scalar curvature problem. This construction allows as to give a characterization of the critical points at infinity of our problem. We recall that the critical points at infinity are the orbits of the gradient flow of J, which remain in $V(p, \varepsilon(s))$, where $\varepsilon(s)$ is a given function tends to zero when s tends $+\infty$ (see [10]).

A crucial property of this gradient flow is that along its flow lines there can be only finitely many isolated blow up points. Such a flow is defined by combining two basic facts. On the one hand, the first one comes from the Morse lemma at infinity which moves points and concentrations as follows: points move according to $-\nabla_{\mathrm{T}} H$, concentrations move so as to decrease the functional J. On the other hand, there is another pseudogradient when the points are very close and the total interaction $\sum \varepsilon_{ij}$ is large with respect to $\sum \frac{1}{\lambda_i^2}$. We need to convex-combine both flows to keep the pseudogradient property, to avoid the creation of new asymptotes and to ensure the property that the flow lines when they leave some $V(p,\varepsilon)$ will loose at least one bubble, that is the flow will never come back to $V(q,\varepsilon)$ for $q \leqslant p$, a fact which is not trivial in scalar curvature problems whose functional's levels on $V(p,\varepsilon)$ are not constant. Some levels of $V(p,\varepsilon)$ might be below some other levels of $V(q,\varepsilon)$ for some $q < p$.

We begin by giving the following main result.

Theorem 5.1. *For* $p \geqslant 1$, *there exists a pseudogradient* W *so that the following holds. There is a constant* $c > 0$ *independent of* $u = \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \in V(p,\varepsilon)$ *so that*

$$\text{(i)} \qquad \bigl(-\nabla J(u), W\bigr) \geqslant c\left(\sum_{i=1}^{p} \frac{|\nabla H(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij}\right);$$

$$\text{(ii)} \qquad \left(-\nabla J(u+\bar{v}), \frac{\partial \bar{v}}{\partial(\alpha_i, a_i, \lambda_i)}(W)\right) \geqslant c\left(\sum_{i=1}^{p} \frac{|\nabla H(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij}\right);$$

(iii) $|W|$ *is bounded. Furthermore, the only case where the maximum of the* λ_i*'s is not bounded is when each point* a_i *is close to a critical point* y_{j_i} *of* H *with* $y_{j_i} \neq y_{j_k}$ *for* $i \neq k$ *and* $\rho(y_{i_1}, \ldots, y_{i_p}) > 0$, *where* $\rho(y_{i_1}, \ldots, y_{i_p})$ *denotes the least eigenvalue of* $M(y_{i_1}, \ldots, y_{i_p})$.

Proof. We start by proving claim (i). For this purpose, let η be a positive constant such that $\eta < \inf_{i \neq j} d(y_i, y_j)$ and for each i, if $d(x, y_i) \leqslant \eta$ then we have $\Delta H(x) \neq 0$. For $u = \sum_{i=1}^{p} \alpha_i \tilde{\delta}_{a_i,\lambda_i} \in V(p,\varepsilon)$ and $\tau = (j_1, \ldots, j_p)$, five cases may occur.

Case 1. $u \in \{u \mid a_i \in B(y_{j_i}, \eta),\ y_{j_i} \neq y_{j_k}$ for $i \neq k$ with $-\Delta H(y_{j_i}) > 0$ and $\rho(\tau) > 0\}$. In this case we have for any $i \neq j$, $|a_i - a_j| > c$ and therefore,

$$\varepsilon_{ij} = \left(\frac{1}{\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2}\right) = \frac{2}{\lambda_i \lambda_j |a_i - a_j|^2}\bigl(1 + o(1)\bigr) = \frac{2G_{ij}}{\lambda_i \lambda_j}\bigl(1 + o(1)\bigr),$$

where G_{ij} denotes the Green's function associated to the operator Ξ defined in $\mathbb{S}^3$. Thus,

$$\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = -\frac{2G_{ij}}{\lambda_i \lambda_j}\bigl(1 + o(1)\bigr).$$

Using the fact that for $u \in V(p,\varepsilon)$ we have $\alpha_i H(a_i) J(u)^{3/2} = 1 + o(1)$ and Proposition 4.2, we derive that

$$\left\langle \nabla J(u), \alpha_i \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i}\right\rangle = 2J(u)\left[c_2 \sum_{j \neq i} \alpha_i \alpha_j \frac{2G_{ij}}{\lambda_i \lambda_j} + \frac{2}{3} \frac{c_1}{H(a_i)} \alpha_i^2 \frac{\Delta H(a_i)}{\lambda_i^2}\right] + O\left(\sum_{j=1}^{p} \frac{1}{\lambda_j^2}\right)$$

$$= 4J(u)^{-2}\left[\frac{c_1}{3}\frac{\Delta H(a_i)}{H(a_i)^3\lambda_i^2} + \sum_{j\neq i}\frac{c_2 G_{ij}}{H(a_i)H(a_j)}\frac{1}{\lambda_i\lambda_j}\right] + O\left(\sum_{j=1}^{p}\frac{1}{\lambda_j^2}\right). \tag{5.1}$$

Let $Z_1 = \sum_{i=1}^{p}\alpha_i\lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i}$, we derive that

$$\langle -\nabla J(u), Z_1\rangle = cT_\Lambda M\Lambda + o\left(\sum_k \frac{1}{\lambda_k^2}\right), \tag{5.2}$$

where M is the matrix already defined in Section 1 and $\Lambda = T_{(\lambda_1,\dots,\lambda_p)}$. Recall that ρ, the least eigenvalue of M, is positive in this case. Then (5.2) becomes

$$\langle -\nabla J(u), Z_1\rangle \geqslant c\left(\sum_k \frac{1}{\lambda_k^2}\right) \geqslant c\left(\sum_k \frac{1}{\lambda_k^2}\right) + c\sum_{i\neq j}\varepsilon_{ij}. \tag{5.3}$$

To complete the construction in this case, we need to define φ a C^∞ function which satisfies $\varphi(t) = 0$ if $t \leqslant 1$ and $\varphi(t) = 1$ if $t \geqslant 2$. We also define

$$Z' = \sum_{i=1}^{p}\varphi\big(\lambda_i|\nabla H(a_i)|\big)\frac{\nabla H(a_i)}{|\nabla H(a_i)|}\frac{1}{\lambda_i}\frac{\partial\tilde{\delta}_i}{\partial a_i}$$

and let $W_1 = CZ_1 + Z'$, where C is a large constant. Using Proposition 4.3 and (5.3), we derive that

$$\langle -\nabla J(u), W_1\rangle \geqslant c\left(\sum_i \frac{|\nabla H(a_i)|}{\lambda_i} + \sum_k \frac{1}{\lambda_k^2} + \sum_{i\neq j}\varepsilon_{ij}\right). \tag{5.4}$$

Case 2. $u \in \{u \mid a_i \in B(y_{j_i}, \eta),\ j_i \neq j_k \text{ for } i \neq k \text{ with } -\Delta H(y_{j_i}) > 0 \text{ and } \rho(\tau) < 0\}$. In this case, we define $e = (e_1, \dots, e_p)$ the eigenvector associated to ρ such that $\|e\| = 1$ with $e_i > 0$ for all i. Let $r > 0$ such that for any $x \in B(e, r) = \{y \in S^{p-1} \mid \|y - e\| \leqslant r\}$, we have $T_x Mx < (1/2)\rho$. Two subcases may occur.

