CONFORMAL METRICS WITH PRESCRIBED BOUNDARY MEAN CURVATURE ON BALLS

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ABSTRACT. We provide a variety of classes of functions which can be realized as the mean curvature on the boundary of the standard n dimensional ball, $n \geq 3$ with respect to some scalar flat metric. Because of the presence of some critical nonlinearity, blow up phenomena occur and existence results are highly nontrivial since one has to overcome topological obstructions. Our approach consists of, on one hand developping a Morse theoretical approach to this problem through a Morse type reduction of the associated Euler Lagrange functional in a neighborhood of its critical points at Infinity and on the other hand extending to this problem some topological invariants introduced by A. Bahri in his study of Yamabe type problems on closed manifolds.
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CONTENTS

1. INTRODUCTION

The celebrated Riemann Mapping Theorem states that any simply connected region in the plane is conformally diffeomorphic to a disk. This theorem cannot be generalized in higher dimensions where very few domains are conformally diffeomorphic to the ball. Nevertheless, one can still ask whether a domain is conformal to a manifold that resembles the ball in two ways: namely, it has zero scalar curvature and its boundary has a constant mean curvature. Escobar studied this problem in

[17]. He showed that most compact manifolds with boundary admit such conformally related metrics. Escobar's problem is equivalent to seeking a smooth positive solution u to the following nonlinear boundary value problem on n-dimensional Riemannian manifold with boundary (M, g), with $n \geq 3$:

$$(1.1) \qquad \begin{cases} -4\dfrac{n-1}{n-2}\Delta_g u + R_g u & = 0 \qquad \text{in } \overset{\circ}{M} \\ \dfrac{2}{n-2}\dfrac{\partial u}{\partial \nu} + H_g u & = c u^{\frac{n}{n-2}} \quad \text{on } \partial M \end{cases}$$

where, $\overset{\circ}{M} = M \setminus \partial M$ denotes the interior of M, R_g is the scalar curvature of M, H_g is the mean curvature of ∂M, ν is the outward unit vector with respect to the metric g, and c is a constant whose sign is uniquely determined by the conformal structure. Indeed, if $\bar{g} = u^{\frac{4}{n-2}} g$, then the metric $\bar{g}$ has zero scalar curvature and the boundary has a constant mean curvature with respect to $\bar{g}$.

In view of the above equation, it is natural to consider the problem of prescribing boundary mean curvature with zero scalar curvature, that is: given a function $H : \partial M \longrightarrow \mathbb{R}$, does there exists a metric g' conformally equivalent to g such that $R_{g'} \equiv 0$ and $H_{g'} \equiv H$? From equation (1.1), the problem is equivalent to finding a smooth positive solution v to the following equation,

$$(1.2) \qquad \begin{cases} -4\dfrac{n-1}{n-2}\Delta_g v + R_g v & = 0 \qquad \text{in } \overset{\circ}{M} \\ \dfrac{2}{n-2}\dfrac{\partial v}{\partial \nu} + H_g v & = H v^{\frac{n}{n-2}} \quad \text{on } \partial M. \end{cases}$$

In this article, we are interested in the case where a non compact group of conformal transformations acts on the equation so that Kazdan-Warner type conditions give rise to obstructions as in the Nirenberg problem (see [27]). The simplest situation is the following one:
Let $\mathbb{B}^n$ be the unit ball in $\mathbb{R}^n$ with Euclidean metric g_0. Its boundary will be denoted by $\partial \mathbb{B}^n = \mathbb{S}^{n-1}$ and will be endowed by the standard metric g_0. Let H be a function on $\mathbb{S}^{n-1}$. In this case, our problem becomes

$$(1.3) \qquad \begin{cases} \Delta u = 0 & \text{and } u > 0 \text{ in } \mathbb{B}^n \\ \dfrac{\partial u}{\partial \nu} + \dfrac{n-2}{2} u = \dfrac{n-2}{2} H u^{\frac{n}{n-2}} & \text{on } \mathbb{S}^{n-1}. \end{cases}$$

Our aim, is to give sufficient conditions on H such that problem (1.3) admits a positive solution. It is easy to see that a necessary condition for solving the problem is that H has to be positive somewhere.
Previously, P. Cherrier [12] studied the regularity question for this equation. He showed that solution of (1.3) which are of class H^1 are also smooth. In [19], Escobar has studied this problem (1.3) on manifolds which are not equivalent to the standard ball. On the ball, sufficient conditions on H in dimensions 3 and 4 are given in [20], and [15], and a perturbative results were obtained in [11]. In [2], the authors of this paper developped a Morse theoretical approach to this problem in the 4-dimensional case providing some multiplicity results under generic conditions on the function H.

Related problems regarding conformal deformations of Riemannian metrics on manifolds with boundary have been studied in [3, 9, 10, 13, 14, 18, 21, 22, 23, 25, 24, 26, 27, 29, 31] and the references therein.

Assume that H is a smooth function having only non degenerate critical points $y_0, y_1, \dots, y_s$ with $\Delta H(y_i) \neq 0$ for each $i = 1, \dots, s$. Let

$$I = \{y_i / \ \nabla H(y_i) = 0 \ \text{ and } \Delta H(y_i) < 0\}.$$

In the first part of this paper, we extend to our problem some topological invariants introduced by A. Bahri [5] in the framework of Yamabe-type problem on closed manifolds, and we use it in order to give partial answer to the boundary Mean curvature problem in dimension $n \geq 7$.

Let Z be a pseudo-gradient of H of Morse-Smale type, that is satisfying that the intersection of the unstable and the stable manifolds of the critical points of H are transverse. We assume that,
$(\mathbf{R_0})$ Assume that $W_s(y_i) \cap W_u(y_j) = \emptyset$ for each critical points y_i, and y_j such that $y_i \in I$ and $y_j \notin I$.
Here $W_s(y_i)$ is the stable manifolds of y_i defined by the set of points $x \in \mathbb{S}^{n-1}$ attracted by y_i through the decreasing flow $\eta(.,x)$ of Z; $W_s(y_i) = \{x \in \mathbb{S}^{n-1} / \eta(s,x) \rightarrow y_i$ when $s \rightarrow +\infty\}$ and $W_u(y_i) = \{x \in \mathbb{S}^{n-1} / \eta(s,x) \rightarrow y_i$ when $s \rightarrow -\infty\}$ denotes the unstable manifolds of y_i. To proceed further we introduce some notations. For y_{i_0} a critical point of H such that $\Delta H(y_{i_0}) < 0$, let

$$X = \overline{W_s(y_{i_0})}.$$

We assume that
$(\mathbf{R_1})$ X is a compact manifold without boundary of dimension $k \in \mathbb{N}^*$.
Let us define

$$B_2(X) = \{\alpha_1 \delta_{x_1} + \alpha_2 \delta_{x_2} / \ \alpha_1 + \alpha_2 = 1 \text{ and } \ x_i \in X\},$$

where δ_x denote the Dirac measure at x. For λ large enough, we introduce the map

$$\begin{aligned} f_\lambda : \ B_2(X) &\longrightarrow \Sigma^+ \\ \alpha_1 \delta_{x_1} + \alpha_2 \delta_{x_2} &\longmapsto \frac{\alpha_1 \tilde{\delta}_{(x_1,\lambda)} + \alpha_2 \tilde{\delta}_{(x_2,\lambda)}}{\left\| \alpha_1 \tilde{\delta}_{(x_1,\lambda)} + \alpha_2 \tilde{\delta}_{(x_2,\lambda)} \right\|}, \end{aligned}$$

where, $\tilde{\delta}_{(a,\lambda)}$ will be defined in section 2 by (2.1). Observe that $B_2(X)$ and $f_\lambda(B_2(X))$ are manifolds in dimension $2k+1$, in the sense that their singularities arise in dimension $2k-1$ and lower, (see [5]).
Let ν^+ be a tubular neighborhood of X in $\mathbb{S}^{n-1}$. We denote by $\nu^+(y)$, the fibre at $y \in X$ of this tubular neighborhood.
For $\varepsilon > 0$, $z_1, z_2 \in X$ such that $z_1 \neq z_2$ and $-\Delta H(z_i) > 0$ for $i = 1, 2$, we introduce the following set

$$\Gamma_{\varepsilon_1} = \Bigg\{ \sum_{i=1}^{2} \frac{\tilde{\delta}_{(z_i+h_i,\lambda_i)}}{H\Big(z_i+h_i\Big)^{\frac{n-2}{2}}} + v/ \ v \in H^1(\mathbb{S}^{n-1}) \text{ satisfying } (V_0),$$

$$\|v-\bar{v}\| < \varepsilon_1,\ |h_1|^2 + |h_2|^2 < \varepsilon_1,\ \lambda_i > \varepsilon_1^{-1} \text{ and } h_i \in \nu^+(z_i) \text{ for } i = 1,2 \Bigg\},$$

where $\bar{v}$ is defined in Proposition 3.6 (see below) and (V_0) is defined in the next section.
For δ a small positive real, the boundary of Γ_{ε_1} (defined by $\|v-\bar{v}\| = \varepsilon_1$ or $\lambda_i = \varepsilon_1^{-1}$ or $|h_1|^2 + |h_2|^2 = \varepsilon_1$), does not intersect $J^{-1}\Big(c_\infty(z_1,z_2)+\delta\Big)$, where J is the Euler-Lagrange functional associated to the problem (1.3) and

$$c_\infty(z_1,z_2) = \left(S_n \sum_{i=1}^{2} \frac{1}{H(z_i)^{n-2}} \right)^{\frac{1}{n-1}}.$$

We then set

$$C_\delta = \Gamma_{\varepsilon_1} \cap J^{-1}\Big(c_\infty(z_1,z_2)+\delta\Big).$$

Observe that for ε_1 and δ small enough, $C_\delta(z_1,z_2)$ is a closed fredholm noncompact manifold without boundary of codimension $2k+2$.
For λ large enough, we define the intersection number (modulo 2) of $W_u\Big(f_\lambda(B_2(X))\Big)$ with C_δ denoted by

$$\tau = W_u\Big(f_\lambda(B_2(X))\Big).C_\delta,$$

where $W_u\Big(f_\lambda(B_2(X))\Big)$ is the unstable manifolds of $f_\lambda(B_2(X))$ for a decreasing pseudo-gradient V for J which is transverse to $f_\lambda(B_2(X))$. By transversality arguments, this number τ is well defined (see [30]).
We then have

Theorem 1.1. *Let $n \geq 7$. Under the assumptions $(\mathbf{R_0})$ and $(\mathbf{R_1})$, if $\tau = 1$, then (1.3) has a solution.*

Besides Theorem 1.1, we have the following result based on another topological invariant denoted by μ. The relation between μ and τ will be explored in paper to come (we will also derive a new proof of the existence of solutions of some Yamabe-type problems on manifolds with boundary based on a deeper study of this invariant).
Let, $C_{y_0}(X)$ the following set

$$C_{y_0}(X) = \{\alpha\delta_{y_0} + (1-\alpha)\delta_x /\ \alpha \in [0,1],\ \ x \in X\}.$$

where δ_x is the Dirac measure at x. For λ large enough, we introduce a map

$$f_\lambda : C_{y_0}(X) \to \Sigma^+$$
$$\alpha\delta_{y_0} + (1-\alpha)\delta_x \to \frac{\alpha\tilde{\tilde{\delta}}_{(y_0,\lambda)} + (1-\alpha)\tilde{\tilde{\delta}}_{(x,\lambda)}}{\|\alpha\tilde{\tilde{\delta}}_{(y_0,\lambda)} + (1-\alpha)\tilde{\tilde{\delta}}_{(x,\lambda)}\|},$$

For λ large enough, we define the intersection number (modulo 2) of $f_\lambda(C_{y_0}(X))$ with $W_s(y_0, y_{i_0})_\infty$

$$\mu = f_\lambda(C_{y_0}(X)) \cdot W_s(y_0, y_{i_0})_\infty,$$

where $W_s(y_0, y_{i_0})_\infty$ is the stable manifold of the critical point at infinity $(y_0, y_{i_0})_\infty$ (see Proposition 4.2 below) for a decreasing pseudo-gradient V for the Euler-Lagrange functional associated to (1.3) which is transverse to $f_\lambda(C_{y_0}(X))$. This number is well defined (see [30]).
We then have the following result,

Theorem 1.2. *Let $n \geq 5$. Under the assumptions* $(\mathbf{R_0})$ *and* $(\mathbf{R_1})$*, if $\mu = 0$, then problem (1.3) has a solution.*

Remark 1.3. To see how to construct an example of functions H satisfying the assumptions of Theorem 1.1 and Theorem 1.2, we refer the reader to Theorem 1 and Theorem 2 of [5] (page 344-346).

Remark 1.4. The choice of X, to be built out of the stable manifold of critical points of H is useful only when we combine Theorems 1.1 and 1.2. Such a construction is necessary for Theorem 1.1 but for Theorem 1.2, X could be general stratified set.

In the second part of this paper, we single out the three dimensional case. Without lose of generality, we can assume that

$$H(y_0) \geq H(y_1) \geq \cdots \geq H(y_s).$$

We then have the following result

Theorem 1.5. *Let $n = 3$. If*
$(\mathbf{H_1})$ $y_1 \in I$.
$(\mathbf{H_2})$

$$\frac{1}{H(y)} > \frac{1}{H(y_0)} + \frac{1}{H(y_1)} \quad \forall y \in I \setminus \{y_0, y_1\},$$

then (1.3) has a solution of Morse index k or $k+1$, where $k = 2 - ind(H, y_1)$.

Beside the above result, we point out that our method enables us to reprove a multiplicity result obtained by Escobar-Garcia in [20] using blow analysis à la Schoen et Yanyan Li. Our approach is completely different from the one used in the above mentioned paper and relies on a suitable use of the method of critical points at infinity of Bahri [4]. Namely, we have

Theorem 1.6. *On $\mathbb{B}^3$, under the assumption that all the solutions of (1.3) are non degenerate, the number of solutions of (1.3) is lower bounded by*

$$\left| 1 - \sum_{y_j \in I} (-1)^{2 - ind(H, y_j)} \right|.$$

We notice that the last statement of Theorem 1.6 regarding the number of solutions can be seen some sort of *Morse Inequalities at Infinity* giving a lower bound to the number of solutions of the problem (1.3) in terms of the topology induced by the critical point at Infinity.

Corollary 1.7. *On $\mathbb{B}^3$, under the the assumption that all the solutions of (1.3) are non degenerate, if*

$$\sum_{y_j \in I} (-1)^{2-ind(H,y_j)} \neq 1$$

then (1.3) has a solution.

We organize the remainder of our paper as follows. In section 2, we recall some preliminaries. In section 3, we perform an expansion of the Euler functional associated to (1.3) and its gradient near critical points at infinity. Section 4 is devoted to the Prescription of the mean curvature on higher dimensional ball. In section 5 we give the characterization of the critical points at infinity in the case of three dimensional ball and lastly in section 6, we give the proofs of Theorems 1.5 and 1.6.

