Riemann Surfaces

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Contents

1 Coverings and fundamental group

1.1 Surfaces

Definition 1.1.1. A **surface** is a topological space S together with a family $(U_i, h_i)_{i\in I}$, where

(a) every $U_i \subset S$ is an open subset,

(b) $h_i : U_i \to \mathbb{R}^2$ is a homeomorphism to an open subset $h_i(U_i) \subset \mathbb{R}^2$,

(c) one has $S = \bigcup_{i\in I} U_i$,

(d) S is a separable Hausdorff space.

Here S being separable means that S has a countable dense subset.

Every h_i is called a **chart** of the surface, the family $(U_i, h_i)_{i\in I}$ is called an **atlas** of the surface.

Examples 1.1.2. (a) $\mathbb{R}^2$ is a surface with the identity map as only chart. The same holds for any open subset of $\mathbb{R}^2$.

(b) The **Riemann number sphere** $\widehat{\mathbb{C}} = \mathbb{C} \sqcup \{\infty\}$ is a surface with the charts $h_1 : \mathbb{C} \to \mathbb{C} \cong \mathbb{R}^2$, $h_1(z) = z$ and $h_2 : \mathbb{C}^\times \sqcup \{\infty\} \to \mathbb{C}$, $h_2(z) = \frac{1}{z}$.

(c) The **torus** $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ is a surface with the charts

$$\begin{aligned} h_{a,b} : U_{a,b} &\to \mathbb{R}^2, \\ (x, y) &\mapsto (x, y), \end{aligned}$$

where $U_{a,b} = \{(x, y) : a < x < a + 1,\ b < y < b + 1\}$ and $(a, b) \in \mathbb{R}^2$.

(d) The condition that S be a Hausdorff space, does not follow from the other axioms as the following example shows: Let $S = \mathbb{C}^\times \sqcup \{a, b\}$, where a, b are two new elements. We call a subset $U \subset S$ *open*, if

(a) $U \cap \mathbb{C}$ is open in $\mathbb{C}$ and

(b) if $a \in U$ or $b \in U$, then there is $\varepsilon > 0$, such that U contains the open disk around zero of radius ε.

(e) The condition of separability does not follows from the other axioms as the following example shows: To every countable ordinal number α we attach a "surface" F_α in a way that $\alpha < \beta \implies F_\alpha \subset F_\beta$.

(i) We set $F_0 = \mathbb{C}$.

(ii) We set $F_{\alpha+1} = F_\alpha \sqcup \{\mathrm{Re}(z) \geq 0\}$, where the topology is defined as follows: we fix a homeomorphism $F_\alpha \xrightarrow{\cong} \mathbb{C} \xrightarrow{\cong} L$, where $L = \{z : \mathrm{Re}(z) < 0\}$ is the left half-plane. Then we give $F_{\alpha+1} \cong \mathbb{C}$ the usual topology of $\mathbb{C}$.

(iii) If λ is a countable limit number, then $F_\lambda = \bigcup_{\alpha<\lambda} F_\alpha$ also is homeomorphic to $\mathbb{C}$.

Let c be the smallest non-countable ordinal number, then

$$F = \bigcup_{\alpha<c} F_\alpha$$

is a topological space which atisfies all axioms of a surface, except for separabilitry.

Definition 1.1.3. Let $p \in S$. A chart (U, h) is called **chart around** p, if $h(p) = 0$.

Remark 1.1.4. Note that a surface S is locally path-connected, i.e., every point $p \in S$ has a path-connected open neighbourhood. This has the following consequence:

If S is connected, then it is path-connected.

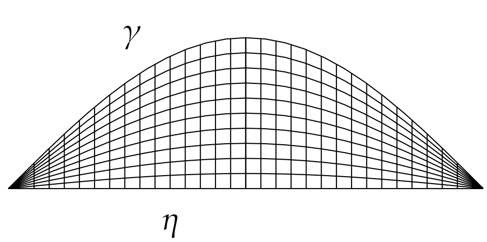
1.2 Paths

Definition 1.2.1. A continuous map $\gamma : [0,1] \to S$ is called a **path** from $\gamma(0)$ to $\gamma(1)$. The space S is called **path-connected**, if to any two points $x, y \in S$ there is a connecting path.