Subcase 2.1. $\frac{\Lambda}{|\Lambda|} \in B(e, r)$. In this case, we define $W_2 = -CZ_1 + Z'$. Using the estimates (5.2), (5.4) and the fact that $\rho(\tau) < 0$, we derive that

$$\langle -\nabla J(u), W_2\rangle \geqslant c\left(\sum_i \frac{|\nabla H(a_i)|}{\lambda_i} + \sum_k \frac{1}{\lambda_k^2} + \sum_{i\neq j}\varepsilon_{ij}\right). \tag{5.5}$$

Subcase 2.2. $\frac{\Lambda}{|\Lambda|} \notin B(e, r)$. In this case, let $y(t) = (1 - t)\Lambda + t|\Lambda|e$, $\Lambda(t) = y(t)/\|y(t)\|$ and we define

$$Z_2 = -\sum_{i=1}^{p}|\Lambda|\alpha_i\lambda_i^2\frac{\partial\tilde{\delta}_i}{\partial\lambda_i}\left[\frac{|\Lambda|e_i - \Lambda_i}{\|y(0)\|} - \frac{y_i(0)}{\|y(0)\|^3}\big(y(0), |\Lambda|e - \Lambda\big)\right].$$

Using Proposition 4.2, we derive that

$$\big\langle -\nabla J(u), Z_2\big\rangle \geqslant -c|\Lambda|^2 \frac{\partial}{\partial t}\big(T_{\Lambda(t)} M \Lambda(t)\big) + o\bigg(\sum_k \frac{1}{\lambda_k^2}\bigg). \tag{5.6}$$

Since $(T_{\Lambda(t)} M \Lambda(t)) = \rho + \frac{(1-t)^2}{\|y(t)\|^2}(T_\Lambda M \Lambda - \rho\|\Lambda\|^2)$, we derive that $\frac{\partial}{\partial t}(T_{\Lambda(t)} M \Lambda(t)) < -c$. Therefore, using (5.4) and (5.6) for $W_2' = C Z_2 + Z'$, we derive

$$\big\langle -\nabla J(u), W_2'\big\rangle \geqslant c\bigg(\sum_i \frac{|\nabla H(a_i)|}{\lambda_i} + \sum_k \frac{1}{\lambda_k^2} + \sum_{i\neq j} \varepsilon_{ij}\bigg). \tag{5.7}$$

Case 3. $u \in \{u \mid a_i \in B(y_{j_i}, \eta),\ \ j_i \neq j_k$ for $i \neq k$ and $\exists j_1, \dots, j_p$ s.t. $-\Delta H(y_{j_i}) < 0\}$. In this case, we can assume without lose of generality, that $1, \dots, q$ are the indices which satisfy $-\Delta H(a_i) < 0$. Let $I = \{i \mid \lambda_i < (1/10)\inf_{k\in\{1,\dots,q\}} \lambda_k\}$. Let also M_I be the matrix defined by the points $(a_i)_{i\in I}$ and ρ_I be the least eigenvalue of M_I. We define

$$Z_3 = -\sum_{i=1}^{p} \alpha_i \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i}.$$

Since for any $i \neq j$ we have $|a_i - a_j| > c$ and using (5.1), we derive that

$$\big\langle -\nabla J(u), Z_3\big\rangle \geqslant c\sum_{k=1}^{q}\bigg(\frac{1}{\lambda_k^2} + \sum_{i\neq k}\frac{G_{ik}}{\lambda_i\lambda_k}\bigg) \geqslant c\sum_{k\notin I}\bigg(\frac{1}{\lambda_k^2} + \sum_{i\neq k}\varepsilon_{ik}\bigg). \tag{5.8}$$

Observe that in the case where $I \neq \emptyset$, we have to add an other vector field since only the indices such as $k \notin I$ appear. Then, if the matrix M_I is positive definite, we define $Z_3' = Z_1/I$, (i.e. the action of Z_1 using only the indices of I). If M_I is not positive definite, we define $Z_3' = Z_2/I$. In both case, we have

$$\big\langle -\nabla J(u), Z_3'\big\rangle \geqslant c\sum_{k\in I}\bigg(\frac{1}{\lambda_k^2} + \sum_{i\neq k,\, i\in I}\varepsilon_{ik}\bigg) - c\sum_{k\in I,\, i\notin I}\varepsilon_{ik}. \tag{5.9}$$

Using (5.4), (5.8) and (5.9) for $W_3 = C Z_3 + Z_3' + m Z'$, where C is a large constant and m is a small constant, we derive that

$$\big\langle -\nabla J(u), W_3\big\rangle \geqslant c\bigg(\sum_i \frac{|\nabla H(a_i)|}{\lambda_i} + \sum_k \frac{1}{\lambda_k^2} + \sum_{i\neq j} \varepsilon_{ij}\bigg). \tag{5.10}$$

Case 4. $u \in \{u \mid a_i \in B(y_{j_i}, \eta),\ \exists i \neq k$ such that $j_i = j_k\}$. In this case, there is at least one $B_i = \{j \mid a_j \in B(y_i, \eta)\}$ which contains at least two indices. Without loss of generality, we can assume that $1, \dots, q$ are the indices such that the set B_i $(1 \leqslant i \leqslant q)$ contains at least two indices. We will decrease the λ_i's for $i \in B_i$ with different speed. For this purpose, let χ be a smooth cutoff function such that $\chi \geqslant 0$, $\chi = 0$ if $t \leqslant r'$ and $\chi = 1$ if $t \geqslant 1$, where r' is a small constant. Set $\bar{\chi}(\lambda_j) = \sum_{i \neq j,\, i, j \in B_k} \chi(\lambda_j/\lambda_i)$. We define

$$Z_4 = \sum_{k=1}^{q} \sum_{j \in B_k} \alpha_j \bar{\chi}(\lambda_j) \lambda_j \frac{\partial \tilde{\delta}_j}{\partial \lambda_j}.$$