2. Preliminaries

In this section, we recall the functional setting and the variational problem and its main features. Problem (1.3) occurs as the Euler equation of the variational functional

$$J(u) = \left(\int_{\mathbb{S}^{n-1}} H u^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{\frac{2-n}{n-1}}$$

defined on $H^1(\mathbb{B}^n)$ equipped with the norm

$$\|u\|^2 = \int_{\mathbb{B}^n} |\nabla u|^2 dv_{g_0} + \frac{n-2}{2} \int_{\mathbb{S}^{n-1}} u^2 d\sigma_{g_0}$$

where dv_{g_0} and $d\sigma_{g_0}$ denote the Riemannian measure on $\mathbb{B}^n$ and $\mathbb{S}^{n-1}$ induced by the metric g_0. We denote by Σ the unit sphere of $H^1(\mathbb{B}^n)$ and we set,
$\Sigma^+ = \{u \in \Sigma / \, u \geq 0\}$.
The exponent $\frac{2(n-1)}{n-2}$ is critical for the Sobolev trace embedding $H^1(\mathbb{B}^n) \to L^q(\mathbb{S}^{n-1})$. This embedding being not compact, the functional J does not satisfy the Palais-Smale condition. For this reason standard variational methods cannot be applied to find critical points of J.

In order to characterize the sequences failing the Palais-Smale condition, we need to introduce some notations.
We will use the notation x for the variables belonging to the unit ball $\mathbb{B}^n$ or to the half space $\mathbb{R}^n_+$ defined by $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n, \; x_n > 0\}$. We will also use the notation $x = (x', x_n)$ for $x \in \mathbb{R}^n_+$.
It will be convenient to perform some stereographic projection in order to reduce the above problem to $\mathbb{R}^n_+$. Let $D^{1,2}(\mathbb{R}^n_+)$ denote the completion of $C^\infty_c(\overline{\mathbb{R}^n_+})$, with respect to the Dirichlet norm. The stereographic projection π_q through an appropriate point $q \in \mathbb{S}^{n-1}$ induces an isometry $i : \, H^1(\mathbb{B}^n) \longrightarrow D^{1,2}(\mathbb{R}^n_+)$ according to the following formula

$$iu(x)=\Big(\frac{2}{|x'|^2+(x_n+1)^2}\Big)^{\frac{n-2}{2}}u\Big(\frac{2x'}{|x'|^2+(x_n+1)^2},\frac{|x'|^2+x_n-1}{|x'|^2+(x_n+1)^2}\Big),$$

where $x'=(x_1,\dots,x_{n-1})$. In particular, we can check that the following relations holds true for every $u\in H^1(\mathbb{B}^n)$,

$$\int_{\mathbb{B}^n}|\nabla u|^2+\frac{n-2}{2}\int_{\mathbb{S}^{n-1}}u^2=\int_{\mathbb{R}^n_+}|\nabla iu|^2 \qquad \text{and}$$
$$\int_{\mathbb{S}^{n-1}}|u|^{2\frac{(n-1)}{n-2}}=\int_{\partial\mathbb{R}^n_+}|iu|^{2\frac{(n-1)}{n-2}}.$$

In the sequel, we will identify the function H and its composition with the stereographic projection π_q. We will also identify a point x of B^n and its image by π_q. These facts will be assumed as understood in the sequel.
For $a\in\partial\mathbb{R}^n_+$ and $\lambda>0$, we define the function:

$$\delta_{a,\lambda}(x)=\bar{c}\frac{\lambda^{\frac{n-2}{2}}}{\Bigg((1+\lambda x_n)^2+\lambda^2|x'-a'|^2\Bigg)^{\frac{n-2}{2}}}$$

where $x\in\mathbb{R}^n_+$, and $\bar{c}$ is chosen such that $\delta_{a,\lambda}$ satisfies the following equation,

$$\begin{cases}\Delta u &=0 \quad \text{and } u>0 \text{ in } \mathbb{R}^n_+\\ -\dfrac{\partial u}{\partial x_n} &=u^{\frac{n}{n-2}} \qquad\qquad \text{on } \partial\mathbb{R}^n_+.\end{cases}$$

Set,

$$\tilde{\delta}_{a,\lambda}=i^{-1}(\delta_{a,\lambda}). \tag{2.1}$$

For $\varepsilon>0$, $p\in\mathbb{N}^*$ and w either a solution of (1.3) or zero, we define

$$V(p,\varepsilon,w)=\begin{cases}u\in\Sigma \text{ s. t } \exists a_1,\dots,a_p\in\mathbb{S}^{n-1},\exists\alpha_0,\alpha_1,\dots,\alpha_p>0 \text{ and}\\ \exists\lambda_1,\dots,\lambda_p>\varepsilon^{-1} \text{ with } \Big\|u-\alpha_0w-\displaystyle\sum_{i=1}^p\alpha_i\tilde{\delta}_{a_i,\lambda_i}\Big\|<\varepsilon,\ \ \varepsilon_{ij}<\varepsilon\ \forall\, i\neq j,\\ \text{and } \Big|J(u)^{\frac{n-1}{n-2}}\alpha_i^{\frac{2}{n-2}}H(a_i)-1\Big|<\varepsilon\ \ \forall i,j=1,\dots,p, \text{ and } |\alpha_0J(u)^{\frac{n-1}{2}}-1|<\varepsilon\end{cases}$$

where,

$$\varepsilon_{ij}=\Bigg(\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j|a_i-a_j|^2\Bigg)^{\frac{2-n}{2}}.$$

If u is function in $V(p,\varepsilon,w)$, one can find an optimal representation, following the ideas introduced in [5] and [6], namely we have

Lemma 2.1. *(see* [5], [6]*) For any $p\in\mathbb{N}^*$, there is $\varepsilon_p>0$ such that if $\varepsilon\le\varepsilon_p$ and $u\in V(p,\varepsilon,w)$, then the following minimization problem*

$$\min\Bigg\{\Big\|u-\sum_{i=1}^p\alpha_i\tilde{\delta}_{(a_i,\lambda_i)}-\alpha_0(w+h)\Big\|,\ \alpha_i>0,\lambda_i>0,a_i\in\mathbb{S}^{n-1},h\in T_w(W_u(w))\Bigg\}$$

has a unique solution $(\bar\alpha, \bar\lambda, \bar a, \bar h)$*. Thus, we can write* u *as follows:*

$$u = \sum_{i=1}^{p} \alpha_i \tilde\delta_{(a_i,\lambda_i)} + \alpha_0(w+h) + v$$

where v *belongs to* $H^1(\mathbb{B}^n) \cap T_w(W_s(w))$ *and satisfies* (V_0)*, where* $T_w(W_u(w))$ *and* $T_w(W_s(w))$ *are the tangent spaces at* w *of the unstable and the stable manifolds of* w*, and* (V_0) *is the following condition:*

$$(V_0): \begin{cases} < v, \varphi_i > & = 0 \quad \text{for } i = 1, \dots, p, \text{ and } \varphi_i = \tilde\delta_i,\ \partial\tilde\delta_i/\partial\lambda_i,\ \partial\tilde\delta_i/\partial a_i, \\ < v, w > & = 0 \\ < v, h > & = 0 \quad \text{for all } h \in T_w(W_u(w)). \end{cases}$$

where, $\tilde\delta_i = \tilde\delta_{a_i,\lambda_i}$ *and* $< ., . >$ *denote the scalar product defined on* $H^1(\mathbb{B}^n)$ *by,*

$$< u, v >= \int_{\mathbb{B}^n} \nabla u \nabla v \, dv_{g_0} + \frac{n-2}{2} \int_{\mathbb{S}^{n-1}} uv \, d\sigma_{g_0}.$$

Notice that Lemma (2.1) is also true if we take $w = 0$ and therefore $h = 0$.
We are ready now to state the characterization of the Palais-Smale sequence failing the P. S. condition, taking on to account the uniqueness result of Li-Zhu [28] and the idea introduced in [5], as follows.

Proposition 2.2. *Let* $(u_k) \subset \Sigma^+$ *be a sequence satisfying* $J(u_k) \to c$*, a positive number and* $\partial J(u_k) \to 0$*. Then, there exist an integer* $p \geq 1$*, a positive sequence* $(\varepsilon_k)_k$ $(\varepsilon_k \to 0)$ *and an extracted subsequence of* (u_k)*, again denoted* u_k *such that* $u_k \in V(p, \varepsilon_k, w)$*, where* w *is either zero or a solution of* (1.3)*.*

Following A. Bahri [4], [5] we set the following definitions and notations

Definition 2.3. *A critical point at infinity* of J on Σ^+ is a limit of a flow line $u(s)$ of the equation:

$$\begin{cases} \frac{\partial u}{\partial s} = -J(u) \\ u(0) = u_0 \end{cases}$$

such that $u(s)$ remains in $V(p, \varepsilon(s), w)$ for $s \geq s_0$.
Here w is either zero or a solution of (1.3) and $\varepsilon(s)$ is some function tending to zero when $s \to \infty$. Using Lemma 2.1, $u(s)$ can be written as:

$$u(s) \ = \sum_{i=1}^{p} \alpha_{i(s)} \, \delta_{(a_i(s), \lambda_i(s))} + \alpha_0(s)(w + h(s)) \ + v(s).$$

Denoting $a_i := \lim_{s\to\infty} a_i(s)$ and $\alpha_i = \lim_{s\to\infty} \alpha_i(s)$, we denote by

$$(a_1, \cdots, a_p, w)_\infty \text{ or } \sum_{i=1}^{p} \alpha_i \, \delta_{(a_i,\infty)} + \alpha_0 \, w$$

such a critical point at infinity. If $w \neq 0$ it is called of *w-type.*

Like a usual critical point, it is associated to a *critical point at infinity* x_∞ of the problem (P_K), which are combination of classical critical points with a $1-$dimensional assymptote, stable and unstable manifolds, $W_s^\infty(x_\infty)$ and $W_u^\infty(x_\infty)$. These manifolds can be easily described once a Morse type reduction is performed, see [5], [8]. The stable amnifold is, as usual, defined to be the set of points attracted by the

asymptote. The unstable one is a shadow object, which is the limit of $W_u(x_\lambda)$, x_λ being the critical point of the reduced problem and $W_u(x_\lambda)$ its associated unstable manifolds. Indeed the flow in this case splits the variable λ from the other variables near x_∞.
In the following defiition, we extend the notation of domination of critical points to critical points at Infinity.

Definition 2.4. z_∞ is said to be dominated by another critical point at infinity z'_∞ if

$$W_u(z'_\infty) \cap W_s(z_\infty) \neq \emptyset.$$

If we assume that the intersection is transverse, then we obtain

$$index(z'_\infty) \geq index(z_\infty) + 1.$$

3. Expansion of the Euler functional

In this section, we will give a useful expansion of the functional J and its gradient near a critical point at infinity. Then, we deal with the v-part of u. In order to simplify the notations, in the remainder we write $\tilde{\delta}_i$ instead of $\tilde{\delta}_{(a_i,\lambda_i)}$.

3.1. **The functional and its gradient.** First we deal with the functional J associated to problem (1.3).

Proposition 3.1. *For $\varepsilon > 0$ small enough and $u = \sum_{i=1}^p \alpha_i \tilde{\delta}_{(a_i,\lambda_i)} + \alpha_0(w+h) + v \in V(p,\varepsilon,w)$, we have the following expansion*

$$J(u) = \frac{S_n \Sigma_{i=1}^p \alpha_i^2 + \alpha_0^2 \|w\|^2}{\left(S_n \Sigma_{i=1}^p \alpha_i^{2\frac{(n-1)}{n-2}} H(a_i) + \alpha_0^{2\frac{(n-1)}{n-2}} \|w\|^2\right)^{\frac{n-2}{n-1}}} \Bigg[1 - \frac{(n-2)}{(n-1)\beta_1} \times$$
$$\sum_{i=1}^p \alpha_i^{2\frac{(n-1)}{n-2}} \begin{cases} \dfrac{c_3 \Delta H(a_i)}{\lambda_i^2} \log \lambda_i & \text{if } n = 3 \\ \dfrac{c_4 \Delta H(a_i)}{\lambda_i^2} & \text{if } n \geq 4 \end{cases} - \frac{c_1}{\gamma} \sum_{i \neq j \geq 1} \alpha_i \alpha_j \varepsilon_{ij}$$
$$- 2\frac{c_1 \alpha_0}{\gamma} \sum_{i=1}^p \alpha_i \frac{w(a_i)}{\lambda_i^{\frac{n-2}{2}}} + \frac{1}{\gamma}\Big(Q_1(v,v) - f_1(v)\Big) + \frac{\alpha_0^2}{\gamma}\Big(Q_2(h,h) + f_2(h)\Big)$$
$$+ o\left(\sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^p \frac{\log \lambda_i}{\lambda_i^2}\right) + O\left(\|v\|^{\inf(3, 2\frac{(n-1)}{n-2})} + \|h\|^{\inf(3, 2\frac{(n-1)}{n-2})}\right)\Bigg],$$

where,

$$S_n = \bar{c}^{\frac{2(n-1)}{n-2}} \int_{\mathbb{R}^{n-1}} \frac{dx}{\left(1+|x|^2\right)^{n-1}}, \qquad c_1 = \bar{c}^{\frac{2(n-1)}{n-2}} \int_{\mathbb{R}^{n-1}} \frac{dx}{\left(1+|x|^2\right)^{\frac{n}{2}}}$$

$$c_3 = \bar{c}^4 \frac{\pi}{2} \qquad c_4 = \frac{\bar{c}^{\frac{2(n-1)}{n-2}}}{2(n-1)} \int_{\mathbb{R}^{n-1}} \frac{|x|^2 dx}{\left(1+|x|^2\right)^{n-1}}$$

a positive constants independents of u and where,

$$Q_1(v,v) = \|v\|^2 - \frac{n\gamma}{(n-2)\beta_1}\left(\sum_{i=1}^{p}\int_{\mathbb{S}^{n-1}} H\left(\alpha_i\tilde{\delta}_i\right)^{\frac{2}{n-2}} v^2 + \int_{\mathbb{S}^{n-1}} H\left(\alpha_0 w\right)^{\frac{2}{n-2}} v^2\right),$$

$$Q_2(h,h) = \|h\|^2 - \frac{n\gamma}{(n-2)\beta_1}\int_{\mathbb{S}^{n-1}} H\left(\alpha_0 w\right)^{\frac{2}{n-2}} h^2,$$

$$f_1(v) = \frac{2\gamma}{\beta_1}\int_{\mathbb{S}^{n-1}} H\left(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i\right)^{\frac{n}{n-2}} v,$$

$$f_2(h) = \frac{2}{\alpha_0}\sum_{i=1}^{p}\alpha_i < \tilde{\delta}_i, h > -\frac{2\gamma}{\alpha_0\beta_1}\int_{\mathbb{S}^{n-1}} H\left(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i\right)^{\frac{n}{n-2}} h,$$

$$\beta_1 = S_n\left(\sum_{i=1}^{p}\alpha_i^{\frac{2(n-1)}{n-2}} H(a_i)\right) + \alpha_0^{2\frac{(n-1)}{n-2}}\|w\|^2, \ \textit{and}\ \gamma = S_n\left(\sum_{i=1}^{p}\alpha_i^2\right) + \alpha_0^2\|w\|^2.$$

Remark 3.2. Regarding the expansion of J, we observe that for $n \geq 7$, any configuration containing a solution w of (1.3) and a collection of critical points y_{j_i} having $-\Delta H(y_{j_i}) > 0$, gives rise to a critical point at infinity of J. This is not true for $n \leq 5$. In dimension 6, we have a balance phenomenon; that is, the self-interaction of the functions failing the Palais-Smale condition and the interaction of one of those functions with the solution w are of the same size.

Proof. Let us recall that

$$J(u) = \frac{\displaystyle\int_{\mathbb{B}^n} |\nabla u|^2 dv_g + \frac{(n-2)}{2}\int_{\mathbb{S}^{n-1}} u^2 d\sigma_g}{\left(\displaystyle\int_{\mathbb{S}^{n-1}} H u^{\frac{2(n-1)}{n-2}} d\sigma_g\right)^{\frac{n-2}{n-1}}}.$$

We need to estimate

$$N(u) = \|u\|^2 = \left\|\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i + v\right\|^2 \text{ and } D^{\frac{n-1}{n-2}} = \int_{\mathbb{S}^{n-1}} H(x) u^{2\frac{n-1}{n-2}}.$$

Using the fact that v satisfies (V_0), We derive,

$$N(u) = \sum_{i=1}^{p}\alpha_i^2\|\tilde{\delta}_i\|^2 + \alpha_0^2\|w\|^2 + \sum_{i\neq j}\alpha_i\alpha_j < \tilde{\delta}_i, \tilde{\delta}_j >$$
$$+ 2\sum_{i=1}^{p}\alpha_i\alpha_0 < \tilde{\delta}_i, w + h > + \|v\|^2 + \alpha_0^2\|h\|^2.$$

An easy computation shows that

$$\|\tilde{\delta}_i\|^2 = \|\delta_i\|^2 = S_n = \bar{c}^{\frac{2(n-1)}{n-2}}\int_{\mathbb{R}^{n-1}}\frac{dx}{\left(1+|x|^2\right)^{n-1}} \tag{3.1}$$

Using Lemma 3.3 below, we derive that

$$
\begin{aligned}
N(u) &= \sum_{i=1}^{p} \alpha_i^2 S_n + \alpha_0^2 \|w\|^2 + c_1 \sum_{i\neq j} \alpha_i\alpha_j \varepsilon_{ij} + 2c_1 \sum_{i=1}^{p} \alpha_i \alpha_0 \frac{w(a_i)}{\lambda_i^{\frac{n-2}{2}}} \\
&+ \alpha_0^2 \|h\|^2 + \|v\|^2 + \sum_{i=1}^{p} \alpha_i\alpha_0 < \tilde{\delta}_i, h > + o\Bigg(\sum_{i\neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2/2}} \Bigg).
\end{aligned} \tag{3.2}
$$

For the denominator, we have

$$
\int_{\mathbb{S}^{n-1}} H\Bigg(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_i + \alpha_0 (w+h) + v \Bigg)^{2\frac{(n-1)}{n-2}} = \int_{\mathbb{S}^{n-1}} H\Bigg(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \Bigg)^{2\frac{(n-1)}{n-2}} \tag{3.3}
$$

$$
\begin{aligned}
&+ \alpha_0^{2\frac{(n-1)}{n-2}} \int_{\mathbb{S}^{n-1}} H\Bigg(w+h \Bigg)^{2\frac{(n-1)}{n-2}} + 2\frac{(n-1)\alpha_0}{n-2} \int_{\mathbb{S}^{n-1}} H\Bigg(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \Bigg)^{\frac{n}{n-2}} (w+h) \\
&+ 2\frac{(n-1)}{n-2} \int_{\mathbb{S}^{n-1}} H\Bigg(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_i + \alpha_0(w+h) \Bigg)^{\frac{n}{n-2}} v + n\frac{(n-1)}{(n-2)^2} \int_{\mathbb{S}^{n-1}} H\Bigg(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \\
&+ \alpha_0(w+h) \Bigg)^{\frac{2}{n-2}} v^2 + 2\frac{(n-1)}{n-2} \int_{\mathbb{S}^{n-1}} H\Bigg(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \Bigg) \Bigg(\alpha_0 (w+h) \Bigg)^{\frac{n}{n-2}} \\
&+ O\Bigg(\int_{\mathbb{S}^{n-1}} \Bigg(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \Bigg)^{\frac{2}{n-2}} . \inf\big((\Sigma \alpha_i \tilde{\delta}_i), w+h \big)^2 \Bigg) + O\Big(\|v\|^{\inf(3, \frac{2(n-1)}{n-2})} \Big).
\end{aligned}
$$

We also have,

$$
\begin{aligned}
\int_{\mathbb{S}^{n-1}} H\Bigg(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \Bigg)^{2\frac{(n-1)}{n-2}} &= \sum_{i=1}^{p} \alpha_i^{2\frac{(n-1)}{n-2}} \int_{\mathbb{S}^{n-1}} H \, \tilde{\delta}_i^{2\frac{(n-1)}{n-2}} \\
+ 2\frac{(n-1)}{n-2} \sum_{i\neq j} \alpha_i^{\frac{n}{n-2}} \alpha_j \int_{\mathbb{S}^{n-1}} H \tilde{\delta}_i^{\frac{n}{n-2}} \tilde{\delta}_j &+ O\Bigg(\sum_{i\neq j} \int_{\mathbb{S}^{n-1}} \tilde{\delta}_i^{\frac{2}{n-2}} \inf(\tilde{\delta}_i \tilde{\delta}_j)^2 \Bigg)
\end{aligned} \tag{3.4}
$$

A computation similar to the one performed in [4] (page 4) shows that,

$$
\int_{\mathbb{S}^{n-1}} \tilde{\delta}_i^{\frac{2}{n-2}} \inf(\tilde{\delta}_i \tilde{\delta}_j)^2 = O\Bigg(\varepsilon_{ij}^{\frac{n-1}{n-2}} \log \varepsilon_{ij}^{-1} \Bigg) \tag{3.5}
$$

and

(3.6)

$$
\int_{\mathbb{S}^{n-1}} \Bigg(\Sigma \alpha_i \tilde{\delta}_i \Bigg)^{\frac{2}{n-2}} . \inf\big((\Sigma \alpha_i \tilde{\delta}_i), w+h \big)^2 = O\Bigg(\|h\|^{\inf(3, 2\frac{(n-1)}{n-2})} \Bigg) + o\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2/2}} \Big).
$$

Using (3.5) and Lemma 3.3, we derive that

$$\int_{\mathbb{S}^{n-1}} H\Big(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i\Big)^{2\frac{(n-1)}{n-2}} = 2\frac{(n-1)}{n-2}\sum_{i\neq j}\alpha_i^{\frac{n}{n-2}}\alpha_j c_1 H(a_i)\varepsilon_{ij} + o\Big(\varepsilon_{ij}\Big) \tag{3.7}$$

$$+\sum_{i=1}^{p}\alpha_i^{2\frac{(n-1)}{n-2}}\Bigg(H(a_i)S_n + \begin{cases}\dfrac{c_3\Delta H(a_i)}{\lambda_i^2}\log\lambda_i + o(\dfrac{\log\lambda_i}{\lambda_i^2}) & \text{if } n=3\\ \dfrac{c_4\Delta H(a_i)}{\lambda_i^2} + o(\dfrac{1}{\lambda_i^2}) & \text{if } n\geq 4\end{cases}\Bigg).$$

Since $h \in T_w(W_u(w))$, we derive that

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} H\Big(w+h\Big)^{2\frac{(n-1)}{n-2}} &= \int_{\mathbb{S}^{n-1}} Hw^{2\frac{(n-1)}{n-2}} + \frac{2(n-1)}{n-2}\int_{\mathbb{S}^{n-1}} Hw^{\frac{n}{n-2}}h\\ &+ \frac{n(n-1)}{(n-2)^2}\int_{\mathbb{S}^{n-1}} Hw^{\frac{2}{n-2}}h^2 + O\Big(\|h\|^{\inf(3,2\frac{(n-1)}{n-2})}\Big)\\ &= \|w\|^2 + \frac{n(n-1)}{(n-2)^2}\int_{\mathbb{S}^{n-1}} Hw^{\frac{2}{n-2}}h^2 + O\Big(\|h\|^{\inf(3,2\frac{(n-1)}{n-2})}\Big).\end{aligned} \tag{3.8}$$

Using again Lemma 3.3, we obtain

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} H\Big(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i\Big)^{\frac{n}{n-2}}(w+h) &= \int_{\mathbb{S}^{n-1}} H\Big(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i\Big)^{\frac{n}{n-2}} h\\ &+c_1\sum_{i=1}^{p}\alpha_i^{\frac{n}{n-2}}\frac{H(a_i)w(a_i)}{\lambda_i^{\frac{n}{n-2}}} + o\Big(\frac{1}{\lambda_i^{\frac{n}{n-2}}}\Big).\end{aligned} \tag{3.9}$$

Since $v \in T_w(W_s(w))$ and $h \in T_w(W_u(w))$, the linear form on v can be written as

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}} H\Big(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i + \alpha_0(w+h)\Big)^{\frac{n}{n-2}} v = \int_{\mathbb{S}^{n-1}} H\Big(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i\Big)^{\frac{n}{n-2}} v\\ &+\int_{\mathbb{S}^{n-1}} H\Big(\alpha_0(w+h)\Big)^{\frac{n}{n-2}} v + O\Big(\sum_{i=1}^{p}\int_{\mathbb{S}^{n-1}}\tilde{\delta}_i^{\frac{2}{n-2}}|w+h||v| + \int_{\mathbb{S}^{n-1}}\tilde{\delta}_i|w+h|^{\frac{2}{n-2}}|v|\Big)\\ &= \frac{\beta_1}{2\gamma}f_1(v) + \alpha_0^{\frac{n}{n-2}}\Big(\int_{\mathbb{S}^{n-1}} Hw^{\frac{n}{n-2}}v + \frac{n}{n-2}\int_{\mathbb{S}^{n-1}} Hw^{\frac{2}{n-2}}hv\Big)\\ &+O\Big(\|v\|\|h\|^{\inf(2,\frac{n}{n-2})}\Big)\end{aligned}$$

$$= \frac{\beta_1}{2\gamma}f_1(v) + O\Big(\|v\|^{\inf(3,2\frac{n-1}{n-2})} + \|h\|^{\inf(3,2\frac{n-1}{n-2})}\Big). \tag{3.10}$$

Furthermore, we have

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} H\Big(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i \quad +\alpha_0(w+h)\Big)^{\frac{2}{n-2}} v^2 &= \int_{\mathbb{S}^{n-1}} H\Big(\sum_{i=1}^{p}\alpha_i\tilde{\delta}_i\Big)^{\frac{2}{n-2}} v^2 \\ &+ \int_{\mathbb{S}^{n-1}} H\Big(\alpha_0(w+h)\Big)^{\frac{2}{n-2}} v^2 + o\big(\|v\|^2+\|h\|^2\big).\end{aligned} \tag{3.11}$$

Combining (3.1),...,(3.11) and the fact that $J(u)^{\frac{n-1}{n-2}}\alpha_i^{\frac{2}{n-2}}H(a_i) = 1+o(1)$, for each i, and $\alpha_0 J(u)^{\frac{n-1}{2}} = 1+o(1)$, the result follows. □

Lemma 3.3. *For $i \in \{1,\dots,p\}$ and $j \neq i$ we have the following estimates,*

$$< \tilde{\delta}_i, \tilde{\delta}_j >= c_1\varepsilon_{ij} + o(\varepsilon_{ij}) \tag{3.12}$$

$$\int_{\mathbb{S}^{n-1}} H\tilde{\delta}_i^{\frac{n}{n-2}}\tilde{\delta}_j = c_1 H(a_i)\varepsilon_{ij} + o\Big(\varepsilon_{ij} + \frac{1}{\lambda_i^2}\Big) \tag{3.13}$$

$$\int_{\mathbb{S}^{n-1}} H\tilde{\delta}_i^{2\frac{n-1}{n-2}} = H(a_i)S_n + \begin{cases} \dfrac{c_3\Delta H(a_i)}{\lambda_i^2}\log\lambda_i + o\Big(\dfrac{\log\lambda_i}{\lambda^2}\Big) & \textit{if } n=3,\\ \dfrac{c_4\Delta H(a_i)}{\lambda_i^2} + o\Big(\dfrac{1}{\lambda_i^2}\Big) & \textit{if } n\geq 4.\end{cases} \tag{3.14}$$

$$< \tilde{\delta}_i, w >= c_1\frac{w(a_i)}{\lambda_i^{\frac{n-2}{2}}} + o\Big(\frac{1}{\lambda_i^{\frac{n-2}{2}}}\Big) \tag{3.15}$$

Proof. The first estimate is similar up to minor modifications to the one given in [4] (page 4), so we will omit it here. The second estimate also is very easy using (3.12), arguing as in [4] (page20).
For the proof of (3.15), using the fact that w is a solution of the problem (1.3) or zero and an easy computation, we derive the desired estimate. We focus now with the estimate (3.14). For $n \geq 4$, the proof is very standard and similar to the analogue in [4] (page21).
For $n = 3$, we have

$$\begin{aligned}\int_{\mathbb{S}^2} H\tilde{\delta}^4 &= \int_{\mathbb{R}^2} H(x)\delta^4(x)\,dx \\ &= \int_{B(a,1)} H(x)\delta^4(x)\,dx + \int_{B(a,1)^c} H(x)\delta^4(x)\,dx \\ &= I_1 + I_2.\end{aligned} \tag{3.16}$$

For the second member we have,

$$I_2 \leq C\int_{B(a,1)^c}\delta^4(x)\,dx \leq C\int_{B(a,1)^c}\frac{\lambda^2 dx}{\Big[1+\lambda^2|x-a|^2\Big]^2}$$

If we set $y = \lambda(x-a)$, we have

$$I_2 \leq C \int_{B(0,\lambda)^c} \frac{dy}{\Bigg[1+|y|^2\Bigg]^2} \leq C_\theta \int_\lambda^{+\infty} \frac{rdr}{\Bigg[1+|r|^2\Bigg]^2}$$