Using Proposition 4.2, we obtain

$$\bigl\langle -\nabla J(u), Z_4 \bigr\rangle = 2J(u) \sum_{k=1}^{q} \sum_{j \in B_k} \alpha_j \bar{\chi}(\lambda_j) \biggl[c_2 \sum_{j \neq i} \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} - \frac{2}{3} J(u)^{\frac{3}{2}} c_1 \alpha_i^2 \frac{\Delta H(a_i)}{\lambda_i^2} \biggr] + o\Biggl(\sum_{r=1}^{p} \frac{1}{\lambda_r^2} + \sum_{i \neq r} \varepsilon_{ir} \Biggr).$$

If $j \in B_k$ with $k \leqslant q$, if $\bar{\chi}(\lambda_j) \neq 0$, then there exists $i \in B_k$ such that $\lambda_j^{-2} = o(\varepsilon_{ij})$ (for η small enough). If $i \notin B_k$ or $i \in B_k$ with λ_i and λ_j are of the same order, then we have $-\lambda_r \frac{\partial \varepsilon_{ij}}{\partial \lambda_r} = \varepsilon_{ij}(1 + o(1))$, for $r = i, j$.

In the case where $i \in B_k$ with $\lambda_i < \lambda_j$, ($\frac{\lambda_i}{\lambda_j} < r'$) we have $\bar{\chi}(\lambda_j) - \bar{\chi}(\lambda_i) \geqslant 1$. Thus,

$$-\bar{\chi}(\lambda_j)\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} - \bar{\chi}(\lambda_i)\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \geqslant -\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} = \varepsilon_{ij}\bigl(1 + o(1)\bigr).$$

We derive now

$$\bigl\langle -\nabla J(u), Z_4 \bigr\rangle \geqslant c \sum_{k=1}^{q} \sum_{j \in B_k, \bar{\chi}(\lambda_j) \neq 0} \Biggl(\frac{1}{\lambda_j^2} + \sum_{i \neq j} \varepsilon_{ij} \Biggr). \tag{5.11}$$

Observe as in the third case, we have to add some terms. Let $\lambda_{i_0} = \inf\{\lambda_i \mid i = 1, \dots, p\}$, two subcases may occur.

Subcase 4.1. There exists j such that $\bar{\chi}(\lambda_j) \neq 0$ and $\frac{\lambda_{i_0}}{\lambda_j} > r'$. In this case, we can make appear in (5.11) the term $1/\lambda_{i_0}^2$ and therefore all the $1/\lambda_i^2$ and the ε_{ik}. Thus, we define in this case $W_4' = CZ_4 + Z'$ where C is a large constant.

Subcase 4.2. For all j, we have $\bar{\chi}(\lambda_j) = 0$ or $\frac{\lambda_{i_0}}{\lambda_j} < r'$. In this case, we define

$$D = \Biggl(\bigl\{ i \mid \bar{\chi}(\lambda_j) = 0 \bigr\} \cup \Biggl(\bigcup_{k=1}^{q} B_k \Biggr)^{c} \Biggr) \cap \biggl\{ i \biggm| \frac{\lambda_i}{\lambda_{i_0}} < 1/r' \biggr\}.$$

For each $i, r \in D$, such that $i \neq r$, we have $a_i \in B(y_{j_i}, \eta)$ and $a_r \in B(y_{j_r}, \eta)$ since the set $\{i \mid \bar{\chi}(\lambda_j) = 0\}$ contains at most one index from each B_j for $j = 1, \dots, q$. Let $u_1 = \sum_{i \in D} \alpha_i \tilde{\delta}_i$. This element has to satisfy one of the three above cases. Thus, we can apply the associated vector field which we will denote Z_4' and we have the estimate

$$\begin{aligned}\left\langle -\nabla J(u), Z_4'\right\rangle \geqslant c \sum_{i\in D}\left(\frac{|\nabla H(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{i,j\in D}\varepsilon_{ij}\right) \\ + O\left(\sum_{k\in D, r\notin D}\varepsilon_{rk}\right) + o\left(\sum_{i\notin D}\frac{1}{\lambda_i^2}\right).\end{aligned} \tag{5.12}$$

For $k \in D$ and $r \notin D$, we have either $r \in B^q := \{i \mid \bar\chi(\lambda_j) \neq 0\} \cup (\bigcup_{j=1}^q B_j)$ or $r \in (B^q)^c$. For the case where $r \in B^q$, the term ε_{kr} appears in (5.11). If $r \in (B^q)^c$ and since $r \notin D$ we deduce that $\lambda_{i_0}/\lambda_r < r'$. Furthermore, we can prove that a_k and a_r are not in the same $B(y,\eta)$ and therefore $|a_k - a_r| > c$. Thus,

$$\varepsilon_{kr} \leqslant \frac{c}{\lambda_k\lambda_r} \leqslant \frac{cr'}{\lambda_k\lambda_{i_0}} = o(\varepsilon_{ki_0}).$$

Since $i_0 \in D$, then from $1/\lambda_{i_0}^2$ we can make appear in (5.11) all the $1/\lambda_i^2$ and ε_{ir} for $i, r \in (B^q)^c$ (we have $|a_i - a_r| > c$). Thus, we derive that

$$\left\langle -\nabla J(u), Z_4'\right\rangle \geqslant c\left(\sum_{i\in D}\frac{|\nabla H(a_i)|}{\lambda_i} + \sum_{i=1}^{p}\frac{1}{\lambda_i^2} + \sum_{i,j\in(B^q)^c}\varepsilon_{ij}\right) + O\left(\sum_{k\in D, r\in B^q}\varepsilon_{rk}\right). \tag{5.13}$$

Using (5.4), (5.11) and (5.13) on the vector field $W_4'' = CZ_4 + Z_4' + mZ'$ where C is a large constant and m is a small constant, we derive

$$\left\langle -\nabla J(u), W_4''\right\rangle \geqslant c\left(\sum_{i}\frac{|\nabla H(a_i)|}{\lambda_i} + \sum_k\frac{1}{\lambda_k^2} + \sum_{i\neq j}\varepsilon_{ij}\right). \tag{5.14}$$

Case 5. $u \in \{u \mid \exists i_1, \dots, i_p$ such that $|a_{i_j} - y| > \eta/2$ for all critical points $y\}$. In this case, we order the λ_i's in an increasing order, $\lambda_1 \leqslant \lambda_2 \leqslant \dots \leqslant \lambda_p$. Let i_1 be such that for any $i < i_1$ we have $|a_i - y_{j_i}| \leqslant \eta/2$ where y_{j_i} is a critical point of H and $|a_{i_1} - y| > \eta/2$, for any critical point y. Since $u = \sum_{i<i_1}\alpha_i\tilde\delta_i$ satisfy one of the four above cases, we can apply the associated vector field which we will denote Z_5 and we have

$$\begin{aligned}\left\langle -\nabla J(u), Z_5\right\rangle \geqslant c\sum_{i<i_1}\left(\frac{|\nabla H(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{j<i_1}\varepsilon_{ij} + O\left(\sum_{j\geqslant i_1}\varepsilon_{ij}\right)\right) \\ + o\left(\sum_{i\geqslant i_1}\frac{1}{\lambda_i^2} + \sum_{k\neq r}\varepsilon_{kr}\right).\end{aligned}$$

We also define

$$Z_5' = \frac{1}{\lambda_{i_1}}\frac{\partial\tilde\delta_i}{\partial a_{i_1}}\frac{\nabla H(a_{i_1})}{|\nabla H(a_{i_1})|} - C'\sum_{i\geqslant i_1}2^i\lambda_i\frac{\partial\tilde\delta_i}{\partial\lambda_i},$$

where C' is a large constant. Using Propositions 4.2 and 4.3 and the fact that $|\nabla H(a_{i_1})| > c$, we deduce that

$$\begin{aligned}\langle -\nabla J(u), Z_5'\rangle &\geqslant \frac{c}{\lambda_{i_1}} + O\biggl(\sum_{i\neq i_1}\varepsilon_{ii_1}\biggr) + cC'\sum_{i\geqslant i_1}\biggl(\sum_{j\neq i}\varepsilon_{ij} + O\biggl(\frac{1}{\lambda_i^2}\biggr) + o\biggl(\sum_{k\neq r}\varepsilon_{kr} + \frac{1}{\lambda_k^2}\biggr)\biggr)\\ &\geqslant \sum_{i\geqslant i_1}\biggl(\frac{c}{\lambda_i} + \sum_{j\neq i}\varepsilon_{ij}\biggr) + o\biggl(\sum_{k\neq r}\varepsilon_{kr} + \frac{1}{\lambda_k^2}\biggr).\end{aligned}$$

Using the vector field $W_5 = Z_5 + CZ_5'$, where C is a large constant, we deduce that

$$\langle -\nabla J(u), W_5\rangle \geqslant c\biggl(\sum_i \frac{|\nabla H(a_i)|}{\lambda_i} + \sum_k \frac{1}{\lambda_k^2} + \sum_{i\neq j}\varepsilon_{ij}\biggr). \tag{5.15}$$

Now, we define the pseudogradient W as a convex combination of W_i for $i = 1, \dots, 5$. W satisfies claims (i) and (iii). The second claim can be obtained once we have (i) arguing as in [11]. □

6. A Morse lemma at infinity

In this section, we use the pseudogradient constructed in Section 5, to derive a *normal form* of the functional and its gradient flow near potential *critical points at infinity*. As a first application of such a *Morse lemma at infinity*, we identify the critical points at infinity of the Euler–Lagrange functional J as well as their Morse indices and their topological contribution to the difference of topology between the level sets of J.

Theorem 6.1. *For $\varepsilon > 0$ sufficiently small given, there exists a change of variables such that for any $u = \sum_{i=1}^p \alpha_i \tilde\delta_{a_i,\lambda_i} + v$ belongs to $V(p,\varepsilon)$, $(a_i, \alpha_i, v) \longrightarrow (a_i', \lambda_i', V)$ where V belongs to a neighborhood of zero in a fixed Hilbert space so that*

$$J\biggl(\sum_{i=1}^p \alpha_i\tilde\delta_{a_i,\lambda_i} + v\biggr) = J\biggl(\sum_{i=1}^p \alpha_i\tilde\delta_{a_i',\lambda_i'}\biggr) + |V|^2.$$

Furthermore, if each a_i belongs to a neighborhood of $y_{k_i} \in \mathcal{F}_\infty$, such that $\rho(y_{k_1}, \dots, y_{k_p}) > 0$, then we can find another change of variables $(a_i, \lambda_i) \to (a_i', \lambda_i')$ such that

$$J\biggl(\sum_{i=1}^p \alpha_i\tilde\delta_{a_i,\lambda_i}\biggr) = \psi(\alpha_i, a_i', \lambda_i') := \frac{S_4^{\frac13}\sum_{i=1}^p \alpha_i^2}{(\sum_{i=1}^p \alpha_i^3 H(a_i'))^{\frac23}}\biggl\{1 - \eta\rho(y_{k_1}, \dots, y_{k_p})\sum_{i=1}^p \frac{1}{\lambda_i'^2}\biggr\},$$

where η is a positive constant.

Theorem 6.2. *For $\varepsilon > 0$ sufficiently small given, there exists a change of variables such that for any $u = \sum_{i=1}^p \alpha_i\tilde\delta_{a_i,\lambda_i} + \alpha_0(w+h) + v$ belongs to $V(p,\varepsilon,w)$, $(a_i, \alpha_i, h, v) \longrightarrow (a_i', \lambda_i', \tilde H, V)$ where each $\tilde H$ and V belongs to a neighborhood of zero in a fixed Hilbert space so that*

$$J\biggl(\sum_{i=1}^p \alpha_i\tilde\delta_{a_i,\lambda_i} + \alpha_0(w+h) + v\biggr)$$

$$= J\left(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_{a_i',\lambda_i'} + \alpha_0 w\right) + |V|^2 - |\tilde{H}|^2$$

$$= \frac{S_4\sum_{i=1}^{p}\alpha_i^2 + \alpha_0^2\|w\|^2}{(S_4\sum_{i=1}^{p}\alpha_i^3 H(a_i) + \alpha_0^3\|w\|^2)^{\frac{2}{3}}}\Bigg[1 - \frac{2}{3}\frac{c_1}{S_4\sum_{i=1}^{p}\alpha_i^3 H(a_i) + \alpha_0^3\|w\|^2}$$

$$\times \sum_{i=1}^{p}\alpha_i^3\frac{\Delta H(a_i)}{\lambda_i^2} - \frac{c_2}{\gamma_1}\sum_{i\neq j}\alpha_i\alpha_j\varepsilon_{ij} - \frac{2c_3\alpha_0}{\gamma_1}\sum_{i=1}^{p}\alpha_i\frac{w(a_i)}{\lambda_i}\Bigg] + |V|^2 - |\tilde{H}|^2,$$

where c_1, c_2, c_3, S_4 *and* γ_1 *are defined in Proposition* 4.4.

Proof of Theorem 6.1. Using Theorem 5.1, the proof is similar to the same argument in [11, Appendix 2] (see also [16]). □

Corollary 6.3. *Assume that* $H \in \mathcal{H}^+$. *Then the only critical points at infinity of* J *in* Σ^+ *correspond to*

$$\sum_{j=1}^{p}\frac{1}{H(y_{i_j})}\tilde{\delta}_{(y_{i_j},\infty)},$$

where $p \geqslant 1$ *and the points* y_{i_j}*'s are critical points of* H *satisfying* $\rho(y_{i_1}, \dots, y_{i_p}) > 0$. *Moreover, the Morse index of such critical points at infinity is equal to* $p - 1 + \sum_{j=1}^{p} 3 - \operatorname{ind}(H, y_{i_j})$, *where* $\operatorname{ind}(H, y_{i_j})$ *is the Morse index of* H *at* y_{i_j}.