Hence

$$I_2 = O(\frac{1}{\lambda^2}). \tag{3.17}$$

It remains now to estimate I_1 which is equal to

$$\int_{B(a,1)} H(x)\delta^4(x)\,dx = H(a)\int_{B(a,1)} \delta^4(x)\,dx + \int_{B(a,1)} \sum_{i=1}^{2} \frac{\partial H(a)}{\partial x_i}(x_i-a)\delta^4(x)\,dx$$
$$+\frac{1}{2}\sum_{i=1}^{2}\frac{\partial^2 H(a)}{\partial x_i^2}\int_{B(a,1)} |x_i-a|^2\delta^4(x)\,dx + o\Bigg(\int_{B(a,1)} |x_i-a|^2\delta^4(x)\,dx\Bigg)$$

By oddness, we have $\displaystyle\int_{B(a,1)} \sum_{i=1}^{2} \frac{\partial H(a)}{\partial x_i}(x_i-a)\delta^4(x)\,dx = 0$. Using (3.1), we derive that

$$I_1 = S_3 H(a) + \frac{1}{2}\sum_{i=1}^{2}\frac{\partial^2 H(a)}{\partial x_i^2}\int_{B(a,1)} |x_i-a|^2\delta^4(x)\,dx + o\Bigg(\int_{B(a,1)} |x_i-a|^2\delta^4(x)\,dx\Bigg).$$

As usual, to estimate

$$\int_{B(a,1)} |x_i-a|^2\delta^4(x)\,dx = \int_{B(a,1)} |x_i-a|^2 \frac{\lambda^2}{\Bigg[1+\lambda^2|x-a|^2\Bigg]^2}\,dx, \tag{3.18}$$

we set $y = \lambda(x-a)$. Thus, (3.18) will be equal to

$$\frac{1}{\lambda^2}\int_{B(0,\lambda)} |y_i|^2 \frac{dy}{\Bigg[1+|y|^2\Bigg]^2} = \frac{1}{2\lambda^2}\int_{B(0,\lambda)} |y|^2 \frac{dy}{\Bigg[1+|y|^2\Bigg]^2} \tag{3.19}$$
$$= \frac{\pi}{\lambda^2}\int_0^\lambda \frac{r^3\,dr}{\Bigg[1+r^2\Bigg]^2} = \frac{\pi}{\lambda^2}\log\lambda\Bigg(1+o(\frac{1}{\log\lambda})\Bigg).$$

Finally

$$o\Bigg(\int_{B(a,1)} |x_i-a|^2\delta^4(x)\,dx\Bigg) = o\Big(\frac{\log\lambda}{\lambda^2}\Big) \tag{3.20}$$

Thus, using (3.19) and (3.20) we derive that

$$I_1 = S_3 H(a) + c_3 \Delta H(a)\frac{\log\lambda}{\lambda^2} + o\Big(\frac{\log\lambda}{\lambda^2}\Big) \tag{3.21}$$

Summing up (3.17) and (3.21), the proof of (3.14) is completed. □

Proposition 3.4. *For any $u = \Sigma_{i=1}^{p}\alpha_i\tilde{\delta}_i \in V(p,\varepsilon)$, we have the following expansion,*

$$< \nabla J(u), \lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i} >= 2J(u)\Bigg[-c_1\sum_{j\neq i}\alpha_j\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} \tag{3.22}$$

$$+J(u)^{\frac{n-1}{n-2}}\frac{n-2}{n-1}\alpha_i^{\frac{n}{n-2}}\begin{cases}\dfrac{c_3\Delta H(a_i)}{\lambda_i^2}\log\lambda_i + o\Big(\displaystyle\sum_{j=1}^{p}\frac{\log\lambda_j}{\lambda_j^2}\Big) & \text{if } n=3\\ \dfrac{c_4\Delta H(a_i)}{\lambda_i^2} + o\Big(\displaystyle\sum_{j=1}^{p}\frac{1}{\lambda_j^2}\Big) & \text{if } n\geq 4\end{cases}\Bigg]+o\Bigg(\sum_{j\neq i}\varepsilon_{ij}\Bigg).$$

Proof. We have,

$$< \nabla J(u), h >= 2J(u)\Bigg[< u,h > -J(u)^{\frac{n-1}{n-2}}\int_{\mathbb{S}^{n-1}}Hu^{\frac{n}{n-2}}h\Bigg].$$

Thus,

$$< \nabla J(u), \lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i} > = 2J(u)\Bigg[< \sum_{j=1}^{p}\alpha_j\tilde{\delta}_j, \lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i} > -J(u)^{\frac{n-1}{n-2}}$$

$$\times\int_{\mathbb{S}^{n-1}}H\Bigg(\sum_{j=1}^{p}\alpha_j\tilde{\delta}_j\Bigg)^{\frac{n}{n-2}}\lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i}\Bigg].$$

Observe that,

$$\int_{\mathbb{S}^{n-1}}H\Bigg(\sum_{j=1}^{p}\alpha_j\tilde{\delta}_j\Bigg)^{\frac{n}{n-2}}\lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i} = \sum_{j=1}^{p}\alpha_j^{\frac{n}{n-2}}\int_{\mathbb{S}^{n-1}}H\tilde{\delta}_j^{\frac{n}{n-2}}\lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i} \tag{3.23}$$

$$+\frac{n}{n-2}\sum_{i\neq j}\int_{\mathbb{S}^{n-1}}H\Big(\alpha_i\tilde{\delta}_i\Big)^{\frac{2}{n-2}}\lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i}(\alpha_j\tilde{\delta}_j) + O\Bigg(\sum_{j\neq i}\int_{\mathbb{S}^{n-1}}\tilde{\delta}_j^{\frac{2}{n-2}}\inf\big(\tilde{\delta}_j,\tilde{\delta}_i\big)^2\Bigg).$$

A computation similar to the one performed in [4], shows that,

$$< \tilde{\delta}_i, \lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i} >= 0 \tag{3.24}$$

$$< \tilde{\delta}_j, \lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i} >= c_1\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + o(\varepsilon_{ij}) \tag{3.25}$$

(3.26)

$$\int_{\mathbb{S}^{n-1}}H\tilde{\delta}_i^{\frac{n}{n-2}}\lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i} = -\frac{n-2}{2(n-1)}\begin{cases}\dfrac{c_3\Delta H(a_i)}{\lambda_i^2}\log\lambda_i + o\Big(\dfrac{\log\lambda_i}{\lambda_i^2}\Big) & \text{if } n=3,\\ \dfrac{c_4\Delta H(a_i)}{\lambda_i^2} + o\Big(\dfrac{1}{\lambda_i^2}\Big) & \text{if } n\geq 4.\end{cases}$$

$$\int_{\mathbb{S}^{n-1}} H\tilde{\delta}_j^{\frac{n}{n-2}}\lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i} = c_1 H(a_j)\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + o\big(\varepsilon_{ij}\big) + o\left(\frac{1}{\lambda_i^{\frac{n}{2}}\lambda_j^{\frac{n-2}{2}}}\right) \tag{3.27}$$

$$\frac{n}{n-2}\int_{\mathbb{S}^{n-1}} H\,\tilde{\delta}_j\tilde{\delta}_i^{\frac{2}{n-2}}\lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i} = c_1 H(a_i)\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + o\big(\varepsilon_{ij}\big) + o\left(\frac{1}{\lambda_i^{\frac{n+2}{2}}\lambda_j^{\frac{n-2}{2}}}\right). \tag{3.28}$$

Using (3.5), (3.23) ... (3.28), and the fact that $J(u)^{\frac{n-1}{n-2}}\alpha_i^{\frac{2}{n-2}}H(a_i) = 1 + o(1)$, for each i, the proposition follows. □

Proposition 3.5. *For any $u = \Sigma_{i=1}^p \alpha_i\tilde{\delta}_i \in V(p,\varepsilon)$, we have the following expansion,*

$$< \nabla J(u), \frac{1}{\lambda_i}\frac{\partial\tilde{\delta}_i}{\partial a_i} >= 2J(u)\Bigg[-c_1\sum_{j\neq i}\frac{\alpha_j}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i} \tag{3.29}$$

$$-J(u)^{\frac{n-1}{n-2}}c_2\alpha_i^{\frac{n}{n-2}}\frac{\nabla H(a_i)}{\lambda_i}\Bigg] + O\left(\sum_{i\neq j}\varepsilon_{ij} + \sum_{j=1}^p\frac{1}{\lambda_j^2}\right).$$

where $c_2 = \frac{(n-2)\bar{c}^{\frac{2(n-1)}{n-2}}}{2(n-1)}\int_{\mathbb{R}^{n-1}}\frac{dx}{(1+|x|^2)^{n-1}}$.

Proof. As in the proof of Proposition 3.4, We have,

$$< \nabla J(u), \frac{1}{\lambda_i}\frac{\partial\tilde{\delta}_i}{\partial a_i} > = 2J(u)\Bigg[< \sum_{j=1}^p\alpha_j\tilde{\delta}_j, \frac{1}{\lambda_i}\frac{\partial\tilde{\delta}_i}{\partial a_i} > -J(u)^{\frac{n-1}{n-2}}$$

$$\times\int_{\mathbb{S}^{n-1}} H\Big(\sum_{j=1}^p\alpha_j\tilde{\delta}_j\Big)^{\frac{n}{n-2}}\frac{1}{\lambda_i}\frac{\partial\tilde{\delta}_i}{\partial a_i}\Bigg].$$

Observe that,

$$\int_{\mathbb{S}^{n-1}} H\left(\sum_{j=1}^p\alpha_j\tilde{\delta}_j\right)^{\frac{n}{n-2}}\frac{1}{\lambda_i}\frac{\partial\tilde{\delta}_i}{\partial a_i} = \sum_{j=1}^p\alpha_j^{\frac{n}{n-2}}\int_{\mathbb{S}^{n-1}} H\tilde{\delta}_j^{\frac{n}{n-2}}\frac{1}{\lambda_i}\frac{\partial\tilde{\delta}_i}{\partial a_i} \tag{3.30}$$

$$+\frac{n}{n-2}\sum_{i\neq j}\int_{\mathbb{S}^{n-1}} H\Big(\alpha_i\tilde{\delta}_i\Big)^{\frac{2}{n-2}}\frac{1}{\lambda_i}\frac{\partial\tilde{\delta}_i}{\partial a_i}(\alpha_j\tilde{\delta}_j) + O\left(\sum_{j\neq i}\int_{\mathbb{S}^{n-1}}\tilde{\delta}_j^{\frac{2}{n-2}}\inf\big(\tilde{\delta}_j,\tilde{\delta}_i\big)^2\right),$$

A computation similar to the one performed in [4], shows that,

$$< \tilde{\delta}_i, \frac{1}{\lambda_i}\frac{\partial\tilde{\delta}_i}{\partial a_i} >= 0 \tag{3.31}$$

$$< \tilde{\delta}_j, \frac{1}{\lambda_i}\frac{\partial\tilde{\delta}_i}{\partial a_i} >= \frac{c_1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i} + o\big(\varepsilon_{ij}\big) \tag{3.32}$$

$$\int_{\mathbb{S}^{n-1}} H\tilde{\delta}_i^{\frac{n}{n-2}}\frac{1}{\lambda_i}\frac{\partial\tilde{\delta}_i}{\partial a_i} = c_2\frac{\nabla H(a_i)}{\lambda_i} + O\Big(\frac{1}{\lambda_i^2}\Big) \tag{3.33}$$

$$\int_{\mathbb{S}^{n-1}} H\tilde{\delta}_j^{\frac{n}{n-2}} \frac{1}{\lambda_i}\frac{\partial \tilde{\delta}_i}{\partial a_i} = c_1 \frac{H(a_j)}{\lambda_i}\frac{\partial \varepsilon_{ij}}{\partial a_i} + o\big(\varepsilon_{ij}\big) \tag{3.34}$$

$$\frac{n}{n-2}\int_{\mathbb{S}^{n-1}} H\tilde{\delta}_i^{\frac{2}{n-2}} \frac{1}{\lambda_i}\frac{\partial \tilde{\delta}_i}{\partial a_i}\tilde{\delta}_j = c_1 \frac{H(a_i)}{\lambda_i}\frac{\partial \varepsilon_{ij}}{\partial a_i} + o\Big(\varepsilon_{ij}\Big) + o\Bigg(\frac{1}{\lambda_i^{\frac{n+2}{2}}\lambda_j^{\frac{n-2}{2}}}\Bigg). \tag{3.35}$$

Combining (3.5), (3.30) ... (3.35), and the fact that $J(u)^{\frac{n-1}{n-2}}\alpha_i^{\frac{2}{n-2}}H(a_i) = 1 + o(1)$, for each i, the proposition follows. □

3.2. **The v-part of u.** In this subsection, we deal with the v-part of u, in order to show that it can be neglected with respect to the concentration phenomenon.
Set

$$E_\varepsilon = \{v \in H^1(\mathbb{B}^n)/v \in T_w(W_s(w)) \text{ satisfying } (V_0), \text{ and } \|v\| < \varepsilon\}$$

Proposition 3.6. *For any $u = \sum_{i=1}^p \alpha_i\tilde{\delta}_i \in V(p,\varepsilon,w)$, there exists a unique $\bar{v} = \bar{v}(\alpha,a,\lambda)$ which minimizes $J(u+v)$ with respect to $v \in E_\varepsilon$ and a unique $\bar{h} = \bar{h}(\alpha,a,\lambda)$ which maximizes $J(u+h)$ with respect to $h \in T_w(W_u(w))$. Moreover, we have the following estimates:*

$$\|\bar{v}\| \le c\left[\sum_{i=1}^p\left(\frac{|\nabla H(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right) + \sum_{i\neq j}\begin{cases}\varepsilon_{ij}^{\frac{n}{2(n-2)}}\log\varepsilon_{ij}^{-\frac{n}{2(n-1)}} & \text{if } n \ge 4,\\ \varepsilon_{ij}\log\varepsilon_{ij}^{-1/2} & \text{if } n = 3.\end{cases}\right]$$

$$\|\bar{h}\| \le c\sum_{i=1}^p \frac{1}{\lambda_i^{\frac{n-2}{2}}}.$$

Before giving the proof of this result, we need to prove the following Lemma adapted from [16]:

Lemma 3.7. *The following Claims hold true:*

(1) *$Q_1(v,v)$ is a positive definite quadratic form in E_ε.*
(2) *$Q_2(h,h)$ is a negative definite quadratic form in $T_w(W_u(w))$.*