Proof. By Theorem 5.1, the only region where the λ_i's are unbounded is the one where each point a_i is close to a critical point y_{j_i}, with $y_{j_i} \neq y_{j_k}$ for $i \neq k$ and $\rho(y_{i_1}, \dots, y_{i_p}) > 0$. In this region, using Theorem 6.1, the normal form of J allows us to split the variables $\tilde{a}_i$ and $\tilde{\lambda}_i$. Then it is easy to see that if $(\tilde{a}_1, \dots, \tilde{a}_p)$ is equal to $(y_{i_1}, \dots, y_{i_p})$, only $\tilde{\lambda}_i$ can move and since $\rho(y_{i_1}, \dots, y_{i_p}) > 0$, in order to decrease the functional J, we have to increase $\tilde{\lambda}_i$. Therefore, we obtain a critical point at infinity only in this region.

In order to compute the Morse index of such critical points at infinity, we observe that this Morse index corresponds to the index of the critical point of the following function:

$$g(\alpha_1, \dots, \alpha_p, a_1, \dots, a_p) := \frac{\sum_{i=1}^{p}\alpha_i^2}{\sum_{i=1}^{p}\alpha_i^3 H(a_i)}.$$

In variables α_i's we have a degenerate critical point $(\bar{\alpha}_1, \dots, \bar{\alpha}_p)$ which satisfies

$$\frac{\bar{\alpha}_i H(a_i)^2}{\bar{\alpha}_j H(a_j)^2} = 1.$$

This critical point has an index equal to $p - 1$ (since the function g is homogeneous in the variables α_i's and the critical point corresponds to a maximum), then

$$g(\alpha_1, \dots, \alpha_p, a_1, \dots, a_p) = \left(\sum_{i=1}^{p}\frac{1}{H(a_i)^2}\right)^{1/3}\left(1 - |Y|^2\right),$$

where Y belongs to $\mathbb{R}^{p-1}$. Thus, the index is equal to $p-1+\sum_{k=1}^{p} 3-\operatorname{ind}(H, y_{i_k})$. $\square$

In order to prove Theorem 6.2, we need to introduce the following result.

Theorem 6.4. *For $p \geqslant 1$, there exists a pseudogradient W' so that the following holds: there is a constant $c>0$ independent of $u=\sum_{i=1}^{p} \alpha_i \tilde{\delta}_i+\alpha_0(w+h) \in V(p, \varepsilon, w)$ so that*

$$\text{(i)} \qquad \bigl(-\nabla J(u), W'\bigr) \geqslant c\left(\sum_{i=1}^{p} \frac{|\nabla H(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^2}+\sum_{i \neq j} \varepsilon_{ij}+|h|^2\right),$$

$$\text{(ii)} \qquad \left(-\nabla J(u+\bar{v}), \frac{\partial \bar{v}}{\partial(\alpha_i, a_i, \lambda_i)}(W')\right) \geqslant c\left(\sum_{i=1}^{p} \frac{|\nabla H(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^2}+\sum_{i \neq j} \varepsilon_{ij}+|h|^2\right).$$

This pseudogradient satisfies the P.S. condition (since the maximum of the λ_i's remains bounded).

The proof of Theorem 6.4 is very similar to the proof of Theorem 5.1, so we will omit it here.

Now, we need to state the following lemma, which proof may be deduced from [11, pp. 354, 355].

Lemma 6.5. *There is a C^1-map which to each (α, a, λ) such that $\sum_{i=1}^{p} \alpha_i \tilde{\delta}_{(a_i, \lambda_i)}+\alpha_0 w$ belongs to $V(p, \varepsilon, w)$ associated $\bar{h}=\bar{h}(\alpha, a, \lambda)$ satisfying*

$$J\left(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_{(a_i, \lambda_i)}+\alpha_0 w+\bar{h}\right)=\max\left\{J\left(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_{(a_i, \lambda_i)}+\alpha_0(w+h)\right),\ h \in T_w W_{\mathrm{u}}(w)\right\}.$$

Proof of Theorem 6.2. Arguing as in [11, Appendix 2], we derive from Theorem 6.4 that for each $u=\sum_{i=1}^{p} \alpha_i \tilde{\delta}_{(a_i, \lambda_i)}+\alpha_0(w+h)$ belongs to $V(p, \varepsilon, w)$ we can find a change of variables $(a, \lambda, h) \rightarrow (\widetilde{a}, \widetilde{\lambda}, \widetilde{h})$ such that

$$J\left(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_{(a_i, \lambda_i)}+\alpha_0(w+h)+\bar{v}\right)=J\left(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_{(\widetilde{a}_i, \widetilde{\lambda}_i)}+\alpha_0(w+\widetilde{h})\right).$$

From Lemma 6.5, we deduce that there is a change of variables $h-\bar{h} \rightarrow H$ such that

$$J\left(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_{(a_i, \lambda_i)}+\alpha_0(w+h)\right)=J\left(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_{(a_i, \lambda_i)}+\alpha_0 w+\bar{h}\right)-|H|^2.$$

Arguing as in [11, Lemma 3.1], we obtain the following estimate:

$$|\bar{h}|=o\left(\sum_{i=1}^{p} \frac{1}{\lambda_i^2}\right).$$

Thus, by the same argument used to prove Theorem 6.1, we can find another change of variable $(a,\lambda)\to(a',\lambda')$ such that

$$J\left(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_{(a_i,\lambda_i)}+\alpha_0 w+\bar{h}\right)=J\left(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_{(a_i',\lambda_i')}+\alpha_0 w\right).$$

The proof of Theorem 6.2 is therefore completed. □

Corollary 6.6. *Let w be a nondegenerate solution of* (P_H). *Then,* $(y_{i_1},\dots,y_{i_p},w)$ *is not a critical point at infinity of J for each* $(y_{i_1},\dots,y_{i_p})$ *critical point of H such that* $M(y_{i_1},\dots,y_{i_p})$ *is definite positive.*

Proof. By Theorem 6.2, we have only compact flow lines in $V(p,\varepsilon,w)$. Since the variable λ_i's remains bounded, hence our result follows. □

7. Morse inequalities at infinity

This section is devoted to the proof of Theorem 1.1 and Corollaries 1.3 and 1.4. Using Theorem 2.1 we assume that the Euler–Lagrange functional has only nondegenerate critical points.