Proof. The second claim follows immediately by the definition of $Q_2(h,h)$ and the fact that $h \in T_w(W_u(w))$. Next, we are going to prove the first claim. We split $T_w(W_u(w))$ into $E_\gamma \oplus F_\gamma$ where E_γ and F_γ are orthogonal in the sense of the scalar product defined before and as well as for the quadratic form associated to w and such that

$$\|v\|^2 - \frac{n}{n-2}\int Hw^{\frac{2}{n-2}}v^2 \ge (1-\gamma)\|v\|^2 \text{ on } F_\gamma \text{ and } \dim(E_\gamma) < \infty, \tag{3.36}$$

where γ is chosen small enough such that $0 < \gamma < \bar{\alpha}/4$, and $\bar{\alpha}$ is the first eigenvalue of

$$\Delta - \frac{n}{n-2}\tilde{\delta}_{(a,\lambda)}^{\frac{2}{n-2}}$$

(α is independent of $\tilde{\delta}_{(a,\lambda)}$).
Now, we split v in two parts $v_1 \in E_\gamma$ and $v_2 \in F_\gamma$ such that, $v = v_1 + v_2$. Thus,

$$\begin{aligned}
Q_1(v,v) &= \|v\|^2 - \frac{n\gamma}{(n-2)\beta_1}\left(\sum_{i=1}^{p}\int_{\mathbb{S}^{n-1}} H\Big(\alpha_i\tilde{\delta}_i\Big)^{\frac{2}{n-2}}v^2 + \int_{\mathbb{S}^{n-1}} H\Big(\alpha_0 w\Big)^{\frac{2}{n-2}}v^2\right)\\
&= \|v_1\|^2 + \|v_2\|^2 - \sum_{i=1}^{p}\frac{n\gamma}{(n-2)\beta_1}\left(\int_{\mathbb{S}^{n-1}} H\Big(\alpha_i\tilde{\delta}_i\Big)^{\frac{2}{n-2}}\Big(v_1^2+v_2^2+2<v_1,v_2>\Big)\right)\\
&- \frac{n\gamma}{(n-2)\beta_1}\int_{\mathbb{S}^{n-1}} H\Big(\alpha_0 w\Big)^{\frac{2}{n-2}}\Big(v_1^2+v_2^2+2<v_1,v_2>\Big)\\
&= \|v_1\|^2 + \|v_2\|^2 - \sum_{i=1}^{p}\frac{n\alpha_i\gamma}{(n-2)\beta_1}\left(\int_{\mathbb{S}^{n-1}} H\tilde{\delta}_i^{\frac{2}{n-2}}\Big(v_1^2+v_2^2\Big)\right)\\
&- \frac{n\gamma}{(n-2)\beta_1}\int_{\mathbb{S}^{n-1}} H\Big(\alpha_0 w\Big)^{\frac{2}{n-2}}\Big(v_1^2+v_2^2\Big) + o\big(\|v_1\|\|v_2\|\big)
\end{aligned}$$

Since $\dim(E_\gamma)<\infty$, then

$$\int \tilde{\delta}_i^{\frac{2}{n-2}}v_1^2 = o\big(\|v\|^2\big)\ \ \forall v_1\in E_\gamma \text{ and } \forall i=1,\dots,p \tag{3.37}$$

(3.36) and (3.37) implies that

$$\begin{aligned}
Q_1(v,v) &= \|v_1\|^2 + (1-\gamma)\|v_2\|^2 - \sum_{i=1}^{p}\frac{n\alpha_i\gamma}{(n-2)\beta_1}\left(\int_{\mathbb{S}^{n-1}} H\tilde{\delta}_i^{\frac{2}{n-2}}v_2^2\right)\\
&- \frac{n\alpha_0\gamma}{(n-2)\beta_1}\int_{\mathbb{S}^{n-1}} Hw^{\frac{2}{n-2}}v_1^2 + o\Big(\|v_1\|\|v_2\| + \|v_1\|^2\Big)
\end{aligned}$$

$$\geq \alpha'\|v_1\|^2 + (1-\gamma)\|v_2\|^2 - \sum_{i=1}^{p}\frac{n\alpha_i\gamma}{(n-2)\beta_1}\left(\int_{\mathbb{S}^{n-1}} H\tilde{\delta}_i^{\frac{2}{n-2}}v_2^2\right) + o\Big(\|v_2\|^2\Big). \tag{3.38}$$

It remains now to study the term in v_2. First, we observe that v is orthogonal to $span\{\tilde{\delta}_i, \lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i}, \frac{1}{\lambda_i}\frac{\partial\tilde{\delta}_i}{\partial a_i},\ 1\leq i\leq p\}$ but not v_2. Furthermore, since v_1 belongs to a finite dimensional space, and for all $f\in\cup_{i\leq p}\{\tilde{\delta}_i, \lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i}, \frac{1}{\lambda_i}\frac{\partial\tilde{\delta}_i}{\partial(a_i)_j}\}$, we have

$$<v_1, f> \leq \|v_1\|_\infty\int|\nabla f|^2 = o(\|v_1\|). \tag{3.39}$$

Now, for all $v_2\in F_\gamma$ there exists $\bar{v_2}\in span\{\tilde{\delta}_i, \lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i}, \frac{1}{\lambda_i}\frac{\partial\tilde{\delta}_i}{\partial(a_i)_j},\ i\leq p,\ j\leq n-1\}^\perp$ such that,

$$v_2 = \bar{v_2} + \sum_{i=1}^{p}A_i\delta_{a_i,\lambda_i} + \sum_{i=1}^{p}B_i\lambda_i\frac{\partial\delta_{a_i,\lambda_i}}{\partial\lambda_i} + \sum_{i=1}^{p}\sum_{j=1}^{n-1}C_{ij}\frac{1}{\lambda_i}\frac{\partial\delta_{a_i,\lambda_i}}{\partial(a_i)_j}, \tag{3.40}$$

Observe that the family of functions $\delta_{a,\lambda}$ is the solution of the Yamabe problem on $\mathbb{R}^n_+$, that is the functional,

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^n_+}|\nabla u|^2 - \frac{n-2}{2(n-1)}\int_{\mathbb{R}^{n-1}}|u|^{2\frac{n-1}{n-2}}$$

has only the family of $\delta_{a,\lambda}$ as critical points. Those critical points are degenerate and of index 1. The nullity space is of dimension n and is generated by the derivative of $\delta_{a,\lambda}$ with respect to λ and a. Furthermore, the set of negativity is generated by the function $\delta_{a,\lambda}$. Thus, on the orthogonal of $span\{\tilde{\delta}_i, \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i}, \frac{1}{\lambda_i}\frac{\partial \tilde{\delta}_i}{\partial (a_i)_j}, \ i \leq p, \ j \leq n-1\}$, the second derivative of I on $\delta_{a,\lambda}$ is positive definite. Therefore, we deduce that there exists $\bar{\alpha} > 0$, such that for any $\bar{v}_2 \in span\{\tilde{\delta}_i, \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i}, \frac{1}{\lambda_i}\frac{\partial \tilde{\delta}_i}{\partial (a_i)_j}, \ i \leq p, \ j \leq n-1\}^{\perp}$, we have

$$\|\bar{v}_2\|^2 - \sum_{i=1}^{p} \frac{n\alpha_i\gamma}{(n-2)\beta_1}\int_{\mathbb{S}^{n-1}} H\tilde{\delta}_i^{\frac{2}{n-2}}\bar{v}_2^2 \geq \frac{\bar{\alpha}}{2}\|\bar{v}_2\|^2 \tag{3.41}$$

Using (3.39)-(3.41) we derive that

$$\begin{aligned}\|v_2\|^2 - \sum_{i=1}^{p} \frac{n\alpha_i\gamma}{(n-2)\beta_1}\int_{\mathbb{S}^{n-1}} H\tilde{\delta}_i^{\frac{2}{n-2}}v_2^2 &= \|\bar{v}_2\|^2 + O\left(\sum_{i=1}^{p}\Big(A_i^2 + B_i^2 + \sum_j C_{ij}^2\Big)\right)\\
&\quad - \sum_{i=1}^{p} \frac{n\alpha_i\gamma}{(n-2)\beta_1}\int_{\mathbb{S}^{n-1}} H\tilde{\delta}_i^{\frac{2}{n-2}}\bar{v}_2^2\\
&\quad + O\left(\|\bar{v}_2\|\Big(|A_i| + |B_i| + \sum_j |C_{ij}|\Big)\right)\\
&= \|\bar{v}_2\|^2 - \sum_{i=1}^{p} \frac{n\alpha_i\gamma}{(n-2)\beta_1}\int_{\mathbb{S}^{n-1}} H\tilde{\delta}_i^{\frac{2}{n-2}}\bar{v}_2^2 + o(\|\bar{v}_1\|)\end{aligned}$$

$$\geq \frac{\bar{\alpha}}{2}\|\bar{v}_2\|^2 + o(\|\bar{v}_1\|). \tag{3.42}$$

Combining (3.38) and (3.42), we get

$$\begin{aligned}Q_1(v,v) &\geq \alpha'\|v_1\|^2 - \gamma\|v_2\|^2 + \frac{\bar{\alpha}}{2}\|\bar{v}_2\|^2 + o(\|\bar{v}_1\|^2 + \|\bar{v}_2\|^2)\\
&\geq \alpha'\|v_1\|^2 + \left(\frac{\bar{\alpha}}{2} - \gamma\right)\|v_2\|^2 + o(\|\bar{v}_1\|^2 + \|\bar{v}_2\|^2).\end{aligned}$$

Since $\gamma < \bar{\alpha}/4$ the claim (1) follows. The proof of Lemma 3.7 is thereby completed. □

Proof of Proposition 3.6 First, we focus on the estimate of $\bar{v}$. Let $v \in E_\varepsilon$ where ε is a fixed small positive constant depending only on p. Using the Lemma 3.7, we derive that

$$\|\bar{v}\| < c\|f_1\| \tag{3.43}$$

where c is a positive constant. It remains now to estimate $\|f_1\|$. We have,

$$
\begin{aligned}
f_1(v) &= \frac{2\gamma}{\beta_1}\int_{\mathbb{S}^{n-1}} H\Big(\sum_{i=1}^{p}\alpha_i\tilde\delta_i\Big)^{\frac{n}{n-2}} v \\
&= \sum_{i=1}^{p}\frac{2\alpha_i^{\frac{n}{n-2}}\gamma}{\beta_1}\int_{\mathbb{S}^{n-1}} H\tilde\delta_i^{\frac{n}{n-2}} v + O\Big(\sum_{i\neq j}\frac{2\alpha_i^{\frac{2}{n-2}}\alpha_j\gamma}{\beta_1}\int_{\mathbb{S}^{n-1}}\tilde\delta_i^{\frac{2}{n-2}}\inf\big(\tilde\delta_i,\tilde\delta_j\big)|v|\Big)
\end{aligned}
$$

Expanding H around a_i, we obtain

$$
f_1(v) = O\Big(\sum_{i=1}^{p}|\nabla H(a_i)|\int_{\mathbb{S}^{n-1}}|x-a_i|\tilde\delta_i^{\frac{n}{n-2}}|v| + \frac{\|v\|}{\lambda_i^2} + \sum_{i\neq j}\int_{\mathbb{S}^{n-1}}\tilde\delta_i^{\frac{2}{n-2}}\inf\big(\tilde\delta_i,\tilde\delta_j\big)|v|\Big)
$$

$$
\le c\|v\|\left[\sum_{i=1}^{p}\Big(\frac{|\nabla H(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^2}\Big)+\sum_{i\neq j}\begin{cases}\varepsilon_{ij}^{\frac{n}{2(n-2)}}\log\varepsilon_{ij}^{-\frac{n}{2(n-1)}} & \text{if } n\geq 4,\\ \varepsilon_{ij}\log\varepsilon_{ij}^{-1/2} & \text{if } n=3.\end{cases}\right] \tag{3.44}
$$

Using (3.43) and (3.44), the estimate of $\|\bar v\|$ follows. For the estimate of $\|\bar h\|$ we use again Lemma 3.7. Since $Q_2(h,h)$ is negative definite quadratic form in $T_w(W_u(w))$, then there exists a unique maximum $\bar h$ in the space of h satisfying

$$
\|\bar h\| \leq c\|f_2\|. \tag{3.45}
$$

Furthermore, we have

$$
\|f_2\| = O\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\frac{n-2}{2}}}\Big). \tag{3.46}
$$

Using (3.45) and (3.46), the estimate of $\bar h$ follows and the proof of the Proposition 3.6 is thereby completed. □

Proposition 3.6 implies the following consequence whose proof is very easy so we omit it.

Corollary 3.8. *For the optimal $(\bar v,\bar h)$ defined in Proposition 3.6, there is a change of variables*

$$
\begin{cases} v-\bar v & \longrightarrow V\\ h-\bar h & \longrightarrow H\end{cases}
$$

such that,

$$J(u) = \frac{S_n \Sigma_{i=1}^p \alpha_i^2 + \alpha_0^2 \|w\|^2}{\left(S_n \Sigma_{i=1}^p \alpha_i^{2\frac{(n-1)}{n-2}} H(a_i) + \alpha_0^{2\frac{(n-1)}{n-2}} \|w\|^2\right)^{\frac{n-2}{n-1}}} \Bigg[1 - \frac{(n-2)}{(n-1)\beta_1} \times$$
$$\sum_{i=1}^p \alpha_i^{2\frac{(n-1)}{n-2}} \begin{cases} \dfrac{c_3 \Delta H(a_i)}{\lambda_i^2} \log \lambda_i & \text{if } n = 3 \\ \dfrac{c_4 \Delta H(a_i)}{\lambda_i^2} & \text{if } n \geq 4 \end{cases} - \frac{c_1}{\gamma} \sum_{i \neq j \geq 1} \alpha_i \alpha_j \varepsilon_{ij}$$
$$- 2 \frac{c_1 \alpha_0}{\gamma} \sum_{i=1}^p \alpha_i \frac{w(a_i)}{\lambda_i^{\frac{n-2}{2}}} + o\left(\sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^p \frac{\log \lambda_i}{\lambda_i^2} \right) \Bigg] + \|V\|^2 - \|H\|^2.$$

Proposition 3.9. *On $\mathbb{B}^3$ let H be a C^2 function and w be a nondegenerate solution of (1.3), then there is no critical points neither critical points at infinity in $V(p, \varepsilon, w)$. That means that we can construct a pseudogradient of J such that the Palais Smale condition is satisfied along the decreasing flow lines*

The proof of this Proposition follows easily from the above corrollary and the fact that $w > 0$ on $\overline{\mathbb{B}^3}$.

4. Prescribing mean curvature on higher dimensional ball

In order to give the proof of the results concerning higher dimensional case, we have to characterize the critical points in this case.

4.1. Characterization of the critical points at infinity.

We give here the characterization of the critical points at infinity in the case of higher dimensional ball.
First, we construct a special pseudo-gradient W for the associated variational problem for which the Palais-Smaile condition is satisfied along the decreasing flow lines, as long as these flow lines do not enter in the neighborhood of critical points y_{j_i} of H such that $-\Delta H(y_{j_i}) > 0$. As a consequence of this construction, we are able to determinate the critical points at infinity associated our problem.

Proposition 4.1. *Let $n \geq 5$. For any $u = \sum_{i=1}^p \alpha_i \tilde{\delta}_i \in V(p, \varepsilon)$ with $p \geq 1$, there exists a pseudo-gradient W so that there is a constant $c > 0$ independent of u such that,*

(1) $< -\nabla J(u), W > \geq c \left(\sum_{i=1}^p \Big(\frac{|\nabla H(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} \Big) + \sum_{i \neq j \geq 1} \varepsilon_{ij} \right)$

(2) $< -\nabla J(u + \bar{v}), W + \frac{\partial \bar{v}}{\partial(\alpha_i, a_i, \lambda_i)}(W) > \geq c \left(\sum_{i=1}^p \Big(\frac{|\nabla H(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} \Big) + \sum_{i \neq j \geq 1} \varepsilon_{ij} \right)$

(3) *W is bounded*

(4) *the only region where the maximum of the λ_i's increases along the flow lines of W is the region where a_i is near a critical point y_{j_i} of H with, $-\Delta H(y_{j_i}) > 0$ and $j_i \neq j_r$ for $i \neq r$.*

Proof. For the sake of simplicity, we can order the λ_i's and we assume that $\lambda_1 \leq \cdots \leq \lambda_p$. Let,

$$\begin{aligned} I_1 =& \Big\{ i \in \{2, \dots, p\} / \ \lambda_i |\nabla H(a_i)| \geq C_1' \Big\} \\ I_2 =& \{1\} \cup \Big\{ i \in \{2, \dots, p\} / \ \lambda_i \leq M \lambda_{j-1}, \ \text{ for each } j \leq i \Big\}, \end{aligned}$$

where C_1' and M are two positive large constants. Set

$$Z_1 = \sum_{i \in I_1} \frac{1}{\lambda_i} \frac{\partial \tilde{\delta}_i}{\partial a_i} \frac{\nabla H(a_i)}{|\nabla H(a_i)|}, \quad Z_2 = -M_1 \sum_{i \notin I_2} 2^i \lambda_i \frac{\partial \tilde{\delta}_i}{\nabla \lambda_i} - m_1 \sum_{i \in I_2} \lambda_i \frac{\partial \tilde{\delta}_i}{\nabla \lambda_i},$$

where M_1 is a large constant and m_1 is a small constant.
Using Proposition 3.5, we derive that

$$\begin{aligned} < -\nabla J(u), Z_1 > \ \ & \geq c \sum_{i \in I_1} \left(\frac{|\nabla H(a_i)|}{\lambda_i} + O\Big(\sum_{j \in I_2} \frac{1}{\lambda_i} \Big| \frac{\partial \varepsilon_{ij}}{\partial a_i} \Big| \Big) \right) \\ & + O\left(\sum_{i \in I_1} \frac{1}{\lambda_i^2} + \sum_{j \notin I_2} \varepsilon_{ij} \right) + o\left(\sum_{k \neq r} \varepsilon_{kr} + \sum_{k=1}^{p} \frac{1}{\lambda_k^2} \right) \end{aligned} \tag{4.1}$$

Using the definitions of ε_{ij} and I_2, we have

$$\frac{1}{\lambda_i} \Big| \frac{\partial \varepsilon_{ij}}{\partial a_i} \Big| = o(\varepsilon_{ij}). \tag{4.2}$$

Using the fact that $i \in I_1$, then (4.1) becomes

$$\begin{aligned} < -\nabla J(u), Z_1 > \ \ & \geq c \sum_{i \in I_1} \left(\frac{|\nabla H(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} \right) + O\left(\sum_{j \notin I_2} \varepsilon_{ij} \right) \\ & + o\left(\sum_{k \neq r} \varepsilon_{kr} + \sum_{k=1}^{p} \frac{1}{\lambda_k^2} \right). \end{aligned} \tag{4.3}$$

For the second vector Z_2, we use Proposition 3.4 and we derive

$$\begin{aligned} < -\nabla J(u), Z_2 > \ \ & \geq c M_1 \sum_{i \notin I_2} \left(\sum_{j \neq i} \varepsilon_{ij} + O\Big(\frac{1}{\lambda_i^2} \Big) + o\Big(\sum_{k \neq r} \varepsilon_{kr} \right. \\ & \left. + \sum_{k=1}^{p} \frac{1}{\lambda_k^2} \Big) \right) + m_1 c \sum_{i \in I_2} \left(\sum_{j \in I_2} \varepsilon_{ij} + O\left(\frac{1}{\lambda_i^2} + \sum_{j \notin I_2} \varepsilon_{ij} \right) \right. \\ & \left. + o\left(\textstyle\sum_{k \neq r} \varepsilon_{kr} + \sum_{k=1}^{p} \frac{1}{\lambda_k^2} \right) \right). \end{aligned} \tag{4.4}$$

Now, we define $Z_3 = Z_1 + Z_2$, using (4.3) and (4.4), we derive that

$$
\begin{aligned}
< -\nabla J(u), Z_3 > \quad & \geq c\sum_{i\in I_1}\left(\frac{|\nabla H(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^2}\right)+c\sum_{j\neq i}\varepsilon_{ij} \\
& +O\left(\sum_{i\notin I_2}\frac{m_1}{\lambda_i^2}+\sum_{i\in I_2}\frac{m_1}{\lambda_i^2}\right)+o\left(\sum_{k\neq r}\varepsilon_{kr}+\sum_{k=1}^{p}\frac{1}{\lambda_k^2}\right).
\end{aligned} \tag{4.5}
$$

To continue the proof, we have to distinguish two case

case1. $I_1\cap I_2\neq\emptyset$. In this case, we can make $1/\lambda_k^2$ appear, for $k\in I_2$ in the lower bound of (4.5) and therefore all the $1/\lambda_k^2$. Notice that for $i\notin I_1$, we have $\lambda_i|\nabla H(a_i)|\leq C_1'$. Thus, if we choose $M_1\leq M$ and $m_1 << M^p$, then (4.5) becomes

$$
< -\nabla J(u), Z_3 > \geq c\left(\sum_{i=1}^{p}\Big(\frac{|\nabla H(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^2}\Big)+\sum_{i\neq j\geq 1}\varepsilon_{ij}\right) \tag{4.6}
$$

case2. $I_1\cap I_2=\emptyset$. In this case, for each $i\in I_2$, the point a_i is close to a critical point y_{k_i} of H. If we suppose that there exist $i,j\in I_2$ such that a_i and a_j in $B(y,\rho)$ for ρ small enough, then $|\nabla H(a_k)|\geq c|y-a_k|$ for $k=i,j$, since y is non degenerate. Therefore, $\lambda_i|a_i-a_j|\leq c$ (we assume that $\lambda_i\leq\lambda_j$). This implies that $\varepsilon_{ij}\geq c\Big(\frac{\lambda_i}{\lambda_j}\Big)^{\frac{n-2}{2}}$, which is a contradiction with the fact that λ_i and λ_j are of the same order. Thus, our suppose is false. We conclude that each neighborhood $B(y,\rho)$ contains at most one point a_i close to y_{k_i} with $k_i\neq k_j$ for $i\neq j$ and $i\in I_2$. Let us now introduce the following subset,

$$
I_3=\{i\in I_2/\ \Delta H(a_i)>0\}.
$$

subcase2.1. $I_3\neq\emptyset$. In this case, we define

$$
Z_4=-\sum_{i\in I_3}\lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i}-M_1\sum_{i\notin I_2}\lambda_i\frac{\partial\tilde{\delta}_i}{\partial\lambda_i}.
$$

Using Proposition 3.4, we derive

$$
\begin{aligned}
< -\nabla J(u), Z_4 > \quad & \geq c\sum_{j\in I_3}\left(\frac{1}{\lambda_i^2}+O\Big(\sum_{j\notin I_2}\varepsilon_{ij}\Big)\right) \\
& +M_1c\sum_{i\notin I_2}\left(\sum_{j\neq i}\varepsilon_{ij}+O\Big(\frac{1}{\lambda_i^2}\Big)\right)+o\left(\sum_{k\neq r}\varepsilon_{kr}+\sum_{k=1}^{p}\frac{1}{\lambda_k^2}\right).
\end{aligned} \tag{4.7}
$$

Observe that for i, j in I_2, we have $|a_i-a_j|\geq c$ and since $n\geq 5$ then

$$
\varepsilon_{ij}=O\Big(\frac{1}{\lambda_i^3}+\frac{1}{\lambda_j^3}\Big) \tag{4.8}
$$

Using (4.3), (4.7) and (4.8), for the vector $Z_5=Z_4+Z_1$, we derive that

$$
< -\nabla J(u), Z_5 > \geq c\left(\sum_{i=1}^{p}\Big(\frac{|\nabla H(a_i)|}{\lambda_i}+\frac{1}{\lambda_i^2}\Big)+\sum_{i\neq j\geq 1}\varepsilon_{ij}\right). \tag{4.9}
$$

subcase2.2. $I_3 = \emptyset$. In this case, we define

$$Z_6 = \sum_{i \in I_2} \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} - M_1 \sum_{i \notin I_2} 2^i \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} + Z_1.$$

Using Proposition 3.4 and (4.3), we derive that

$$< -\nabla J(u), Z_6 >\geq c \Bigg(\sum_{i=1}^{p} \Big(\frac{|\nabla H(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} \Big) + \sum_{i \neq j \geq 1} \varepsilon_{ij} \Bigg). \tag{4.10}$$

The vector W will be built as a convex combination of Z_3, Z_5 and Z_6 and using (4.6), (4.9) and (4.10), the proof of claim (1) is completed.
From the construction, W is bounded and we have $|d\lambda_i(W)| \leq c\lambda_i$ for each i. Observe also that the only case where the maximum of the λ_i's increase, is when $I_2 = \{1, \dots, p\}$ and $I_1 = I_3 = \emptyset$. In this case we have a_i is close to a critical point y_{j_i} of H with $j_i \neq j_k$ for $i \neq k$ and $-\Delta H(y_{j_i}) > 0$ for each i. Hence claim (4) follows.
Finally, arguing as in Appendix B of [8], claim (2) follows from claim (1) and Proposition 2.3.5. □

Using Proposition 4.1 and Corollary 3.8, and arguing as in [5] and [8], we can deduce the following two Propositions.

Proposition 4.2. *Let $n \geq 5$. Assume that J does not have any critical point in Σ^+. Then, the only critical point at infinity of J in $V(p, \varepsilon)$ for ε small enough correspond to $\sum_{j=1}^{p} \tilde{\delta}_{(y_{i_j}, \infty)}$, where y_{i_j} is a critical points of H satisfying $-\Delta H(y_{i_j}) > 0$ and $j_i \neq j_k$ for $i \neq k$. Moreover, the Morse index of such critical points at infinity, is equal to $p - 1 + \sum_{j=1}^{p} n - 1 - ind(H, y_{i_j})$, where $ind(H, y_{i_j})$ is the Morse index of H at y_{i_j}.*

Proposition 4.3. *Let $n \geq 7$. Let w be a nondegenerate solution of (1.3). Then $(y_{i_1}, \dots, y_{i_p}, w)$ is a critical points at infinity for each $(y_{i_1}, \dots, y_{i_p})$ critical point of H such that $\Delta H(y_{i_j}) < 0$ and $y_{i_j} \neq y_{i_k}$ for each $j \neq k$ belong to $\{1, \dots, p\}$.*

4.2. **Proof of Theorem 1.1.** In order to give here the proof of Theorem 1.1, we have to give some notations and definitions that will be useful later.

In the sequel, we denote by $\mathcal{A}$ the set of w such that w is a critical point or a critical point at infinity of J in Σ^+ not containing y_0 in its description. We also denote by $\mathcal{A}_q$ the subset of $\mathcal{A}$ such that the Morse index of the critical points (at infinity) is equal to q.

Definition 4.4. Let $\mathcal{F}$ be the family of pseudo-gradient V associated to J satisfying the following properties

1. the set of critical points at infinity of J on Σ^+ do not change if we replace $-\nabla J$ by the pseudo-gradient V,
2. the pseudo-gradient V is transverse to $f_\lambda\big(B_2(X)\big)$,
3. for any $w \in \mathcal{A}$, we have $(y_0, w)_\infty$ is a critical point at infinity satisfying

$$\begin{array}{lll} i((y_0, w)_\infty, w) & = 1 & \forall w \in \mathcal{A} \\ i((y_0, w)_\infty, w') & = 0 & \forall w' \in \mathcal{A},\ w' \neq w,\ ind(w') = ind(w) \\ i((y_0, w)_\infty, (y_0, w')_\infty) & = i(w, w') & \forall w' \in \mathcal{A},\ ind(w') = ind(w) - 1 \end{array}$$

where $i(w,w')$ is the Morse-Smale boundary operator used in [5] and defined in [30].

Definition 4.5. Let V a decreasing pseudo-gradient for J. We denote by $\varphi(s,.)$ the associated flow. A critical point at infinity z_∞ is said to be dominated by $f_\lambda\big(B_2(X)\big)$, if

$$\overline{\bigcup_{s\geq 0}\varphi\Big(s, f_\lambda\big(B_2(X)\big)\Big)}\cap W_s(z_\infty)\neq\emptyset. \tag{4.11}$$

Near the critical points at infinity, a Morse Lemma can be completed (see Corollary 3.8 and Proposition 4.3) so that the usual Morse theory can be extended and the intersection can be assumed to be transverse. Thus, arguing as in sections 7 and 8 of [7]), (4.11) is equivalent to

$$\bigcup_{s\geq 0}\varphi\Big(s, f_\lambda\big(B_2(X)\big)\Big)\cap W_s(z_\infty)\neq\emptyset. \tag{4.12}$$

Definition 4.6. A critical point at infinity z_∞ is said to be dominated by another critical point at infinity z'_∞ if

$$W_u(z'_\infty)\cap W_s(z_\infty)\neq\emptyset. \tag{4.13}$$

If we add that the intersection is transverse, then $ind(z'_\infty)\geq ind(z_\infty)+1$.

For $V\in\mathcal{F}$ and $w_{2k+1}\in\mathcal{A}_{2k+1}$, we denote by

$$\big(y_0, w_{2k+1}\big)_\infty .\, C_\gamma \tag{4.14}$$

the intersection number (modulo 2) of $W_u\big((y_0,w_{2k+1})_\infty\big)$ and C_γ (see section 1).

In order to compute this intersection number, one can perturb V (not necessary in $\mathcal{F}$) so as to bring $W_u\big((y_0,w_{2k+1})_\infty\big)\cap C_\gamma$ to be transverse. This number is the same for all such small perturbations (as in degree theory). Observe that the dimension of $W_u\big((y_0,w_{2k+1})_\infty\big)$ is equal to $2k+2$ and the codimension of C_γ is $2k+2$. Then, $\big(y_0,w_{2k+1}\big)_\infty .\, C_\gamma$ is also well defined, since the closure of $W_u\big((y_0,w_{2k+1})_\infty\big)$ only adds to $W_u\big((y_0,w_{2k+1})_\infty\big)$, the unstable manifolds of critical points of Morse index less than equal to $2k+1$. These manifolds are of dimension $2k+1$ at most. Since the codimension of C_γ is $2k+2$, these manifolds can be assumed to avoid C_γ.