Proof of Theorem 1.1. Let $\mathcal{H}_\infty$ be the set of critical points at infinity of J and let L_0 their maximal Morse index. We define the following:

$$X:=\overline{\bigcup_{z\in\mathcal{H}_\infty}W_{\mathrm{u}}(z)},$$

where $W_{\mathrm{u}}(z)$ is the unstable manifold of the critical point at infinity z.

By a theorem of A. Bahri and P. Rabinowitz [14], we have that

$$X=\bigcup_{z\in\mathcal{H}_\infty}W_{\mathrm{u}}(z)\cup\bigcup_{\{y\text{ dominated by a point }z\in\mathcal{H}_\infty\}}W_{\mathrm{u}}(y).$$

Therefore, X is a stratified set of top dimension L_0, which is contractible in Σ^+. Let U be such a contraction which is a stratified set of top dimension L_0+1. U can also be deformed using the flow of $-J$. For dimension's reason, the stable manifold of any critical point of Morse index $\geqslant L_0+2$ can be avoided during such a deformation, see e.g. [44]. Therefore, U is deformed onto some set

$$Z:=\overline{\bigcup_{\mathrm{ind}(z)\leqslant L_0+1}W_{\mathrm{u}}(z)},$$

where z is a critical point or critical point at infinity and $m(z)$ denotes its Morse index. Again as above, it follows from a theorem of A. Bahri and P. Rabinowitz [14], that

$$Z=\bigcup_{z_\infty\in X}W_{\mathrm{u}}(z_\infty)\cup\bigcup_{\{x\text{ critical point dominated by }X\}}W_{\mathrm{u}}(x).$$

Recall that x is said to be dominated by X if $W_{\mathrm{u}}(x) \cap W_{\mathrm{s}}(X) \neq \emptyset$.

We prove now the following proposition.

Proposition 7.1. *Assuming that $H \in \mathcal{H}$ such that all the critical points of J having their Morse index $\leqslant L_0 + 1$ are nondegenerate, it holds*:

$$\#\{\textit{critical points of Morse index } \leqslant L_0 + 1\} \geqslant \left| 1 - \sum_{s=1}^{L_0} \sum_{\tau_s = (i_1, \ldots, i_s)/\rho(\tau_s) > 0} (-1)^{3s - 1 - \sum_{j=1}^{s} k_{i_j}} \right|.$$

Proof. Since X is contractible in Z, we have from the exact sequence in homology that

$$H_k(X) \to H_k(Z) \to H_k(Z, X) \to H_{k-1}(X) \to H_{k-1}(Z),$$

where $H_k(X) := H_k(X, \mathbb{Q})$ is the kth homology group with rational coefficients. Therefore, it follows that

$$\sum_{j=0}^{L_0+1} (-1)^j \big(\dim H_j(Z, X) + \dim H_j(X)\big) = 1,$$

which then implies

$$\sum_{j=0}^{L_0+1} \dim H_j(Z, X) \geqslant \left| 1 - \sum_{j=0}^{L_0} (-1)^j \dim H_j(X) \right|. \tag{7.1}$$

Now we claim the following lemma.

Lemma 7.2. *Under the assumption that all the critical points are nondegenerate, it holds*:

$$\sum_{j=1}^{L_0+1} \dim H_j(Z, X) \leqslant \#\{\textit{critical points of index } \leqslant L_0 + 1\}.$$

The critical points in the above estimate are those in Z but not in X.

Proof. The idea of the proof is that the pair (Z, X) is built by adding to X the unstable manifold of other critical points of index $\leqslant L_0 + 1$. Namely each time we add one of these unstable manifolds, starting from X and going with increasing index. At each step the new object we obtain has a total dimension of homology increased at most by one. Therefore the total homology of (Z, X) has its dimension upperbound by the number of critical points of index $\leqslant L_0 + 1$, not dominated by X. In the following we give a detailed proof of these facts.

First, we decompose Z as follows: $Z := X + X'$ where $X' := \bigcup W_{\mathrm{u}}(z')$ such that $W_{\mathrm{u}}(z_\infty) \cap W_{\mathrm{s}}(z') = \emptyset$, where z_∞ is any critical point at infinity of J. That is X' is the union of the unstable manifold of all the critical points in Z not dominated by X.

Taking the top index in X', we choose a corresponding critical point z'. Then we write $Z = X \cup X'' \cup W_{\mathrm{u}}(z')$, and observe that by excision we have

$$H_*(Z, X \cup X'') = H_*\big(W_{\mathrm{u}}(z'), \partial W_{\mathrm{u}}(z')\big) = H_*\big(D^{\operatorname{index} z'}, S^{\operatorname{index} z'-1}\big),$$

where $D^{\operatorname{index} z'}$ and $S^{\operatorname{index} z'-1}$ are respectively the ball and the sphere of dimension index z' and index $z' - 1$.

Now we claim the following.

Claim.

$$\dim H_*(Z, X) \leqslant \dim H_*(X \cup X'', X) + 1. \tag{7.2}$$

Proof. Indeed, we have

$$X \subset X \cup X'' \subset Z \quad \text{and} \quad H_*\big(W_{\mathrm{u}}(z'), \partial W_{\mathrm{u}}(z')\big) = H_*\big(S^{\operatorname{index} z'}\big).$$

On the another hand, by the exact sequence in homology we have

$$H_k(X \cup X'', X) \to H_k(Z, X) \to H_k(Z, X \cup X'') \to H_{k-1}(X \cup X'', X).$$

From our assumption that all the critical points are nondegenerate, we have that

$$\dim H_k(Z, X \cup X'') = \begin{cases} 0 & \text{if } k \neq \operatorname{index} z', \\ 1 & \text{if } k = \operatorname{index} z'. \end{cases}$$

It follows from the above sequence in homology that for $k \neq \operatorname{index} z'$, the homomorphism

$$H_k(X \cup X'', X) \to H_k(Z, X)$$

is onto, hence $\dim H_k(X \cup X'', X) \geqslant \dim H_k(Z, X)$, while for $k = \operatorname{index} z'$, we have that

$$\dim H_k(X \cup X'', X) \geqslant \dim H_k(Z, X) - 1.$$

Our claim follows. □

Now we set $X \subset Z' := X \cup X''$ and observe that $\overline{Z'} = Z'$ since we have removed from Z its top index.

The following lemma shows that we can actually continue our induction process.

Lemma 7.3. *Let $\tilde{X} := \overline{\bigcup W_{\mathrm{u}}(z)}$ be a union of unstable manifolds as above and z_1 be a critical point such that $W_{\mathrm{u}}(z_1)$ is a part of $\tilde{X}$ of top dimension. Then for $\tilde{X}_1 := \tilde{X} \setminus W_{\mathrm{u}}(z_1)$, it holds*

$$\chi(\tilde{X}) = \chi(\tilde{X}_1) + (-1)^{\dim W_{\mathrm{u}}(z_1)},$$

where χ denotes the Euler characteristic.

Proof. Notice that if the difference between the Morse index of z_1 and the Morse indices of the other critical points is bigger than 2, we may assume by transversality that $\tilde{X}_1 \cap \overline{W_{\mathrm{u}}(z_1)} = \emptyset$. It follows then that

$$\chi(\tilde{X}) = \chi(\tilde{X}_1) + (-1)^{\dim W_{\mathrm{u}}(z_1)}.$$

Otherwise observe that $W_{\mathrm{u}}(z_1)$ is diffeomorphic to an open disk, and that $\overline{W_{\mathrm{u}}(z_1)} \setminus W_{\mathrm{u}}(z_1) \subset \tilde{X}_1$. Taking a collar neighborhood of $\overline{W_{\mathrm{u}}(z_1)} \setminus W_{\mathrm{u}}(z_1)$ in $\overline{W_{\mathrm{u}}(z_1)}$ we denote it by V. We claim that

$$V \cap \overline{\tilde{X}_1} = \overline{W_{\mathrm{u}}(z_1)} \setminus W_{\mathrm{u}}(z_1).$$

Indeed, assuming that $V \cap \overline{\tilde{X}_1}$ is larger than $\overline{W_{\mathrm{u}}(z_1)} \setminus W_{\mathrm{u}}(z_1)$ we argue as follows: since $\tilde{X}_1$ is a union of unstable manifolds so is also $\overline{\tilde{X}_1}$ by [14, Proposition 7.24]. Therefore $V \cap \overline{\tilde{X}_1}$ will be due to one of these unstable manifolds of $\overline{\tilde{X}_1}$. That manifold is a manifold of $\tilde{X}$, therefore has dimension $\leqslant \dim W_{\mathrm{u}}(z_1)$. Furthermore it has to intersect $W_{\mathrm{u}}(z_1)$ not on his boundary, this forces it to be $W_{\mathrm{u}}(z_1)$ which is not a part of $\tilde{X}_1$ neither $\overline{\tilde{X}_1}$, a contradiction! Thus, $V \cap \overline{\tilde{X}_1} = \overline{W_{\mathrm{u}}(z_1)} \setminus W_{\mathrm{u}}(z_1)$ and $V \cup \tilde{X}_1$ retracts by deformation onto $\tilde{X}_1$.

We then consider the pair $(V \cup \tilde{X}_1, \overline{W_{\mathrm{u}}(z_1) \setminus V})$. Observe that the union is $\tilde{X}$, the intersection is a sphere of dimension $\dim W_{\mathrm{u}}(z_1) - 1$. Moreover, $\overline{W_{\mathrm{u}}(z_1) \setminus V}$ is a disk that we denote by D. It follows then that the pair $(V \cup \tilde{X}_1, \overline{W_{\mathrm{u}}(z_1) \setminus V})$ is an excessive pair, see for example Dold [29, p. 47]. Writing then the Mayer–Vietoris sequence of this pair we obtain

$$H_l\big(S^{\dim W_{\mathrm{u}}(z)-1}\big) \to H_l(V \cup \tilde{X}_1) \oplus H_l(D) \to H_l(\tilde{X}).$$

Therefore, it follows (see also Dold [29, pp. 104, 105]), that

$$\chi(V \cup \tilde{X}_1) + \chi(D) = \chi(\tilde{X}) + \chi\big(S^{\dim W_{\mathrm{u}}(z_1)-1}\big).$$

Hence

$$\chi(\tilde{X}_1) + (-1)^{\dim W_{\mathrm{u}}(z_1)} = \chi(\tilde{X}).$$

Therefore Lemma 7.3 is proved. □

Now it is clear that from Lemma 7.3 and the claim (7.2), it follows Lemma 7.2. □

Now Proposition 7.1 follows by formula (7.1) and Lemma 7.2. Indeed, it follows by (7.1) that

$$\left| 1 - \sum_{j=0}^{L_0} (-1)^j \dim H_j(X) \right| \leqslant \sum_{j=0}^{L_0+1} \dim H_j(Z, X).$$

Now observe that

$$\sum_{j=0}^{L_0} (-1)^{L_0-1} \dim H_j(X) = \chi(X) \quad \text{(the Euler characteristic of } X\text{)}.$$

Since by [14, Proposition 7.24],

$$X = \bigcup_{z_\infty} W_{\mathrm{u}}(z_\infty) \cup \bigcup_{\{y \text{ dominated by } z_\infty \in \mathcal{H}_\infty\}} W_{\mathrm{u}}(y),$$

it follows then that $\chi(X)$ is equal to

$$\sum_{k_i \text{ index of } z^i_\infty \in \mathcal{H}_\infty} (-1)^{k_i} + \sum_{m(i) \text{ index of } y^i \in X, \text{ critical point}} (-1)^{m(i)}.$$

Therefore,

$$\left| 1 - \sum_{k_i \text{ index of } z^i_\infty \in \mathcal{H}_\infty} (-1)^{k_i} \right| \leqslant \sum_{j=1}^{L_0+1} \dim H_j(Z, X) + \#\{\text{critical point in } X\}.$$

Using now the upper bound of Lemma 7.2 on the homology on the right-hand side of the above formula, our claim in Proposition 7.1 follows. □

Once Proposition 7.1 is proved, Theorem 1.1 as well as Corollaries 1.3 and 1.4 follow immediately. □

8. Proof of the existence results

In this section we prove Theorems 1.5, 1.6. For that purpose, we need the following lemma whose proof is very similar to the proof of [12, Corollary B.3] (see also [11]).

Lemma 8.1. *Let $a_1, a_2 \in \mathbb{S}^3$, $\alpha_1, \alpha_2 > 0$ and λ large enough. For $u = \alpha_1 \tilde{\delta}_{(a_1,\lambda)} + \alpha_2 \tilde{\delta}_{(a_2,\lambda)}$, we have*

$$J\left(\frac{u}{\|u\|}\right) \leqslant S_4^{1/3} \left(\frac{1}{H(a_1)^2} + \frac{1}{H(a_2)^2}\right)^{1/3} \bigl(1 + o(1)\bigr).$$

Proof of Theorem 1.5. Arguing by contradiction, we assume that J has no critical points in Σ^+. Let

$$c_\infty(y_0, y_1) = S_4^{\frac{1}{3}} \left(\frac{1}{H(y_0)^2} + \frac{1}{H(y_1)^2}\right)^{\frac{1}{3}}.$$

Observe that under the assumption (H$_2$), (y_0, y_1) is not a critical point at infinity of J. Using Corollary 6.3 and the assumption of the Theorem 1.5, it follows that the only critical points at infinity of J under the level $c_1 = c_\infty(y_0, y_1) + \varepsilon$, for ε small enough, correspond to $\tilde{\delta}_{(y_0,\infty)}$ and $\tilde{\delta}_{(y_1,\infty)}$.