Now, for $V\in\mathcal{F}$ and $w_{2k+1}\in\mathcal{A}_{2k+1}$, we denote by

$$f_\lambda\big(B_2(X)\big).\, w_{2k+1} := f_\lambda\big(B_2(X)\big).\, W_s\big(w_{2k+1}\big), \tag{4.15}$$

the intersection number of $f_\lambda\big(B_2(X)\big)$ and $W_s\big(w_{2k+1}\big)$. We notice that the dimension of $f_\lambda\big(B_2(X)\big)$ is equal to $2k+1$ and the codimension of $W_s\big(w_{2k+1}\big)$ is $2k+1$. Then the intersection number defined in (4.15) is well defined since V is transverse to $f_\lambda\big(B_2(X)\big)$ outside $f_\lambda\big(B_1(X)\big)$, which cannot dominate critical points of Morse index $2k+1$. Furthermore, the closure of $W_s\big(w_{2k+1})\big)$ only adds to $W_s\big(w_{2k+1})\big)$, the stable manifolds of critical points of Morse index larger than equal to $2k+2$. Since $f_\lambda\big(B_2(X)\big)$ is of codimension $2k+1$, these manifolds can be assumed to avoid it.

Lastly, for each $V \in \mathcal{F}$, we set

$$I(V) = \tau - \sum_{w_{2k+1} \in \mathcal{A}_{2k+1}} \left(\left((y_0, w_{2k+1})_\infty . \, C_\gamma \right) \left(f_\lambda(B_2(X)) . \, w_{2k+1} \right) \right). \tag{4.16}$$

We notice that (4.16) was first introduced by Bahri in [5], where he proved that $I(V)$ is independent on $V \in \mathcal{F}$. Precisely, he showed that $I(V) = 0$ for each $V \in \mathcal{F}$ for the scalar curvature problem on $\mathbb{S}^n$ with $n \geq 7$. We will prove that the same holds for the mean curvature in $\mathbb{B}^n$.

In order to prove Theorem 1.1, we have to start by proving the following results.

Proposition 4.7. *Let $n \geq 7$. Let z_1, z_2 in X satisfying $(\mathbf{R_1})$, such that $-\Delta H(z_i) > 0$ for $i = 1, 2$, and $z_1 \neq z_2$. If we assume*

(1) $J\left(\dfrac{\tilde{\delta}_{(z_1,\lambda)}}{H(z_1)^{\frac{n-2}{2}}} + \dfrac{\tilde{\delta}_{(z_2,\lambda)}}{H(z_2)^{\frac{n-2}{2}}} \right) \geq C_\infty(z_1, z_2) + \varepsilon,$

(2) $\dfrac{\partial}{\partial \mu} J\left(\dfrac{\tilde{\delta}_{(z_1,\mu)}}{H(z_1)^{\frac{n-2}{2}}} + \dfrac{\tilde{\delta}_{(z_2,\mu)}}{H(z_2)^{\frac{n-2}{2}}} \right)\Bigg|_{\mu=\lambda} < 0,$

then $I(V) = 0$ for any $V \in \mathcal{F}$.

Proof. First, we notice that a topological argument used in [5] (page 358-369), show that $I(V)$ is constant for any $V \in \mathcal{F}$. We want to extends this argument to our framework. Let $\varepsilon > 0$ and $H_\varepsilon = 1 + \varepsilon H$. Let J_ε be the associated variational problem, that is as ε tend to zero, j_ε tends to J_0 in the C^1 sense, where J_0 is the functional defined replacing H by 1 in the expansion of J.
Using Proposition 3.1, we derive that

$$J_\varepsilon\left(\alpha_1 \tilde{\delta}_{(a_1,\lambda)} + \alpha_2 \tilde{\delta}_{(a_2,\lambda)} \right) \leq 2 S_n^{1/n-1} \left(1 - \frac{c}{\lambda^{n-2}} + O(\varepsilon) \right), \tag{4.17}$$

where c is independent of ε. Thus, we can assume that ε is so small that all critical points at infinity of J_ε (of two masses or more) are above $f_\lambda\big(B_2(X)\big)$ since $2S_n^{1/n-1}$ is the level to which a critical point at infinity of two masses of H_ε converges when ε tends to zero. On the other hand, for ε small enough, $C_\gamma(z_1, z_2)$ is above $(2S_n^{1/n-1} + \gamma/2)$. Hence,

$$W_u^\varepsilon\Big(f_\lambda(B_2(X)) \Big) . \, C_\gamma\big(z_1, z_2\big) = 0. \tag{4.18}$$

Observe that, decreasing λ we can complete an homotopy of $f_\lambda(B_2(X))$ that increase the intersection of any masses and therefore, remains below $C_\gamma(z_1, z_2)$. Thus, for each $\mu \in [1, \lambda]$, we have

$$W_u^\varepsilon\Big(f_\mu(B_2(X)) \Big) . \, C_\gamma\big(z_1, z_2\big) = 0. \tag{4.19}$$

Regarding (4.16), in order to compute $I(V)$, we have to compute $f_\lambda(B_2(X)) . \, w_{2k+1}$ for any $w_{2k+1} \in \mathcal{A}_{2k+1}$. Let

$$F = \bigcup_{\mu=1}^{\lambda} f_\mu\big(B_2(X)\big).$$

We can assume that F is a compact manifold in dimension $2k+2$. The singularity of F is $\cup_{\mu=1}^{\lambda} f_\mu(B_1(X))$ which is of dimension less than $k+1$. This singularity cannot dominate w_{2k+1}. We deduce that $F \cap \overline{W}_s(w_{2k+1})$ is a compact manifold of dimension one. Thus, the cardinal of $\partial\Big(F \cap \overline{W}_s(w_{2k+1})\Big)$ is equal to zero, where ∂ is the same operator defined in Milnor [30]. Observe that

$$\partial F = f_1\big(B_2(X)\big) + f_\lambda\big(B_2(X)\big). \tag{4.20}$$

It follows that

$$f_\lambda\big(B_2(X)\big).\, w_{2k+1} = f_1\big(B_2(X)\big).\, w_{2k+1} + F.\, \partial^{-1}\big(W_s(w_{2k+1})\big). \tag{4.21}$$

Along this homotopy, the trace of $f_\mu\big(B_2(X)\big)$ might intersect $\partial^{-1}\big(W_s(w_{2k+1})\big)$ for some values, where $\partial^{-1}\big(W_s(w_{2k+1})\big)$ is made of stable manifolds of critical points of Morse index $2k+2$. Therefore, The topological argument of [5] (see pages 358-369) applies. Thus the invariant $I(V)$ remains unchanged for $V \in \mathcal{F}$. For $\mu = 1$, at the end of homotopy, $B_2(X)$ is mapped onto a single function and then $f_1\big(B_2(X)\big).\, w_{2k+1} = 0$. Hence, $I(V)$ at the end of the homotopy is equal to zero and the result follows. □

Proof of Theorem 1.1 Arguing by contradiction, we may assume that J has no critical points in Σ^+. It follows from Proposition 4.7 that $\mathcal{A}_{2k+1} = \emptyset$. Therefore, combining (4.16), Proposition 4.7 and the fact that $\tau = 1$, we derive a contradiction. The proof of our result is thereby completed. □

4.3. **Proof of Theorem 1.2.** Arguing by contradiction, we assume that (1.3) has no solution. Let

$$u = \alpha\tilde{\delta}_{(y_0,\lambda)} + (1-\alpha)\tilde{\delta}_{(x,\lambda)} \in f_\lambda(C_{y_0}(X)),$$

the action of the flow of the pseudo-gradient defined in the Proposition 4.1, is essentially on α. Three case may occur,

If $\alpha < 1/2$, then u goes to $\overline{W_u(y_0)_\infty} \equiv \{y_0\}$.

If $\alpha > 1/2$, then u goes to $\overline{W_u(y_{i_0})_\infty} \equiv X_\infty$.

If $\alpha = 1/2$, observe that only x can move and then y_0 remains one of the points of concentration of u and x goes to $W_s(y_i)$, where y_i is a critical point of H dominated by y_{i_0} (see definition 4.6 below). Thus, u goes to

$$W_u(y_0, y_{i_0}) \cup \left[\bigcup_{y_i \in X\setminus\{y_{i_0}\}} W_u(y_0, y_i)\right].$$

We denote $\sigma = \bigcup_{y_i \in X\setminus\{y_{i_0}\}} W_u(y_0, y_i)$, where σ is a manifold in dimension at most $k_0 - 1$ ($k_0 = 2 - ind(H, y_{i_0})$). Therefore, $X \cup W_u(y_0, y_{i_0}) \cup \sigma$ contains a strong retract of $f_\lambda(C_{y_0}(X))$. Since $\mu = 0$, this strong retract does not intersect $W_u(y_0, y_{i_0})$, and thus it is contained in $X \cup \sigma$. It can be written as $X \cup Z$ where $Z \subset \sigma$ is a stratified

set (see [8], Proposition 7.24), and hence of dimension at most $k_0 - 1$. Therefore, $H_*(X \cup Z) = 0$ for all $* \in \mathbb{N}^*$, since $f_\lambda(C_{y_0}(X))$ is a contractible set.
Using the exact homology sequence of $(X \cup Z, X)$, we have

$$... \longrightarrow H_{k_0+1}(X \cup Z) \longrightarrow H_{k_0+1}(X \cup Z, X) \longrightarrow H_{k_0}(X) \longrightarrow H_{k_0}(X \cup Z) \longrightarrow ...$$

Since $H_*(X \cup Z) = 0$ for all $* \in \mathbb{N}^*$, then $H_{k_0}(X) = H_{k_0+1}(X \cup Z, X)$. If $(X \cup Z, X)$ is a stratified set of dimension at most k_0, then $H_{k_0+1}(X \cup Z, X) = 0$ and therefore $H_{k_0}(X) = 0$ a contradiction. Then the result follows. $\square$

5. Characterization of critical points at infinity on three dimensional ball

We give here, the characterization of the critical points in the case of three dimensional ball. First, we claim that there is no critical points in $V(p, \varepsilon)$ with $p \geq 2$. Second, we construct in $V(1, \varepsilon)$ a special pseudogradient W for which the Palais-Smaile condition is satisfied along the decreasing flow lines, as long as these flow lines do not enter in the neighborhood of critical points y_i of H such that $\Delta H(y_i) < 0$. Finally, if w is a nondegenerate solution of (1.3), we prove that there is no critical points at infinity in $V(p, \varepsilon, w)$.

Proposition 5.1. *Let $n = 3$. For $\varepsilon > 0$ small enough and $u = \sum_{i=1}^p \alpha_i \tilde{\delta}_{(a_i,\lambda_i)} \in V(p, \varepsilon)$, we have the following expansion*

$$J(u) = S_3^{\frac{1}{2}} \frac{\Sigma_{i=1}^p \alpha_i^2}{\left(\Sigma_{i=1}^p \alpha_i^4 H(a_i) \right)^{\frac{1}{2}}} \Bigg[1 - \frac{1}{2S_3 \left(\sum_{i=1}^p \alpha_i^4 H(a_i) \right)} c_3 \sum_{i=1}^p \alpha_i^4 \frac{\Delta H(a_i)}{\lambda_i^2} \log \lambda_i$$
$$- \frac{c_1}{S_3 \left(\sum_{i=1}^p \alpha_i^2 \right)} \sum_{i \neq j} \alpha_i \alpha_j \varepsilon_{ij} + f(v) + Q(v, v) + o\left(\sum_{i \neq j} \varepsilon_{ij} + \frac{\log \lambda_i}{\lambda_i^2} + \|v\|^3 \right) \Bigg],$$

where

$$Q(v, v) = \frac{1}{S_3 \left(\sum_{i=1}^p \alpha_i^2 \right)} \|v\|^2 - \frac{3}{S_3 \left(\sum_{i=1}^p \alpha_i^4 H(a_i) \right)} \int_{\mathbb{S}^2} H \left(\sum_{i=1}^p \alpha_i \tilde{\delta}_i \right)^2 v^2,$$

$$f(v) = -\frac{2}{S_3 \left(\sum_{i=1}^p \alpha_i^4 H(a_i) \right)} \int_{\mathbb{S}^2} H \left(\sum_{i=1}^p \alpha_i \tilde{\delta}_i \right)^3 v,$$

$$S_3 = \bar{c}^4 \int_{\mathbb{R}^2} \frac{dx}{\left(1 + |x|^2\right)^2}, \qquad c_1 = \bar{c}^4 \int_{\mathbb{R}^2} \frac{dx}{\left(1 + |x|^2\right)^{\frac{3}{2}}}, \quad c_3 = \bar{c}^4 \frac{\pi}{2}, \quad c_4 = \frac{\bar{c}^4}{4} \int_{\mathbb{R}^2} \frac{|x|^\beta dx}{\left(1 + |x|^2\right)^2}$$

Proof. Proposition 5.1 is a particular case of Proposition 3.1 for $n = 3$, so we omit its proof here. □

Proposition 5.2. *There is a C^1-map which to each (α, a, λ) such that For any $u = \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \in V(p, \varepsilon)$, associates a unique $\bar{v} = \bar{v}(\alpha, a, \lambda)$ satisfying*

$$J\left(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_i + \bar{v}\right) = \min\left\{ J\left(\sum_{i=1}^{p} \alpha_i \tilde{\delta}_i + v\right) v \text{ satisfies } (V_0)\right\}$$

Moreover, we have the following estimates:

$$\|\bar{v}\| \leq c\left[\sum_{i=1}^{p}\left(\frac{|\nabla H(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right) + \sum_{i\neq j} \varepsilon_{ij} \log \varepsilon_{ij}^{-1/2}\right].$$

Proof. The proof of Proposition 5.2 is similar to the proof of Proposition 3.6 for $n = 3$, so we will omit it. □

Proposition 5.3. *Let $n = 3$. For $p \geq 2$, there exists a pseudo-gradient W so that; there is a constant $c > 0$ independent of u such that,*

(1) $< -\nabla J(u), W > \geq c\left(\sum_{i\neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{|\nabla H(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right)$

(2) $< -\nabla J(u + \bar{v}), W + \frac{\partial \bar{v}}{\partial(\alpha_i, a_i, \lambda_i)}(W) > \geq c\left(\sum_{i\neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{|\nabla H(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right)$

for all $u = \sum_{i=1}^{p} \alpha_i \tilde{\delta}_i \in V(p, \varepsilon)$. Furthermore, $|W|$ is bounded and the λ_i's decrease along the flow lines.