Let Z be a pseudogradient of H of Morse–Smale type. Set

$$X = \overline{W}_{\mathrm{s}}(y_1) = W_{\mathrm{s}}(y_1) \cup W_{\mathrm{s}}(y_0),$$

where $W_s(y_i)$ is the stable manifold of y_i for Z. Then X is a compact manifold in dimension k_1 without boundary, where k_1 is the coindex of y_1 as a critical point of H, in particular X is not contractible.

The unstable manifolds at infinity for such critical points at infinity, $W_u(y_i)_\infty$, $i = 0, 1$, can be described as the product of $W_s(y_i)$ (for a decreasing pseudogradient of H) by $[A, \infty[$ domain of the variable λ, for some positive number A large enough. Since J has no critical points in Σ^+, it follows from [14, Proposition 7.24 and Theorem 8.2], that $J_{c_1} = \{u \in \Sigma^+ \mid J(u) \leqslant c_1\}$ retract by deformation onto $X_\infty = W_u(y_1)_\infty \cup W_u(y_0)_\infty$, which can be parameterized by $X \times [A, +\infty[$.

Now, we claim that X_∞ is contractible in J_{c_1}. Indeed, let

$$\begin{aligned} f : [0,1] \times X_\infty &\to \Sigma^+ \\ (t, x, \lambda) &\mapsto \frac{t\tilde{\delta}_{(y_0,\lambda)} + (1-t)\tilde{\delta}_{(x,\lambda)}}{\|t\tilde{\delta}_{(y_0,\lambda)} + (1-t)\tilde{\delta}_{(x,\lambda)}\|}, \end{aligned}$$

f is continuous and satisfies

$$f(0, x, \lambda) = \frac{1}{S_4^{1/3}}\tilde{\delta}_{(x,\lambda)} \quad \text{and} \quad f(1, x, \lambda) = \frac{1}{S_4^{1/3}}\tilde{\delta}_{(y_0,\lambda)}.$$

Furthermore, using Lemma 8.1, we deduce that

$$J\big(f(t,x,\lambda)\big) \leqslant \left(S\left(\frac{1}{H(y_0)^2} + \frac{1}{H(x)^2}\right)\right)^{\frac{1}{3}}\big(1 + o(1)\big).$$

Since $H(x) \geqslant H(y_1)$ for any $x \in X$, it follows from the above estimates that $J(f(t,x,\lambda)) < c_1$ for any $(t, x, \lambda) \in [0,1] \times X \times [A, \infty[$.

Thus, the contraction f is performed under the level c_1. We derive that X_∞ is contractible in J_{c_1}, which retracts by deformation on X_∞, therefore X_∞ is contractible leading to the contractibility of X which is a contradiction. Now, we are going to show that such a critical point has a Morse index equal to k_1 or $k_1 + 1$. Using a dimension argument and since $f([0,1], X_\infty)$ is a manifold in dimension $k_1 + 1$, we derive that the Morse index of such a critical point is $\leqslant k_1 + 1$. Now, arguing by contradiction, we assume that the Morse index is $\leqslant k_1 - 1$. Perturbing J if necessary, we may assume that all the critical points of J are nondegenerate and have their Morse index $\leqslant k_1 - 1$. Such critical points do not change the homological group in dimension k_1 of the level sets of J. Now let $c_\infty(y_1) = S_4^{1/3} H(y_1)^{-2/3}$ and let ε be a small positive real. Since X_∞ defines a homological class in dimension k_1 which is not trivial in J_{c_1} but trivial in $J_{c_\infty(y_1)+\varepsilon}$, a contradiction. The proof of Theorem 1.5 is thereby completed. □

Proof of Theorem 1.6. We notice that the assumption (H_4) implies that (y_0, y_1) is a critical point at infinity of J. Now, arguing by contradiction, we assume that (P_H) has no solution. We claim that $f_\lambda(C_{y_0}(X))$ retracts by deformation on $X \cup W_u(y_0, y_1)_\infty$. Indeed, let

$$u = \alpha\tilde{\delta}_{(y_0,\lambda)} + (1-\alpha)\tilde{\delta}_{(x,\lambda)} \in f_\lambda\big(C_{y_0}(X)\big),$$

the action of the flow of the pseudogradient W defined in the proof of Theorem 5.1, is essentially on α. Three cases may occur:

- If $\alpha < 1/2$, the flow of W brings α to zero and thus u goes to $\overline{W_{\mathrm{u}}(y_0)}_\infty \equiv \{y_0\}$.
- If $\alpha > 1/2$, the flow of W brings α to 1 and thus u goes to $\overline{W_{\mathrm{u}}(y_1)}_\infty \equiv X_\infty$.
- If $\alpha = 1/2$, observe that only x can move and then y_0 remains one of the points of concentration of u and x goes to $W_{\mathrm{s}}(y_i)$, where $i = 0$ or $i = 1$. If $y_i = y_1$, then u goes to $W_{\mathrm{u}}(y_0, y_1)_\infty$. If $y_i = y_0$, then there exists $s_0 \geqslant 0$ such that $x(s_0)$ is close to y_0. Thus, using Lemma 8.1, we have the following inequality

  $$J\big(u(s_0)\big) \leqslant c_\infty(y_0, y_0) + \gamma := c_2,$$

  where $c_\infty(y_0, y_0) = S_4^{\frac{1}{3}}(\frac{2}{H(y_0)^2})^{\frac{1}{3}}$ and where γ is a positive constant small enough. Since $H(y_0) \geqslant H(y_1)$, it follows from Corollary 6.3, that J_{c_2} retracts by deformation on $W_{\mathrm{u}}(y_0)_\infty \equiv \{y_0\}$ and thus u goes to $W_{\mathrm{u}}(y_0)_\infty$.

Therefore, $f_\lambda(C_{y_0}(X))$ retracts by deformation on $X \cup W_{\mathrm{u}}(y_0, y_1)_\infty$.

Since $\mu(y_i) = 0$, it follows that this strong retract does not intersect $W_{\mathrm{u}}(y_0, y_1)_\infty$ and thus it is contained in X_∞. This implies that $H_*(X_\infty) = 0$ for all $* \in \mathbb{N}^*$ leading to the contractibility of X. This yields a contradiction since X is a manifold of dimension k without boundary. Then (P_H) admits a solution. Using the same arguments as used in the proof of Theorem 1.5, we derive easily that the Morse index of the solution provided is equal to k_1 or $k_1 + 1$. Thus, our result follows. □

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