Proof. For sake of simplicity, we can assume without any inconvenient that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p$. Let $N = \{i/\lambda_i |\nabla H(a_i)| \geq 1\}$ and we set

$$Z_1 = \sum_{i=2}^{p} 2^i \alpha_i \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i}, \qquad Z_2 = \sum_{i\in N} \frac{1}{\lambda_i} \frac{\partial \tilde{\delta}_i}{\partial a_i} \frac{\nabla H(a_i)}{|\nabla H(a_i)|}.$$

Using Propositions 3.4 and 3.5, we derive that

$$< -\nabla J(u), Z_1 > \geq c \sum_{i\neq j} \varepsilon_{ij} + O\left(\sum_{i=2}^{p} \frac{\log \lambda_i}{\lambda_i^2}\right) + o\left(\frac{\log \lambda_1}{\lambda_1^2}\right) + o\left(\sum_{i\neq j} \varepsilon_{ij}\right) \tag{5.1}$$

$$< -\nabla J(u), Z_2 > \geq c \sum_{i\in N} \frac{|\nabla H(a_i)|}{\lambda_i} + O\left(\sum_{i\neq j} \varepsilon_{ij}\right) + O\left(\sum_{i\in N} \frac{1}{\lambda_i^2}\right) \tag{5.2}$$

For any critical point y of H, we set $\mu > 0$ such that if $d(a, y) \leq 2\mu$ then $|\Delta H(a)| > c > 0$ Two cases may occur,

Case1. $\lambda_2 \leq \lambda_1^2$ or $d(a_1, y) > \mu$ for any critical point y of H.

In this case, we set $W_1 = MZ_1 + Z_2$ where M is a large constant. Observe that in the case where $d(a_1, y) > \mu$, we can appear $1/\lambda_1$ in the lower bound of (5.2) and therefore all the $1/\lambda_i$'s. Combining (5.1) and (5.2), we derive

$$< -\nabla J(u), W_1 > \geq c\Bigg(\sum_{i=1}^{p}\Bigg(\frac{|\nabla H(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\Bigg) + \sum_{i\neq j}\varepsilon_{ij}\Bigg) \tag{5.3}$$

In the other case, that is $\lambda_2 \leq \lambda_1^2$, we can prove easily that $1/\lambda_1^2 = o(\varepsilon_{12})$. Therefore we can also obtain (5.3) in this case.

Case2. $\lambda_2 \geq \lambda_1^2$ and $d(a_1, y) \leq 2\mu$ for any critical point y of H.

In this case, we set Z_3=sign$\Big(-\Delta H(y)\Big)\lambda_1 \dfrac{\partial\tilde{\delta}_1}{\partial\lambda_1}$, that is, we increase λ_1 if $-\Delta H(y) > 0$ otherwise we decrease it. Observe that

$$< -\nabla J(u), Z_3 > \geq c\Bigg(\frac{1}{\lambda_1^2} + O\Bigg(\sum_{i\neq j}\varepsilon_{ij}\Bigg)\Bigg).$$

We define $W_2 = MZ_1 + Z_3 + mZ_2$ where M is a large constant and m is a small constant, we derive that

$$\begin{aligned} < -\nabla J(u), W_2 > &\geq cM\sum_{i\neq j}\varepsilon_{ij} + o(1/\lambda_1^2) + \frac{c}{\lambda_1^2} + O\Bigg(\sum_{i\neq j}\varepsilon_{ij}\Bigg) + m\sum_{i\in N}\frac{|\nabla H(a_i)|}{\lambda_i} \\ &\quad + O\Bigg(m\sum_{i\neq j}\varepsilon_{ij}\Bigg) + O\Big(\frac{m}{\lambda_1^2}\Big) \\ &\geq c\Bigg(\sum_{i=1}^{p}\Bigg(\frac{|\nabla H(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\Bigg) + \sum_{i\neq j}\varepsilon_{ij}\Bigg) \end{aligned}$$

The pseudo-gradient W will be built as a convex combination of W_1 and W_2 and W will satisfy the first estimate. Arguing as in Appendix B of [8], we derive that

$$< -\nabla J(u+\overline{v}), W + \frac{\partial\overline{v}}{\partial(\alpha_i, a_i, \lambda_i)}(W) > \geq$$

$$< -\nabla J(u), W > + o\Bigg(\sum_{i=1}^{p}\Bigg(\frac{|\nabla H(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\Bigg) + \sum_{i\neq j}\varepsilon_{ij}\Bigg) \tag{5.4}$$

and therefore the Proposition 5.3 follows under (5.4). □

Proposition 5.4. *Let $n = 3$. For any $u = \alpha\tilde{\delta}_{(a,\lambda)} + v \in V(1, \varepsilon)$, There exists a pseudo-gradient W so that; there is a constant $c > 0$ independent of u such that,*

(1) $< -\nabla J(u), W > \geq c\Bigg(\dfrac{|\nabla H(a)|}{\lambda} + \dfrac{1}{\lambda^2}\Bigg)$
(2) $< -\nabla J(u+\overline{v}), W + \dfrac{\partial\overline{v}}{\partial(\alpha, a, \lambda)}(W) > \geq c\Bigg(\dfrac{|\nabla H(a)|}{\lambda} + \dfrac{1}{\lambda^2}\Bigg)$
(3) *W is bounded*
(4) *the only region where λ increases along the flow lines of W is the region where a is near a critical point y of H with, $-\Delta H(y) > 0$.*

Proof. Let $\rho > 0$ be such that, for any critical point y of H, if $d(x, y) \leq 2\rho$ then $|\Delta H(x)| > c > 0$. Three case may occur,

case1. $d(a, y) > \rho$ for any critical point y. In this case we have, $|\nabla H(a)| > c > 0$. Set

$$W_1 = \frac{1}{\lambda}\frac{\partial\tilde{\delta}}{\partial a}\frac{\nabla H(a)}{|\nabla H(a)|}.$$

From Proposition 3.5, we have

$$< -\nabla J(u), W_1 >\geq c\frac{|\nabla H(a)|}{\lambda} + O\Big(\frac{1}{\lambda^2}\Big) \geq c\frac{|\nabla H(a)|}{\lambda} + \frac{c}{\lambda^2}$$

case2. $d(a, y) \leq 2\rho$ where y is a critical point of H with $-\Delta H(y) < 0$. Set

$$W_2 = -\lambda\frac{\partial\tilde{\delta}}{\partial\lambda} + m\varphi(\lambda|\nabla H(a)|)W_1$$

where, m is a small constant and φ is a C^∞ function which satisfies $\varphi(t) = 1$ if $t \geq 2$ and $\varphi(t) = 0$ if $t \leq 1$. Using Propositions 3.4 and 3.5, we derive that,

$$\begin{aligned} < -\nabla J(u), W_2 > &\geq \frac{c}{\lambda^2} + cm\left(\frac{|\nabla H(a)|}{\lambda} + O\Big(\frac{1}{\lambda^2}\Big)\right) \\ &\geq c\frac{|\nabla H(a)|}{\lambda} + \frac{c}{\lambda^2}. \end{aligned}$$

case3. $d(a, y) \leq 2\rho$ where y is a critical point of H with $-\Delta H(y) > 0$. Set

$$W_3 = \lambda\frac{\partial\tilde{\delta}}{\partial\lambda} + m\varphi\Big(\lambda|\nabla H(a)|\Big)W_1$$

We obtain the same equality as in case2.
Hence, W will be built as a convex combination of W_1, W_2 and W_3. The proof of (1) is thereby completed. Claims (3) and (4) can be derived from the definition of W. The claim (2) can be obtained using the claim (1.3) and arguing as in [5] and [8]. □

Using Proposition 5.4 and Corollary 3.8, and arguing as in [5] and [8], we can deduce the following two Propositions.

Proposition 5.5. *On $\mathbb{B}^3$ the only critical point at infinity of J in $V(1, \varepsilon)$ for ε small enough correspond to $\tilde{\delta}_{y,\infty}$, where y is a critical point of H such that $-\Delta H(y) > 0$. Moreover, such critical point at infinity has a Morse index equal to $2 - ind(H, y)$.*

6. Prescribing mean curvature on three dimensional ball

This section is devoted to the proof of the existence and multiplicity results concerning three dimensional ball. Before providing the proof of Theorem 1.5, we state the following Lemma, the proof of which is very similar to that of [6].

Lemma 6.1. *For $u = \alpha_1\tilde{\delta}_{(a_1,\lambda)} + \alpha_2\tilde{\delta}_{(a_2,\lambda)}$, we have*

$$J(u) \leq \left(S_3\Big(\frac{1}{H(y_0)} + \frac{1}{H(x)}\Big)\right)^{1/2}\Big(1 + o(1)\Big) := C_1(a_1, a_2)(1 + o(1)).$$

Proof of Theorem 1.5 Arguing by contradiction, we suppose that J has no critical points in Σ^+. Let

$$C_\infty = S_3^{1/2}\left(\frac{1}{H(y_0)}+\frac{1}{H(y_1)}\right)^{1/2}.$$

Using Proposition 5.5 and the assumptions of Theorem 1.5, we derive that the only critical points at infinity of J under the level $C_1 = C_\infty + \varepsilon$ for ε small enough correspond to $\tilde{\delta}_{(y_0,\infty)}$ and $\tilde{\delta}_{(y_1,\infty)}$. The unstable manifolds at infinity $W_u(y_0)_\infty$ and $W_u(y_1)_\infty$ of these critical points can be described, by Proposition 5.5, as the product of $W_s(y_0)$ and $W_s(y_1)$ (for a pseudo-gradient of H) with $[A,+\infty[$ (the domain of the variable λ) for some sufficiently large positive number A. Let

$$X = \overline{W}_s(y_1) = W_s(y_0)\cup W_s(y_1).$$

Observe that X is a compact manifold without boundary of dimension k.
Since J has no critical points in Σ^+, it follows that $J_{C_1} = \{u\in\Sigma^+/\, J(u)\le C_1\}$ retracts by deformation on $X_\infty = W_u(y_0)_\infty \cup W_u(y_1)_\infty$ (see section 7 and 8 of [7]), which can be parameterized by $X\times[A,+\infty[$.

On the other hand, X_∞ is contractible in J_{C_1}. Indeed, let

$$\begin{aligned} h : [0,1]\times X_\infty &\longrightarrow \Sigma^+\\ (t,x,\lambda) &\longmapsto \frac{t\tilde{\delta}_{(y_0,\lambda)}+(1-t)\tilde{\delta}_{(x,\lambda)}}{\left\|t\tilde{\delta}_{(y_0,\lambda)}+(1-t)\tilde{\delta}_{(x,\lambda)}\right\|}\end{aligned}$$

Observe that h is continuous, $h(0,x,\lambda) = \dfrac{1}{S_3}\tilde{\delta}_{(x,\lambda)} \in X_\infty$ and $h(1,x,\lambda) = \dfrac{1}{S_3}\tilde{\delta}_{(y_0,\lambda)}$. Furthermore, using Lemma 6.1, we have

$$J\left(\frac{t\tilde{\delta}_{(y_0,\lambda)}+(1-t)\tilde{\delta}_{(x,\lambda)}}{\left\|t\tilde{\delta}_{(y_0,\lambda)}+(1-t)\tilde{\delta}_{(x,\lambda)}\right\|}\right) \le \left(S_3\Big(\frac{1}{H(y_0)}+\frac{1}{H(x)}\Big)\right)^{1/2}\Big(1+o(1)\Big).$$

Using the fact that $H(x)\ge H(y_1)$ for each $x\in X$, we derive that the contraction h is performed under the level C_1. Therefore, X_∞ is contractible leading to the contractibility of X, which is a contradiction since X is a manifold in dimension k without boundary. Hence (1.3) has a solution.

It remains to compute the Morse index of this solution. Using a dimension argument, since $h([0,1]\times X_\infty)$ is a manifold in dimension k +1, then the Morse index of the solution provided by Theorem 1.5 is less or equal than $k+1$. On the other hand, assume that the Morse index of the solution is less or equal than $k-1$. Perturbing J if necessary, we may assume that all the critical points of J are nondegenerate and have their Morse index $\le k-1$, and hence do not change the k-dimensional homology group of the level sets of J . Since X_∞ defines a nontrivial homology class in dimension k in $J_{C_\infty(y_1)+\varepsilon}$, but a trivial one in J_{C_1} , our result follows. □

Proof of Theorem 1.6 First, observe that we can assume that the solutions of (1.3) are of finite number, otherwise we are done. Let $\mathcal{H}$ denote the set of critical points of J.

Let $J^\beta := \{u \in \Sigma^+/ \ J(u) < \beta\}$. It is well established that if z is the only critical point of J in the set $J^{c+\varepsilon} \setminus J^{c-\varepsilon}$, assume that there are also no critical point at infinity, then $J^{c+\varepsilon}$ retracts by deformation on $J^{c-\varepsilon} \cup W_u(z)$, where $W_u(z)$ denotes the unstable manifold of the critical point.
Now, we claim that we have the same argument if z is a critical point at infinity in $J^{c+\varepsilon} \setminus J^{c-\varepsilon}$. Indeed, under the assumptions of Theorem 1.6, the critical points at infinity of J correspond to $\delta(y_j, \infty)$ where $y_j \in I$ and $j \in \{0, \dots, s\}$. In the neighborhood of such critical points at infinity, we have (see Corollary 3.8)

$$J\Big(\alpha \ \delta_{(a,\lambda)} + v\Big) = \frac{S_3^{1/2}}{H(\tilde{a})^{1/2}}\Big(1 - c\frac{\Delta H(y_j)\log\tilde{\lambda}}{\tilde{\lambda}^2}\Big) + \|V\|^2.$$

Observe that we have a critical point $\bar{y}$ in the a-variable. Therefore, the unstable manifold of (z_∞) can be defined as $W_u(\bar{y}) \times [A, \infty)$, where $W_u(\bar{y})$ is the unstable manifold of $\bar{y}$. We deduce that, $J^{c+\varepsilon}$ retracts by deformation on $J^{c-\varepsilon} \cup W_u(z_\infty)$.
Let c_0 be large such that $\mathcal{H} \cup I \subset J^{c_0}$. Without loss of generality, we assume that there exist $0 < c_1 < \cdots < c_s < c_0$ such that each level c_j contains only one element of $\mathcal{H} \cup I$. It follows from the above that J^{c_0} retracts by deformation on

$$\bigcup_{z \in \mathcal{H}} W_u(z) \cup \bigcup_{z_\infty \in I} W_u(z_\infty).$$

Now, using the Euler-Poincare characteristic (denoted by χ), it holds that

$$1 = \chi(J^{c_0}) = \sum_{z \in \mathcal{H}} (-1)^{m(z)} + \sum_{z_\infty \in I} (-1)^{m(z_\infty)},$$

where $m(z)$ (res. $m(z_\infty)$) denotes the Morse index of z (res. z_∞). Hence

$$\Big|1 - \sum_{z_\infty \in I} (-1)^{m(z_\infty)}\Big| \le \Big|\sum_{z \in \mathcal{H}} (-1)^{m(z)}\Big| \le \sharp\mathcal{H},$$

the proof of Theorem 1.6 is thereby completed. □

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