Definition 1.2.2. Two paths $\gamma, \eta : I = [0,1] \to S$ are called **homotopic** (with fixed ends), if there is a continuous map $h : I \times I \to S$ with

- $h(s,0) = \gamma(0)$, $h(s,1) = \gamma(1)$ fuer alle $s \in I$, sowie
- $h(0,t) = \gamma(t)$ and $h(1,t) = \eta(t)$ fuer alle $t \in I$.

If γ and η are homotopic, they have the same ends, so $\gamma(0) = \eta(0)$ and $\gamma(1) = \eta(1)$.



homotopy of paths

Lemma 1.2.3. *Homotopy of paths is an equivalence relation.*

Proof. The relation $\gamma \sim \gamma$ is clear. Further it is clear that $\gamma \sim \eta$ implies $\eta \sim \gamma$, since if h is the first homotopy then $\tilde{h}(s,t) = h(1-s,t)$ is the second. Finally assume $\gamma \sim \eta$ and $\eta \sim \tau$ with homotopies h and h', then $\tilde{h}$, given by

$$\tilde{h}(s,t) = \begin{cases} h(2s,t) & 0 \le s \le \frac{1}{2}, \\ h'(2s-1,t) & \frac{1}{2}, s \le 1 \end{cases}$$

is a homotopy of γ and τ. □

Definition 1.2.4 (Composition of paths)**.** Let γ, η be paths with $\gamma(1) = \eta(0)$. Then the path $\gamma \cdot \eta$ is defined by

$$\gamma \cdot \eta(t) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2}, \\ \eta(2s-1) & \frac{1}{2} < t \le 1. \end{cases}$$

If γ is a path from x to y and η from y to z, then $\gamma \cdot \eta$ is a path from x to z.

Lemma 1.2.5. *Composition of paths is well-defined and associative on homotopy classes.*

Proof. The first assertion is that $\gamma \sim \gamma'$ and $\eta \sim \eta'$ implies that $\gamma \cdot \eta$ is homotopic to $\gamma' \cdot \eta'$. Constructing a homotopy to do this job is left to the readeras an exercise.

The second assertion means that $(\alpha \cdot \beta) \cdot \gamma$ is homotopic to $\alpha \cdot (\beta \cdot \gamma)$. These two paths have the same image, so the homotopy only has to perform a change of parameters. The homotopy

$$h(s,t) = \begin{cases} \alpha\left((2+2s)t\right) & 0 \le t \le \frac{1}{2+2s}, \\ \beta\left(4t-(1+s)\right) & \frac{1}{2+2s} < t \le \frac{1}{2+2s} + \frac{1}{4}, \\ \gamma\left((4-2s)t-(3-2s)\right) & \frac{1}{2+2s} + \frac{1}{4} < t \le 1 \end{cases}$$

does this. □

Definition 1.2.6. The path $\check{\gamma}(t) = \gamma(1-t)$ is called the **reverse path** of γ.

A path $c : [0,1] \to S$ with $c(0) = c(1)$ is called a **closed path**.

We write $\pi_1(S,a)$ for the set of all homotopy classes of closed paths with endpoint $a \in S$.

Proposition 1.2.7. *The set $\pi_1(S,a)$ is a group with composition of paths as multiplication.*

If S is connected, then for every $b \in S$ the group $\pi_1(S,b)$ is isomorphic with $\pi_1(S,a)$. The group $\pi_1(S,a)$ is called the **fundamental group** *of S in the basepoint a.*

Proof. The law of associativity is satisfied by Lemma 1.2.5. The neutral element is given by the constant path. The inverse to γ is given by the reverse path $\check{\gamma}$. If finally η is a path connecting a to b, then

$$\gamma \mapsto \eta \cdot \gamma \cdot \check{\eta}$$

is an isomorphism of the fundamental groups $\pi_1(S,a) \xrightarrow{\cong} \pi_1(S,b)$. □

Definition 1.2.8. A connected surface S is called **simply connected**, if $\pi_1(S,a) = \{1\}$ for one and therefore all $a \in S$.

Proposition 1.2.9. *If $S \subset \mathbb{C}$ is a star-shaped domain, then S is simply-connected.*

Proof. Let $a \in S$ be a point, from which one sees all of S. Then for a closed path γ with end-point a the map

$$h(s,t) = (1-s)\gamma(t) + sa$$

is a homotopy to the constant path. □

Proposition 1.2.10. *The number-sphere $\widehat{\mathbb{C}}$ is simply connected.*

Proof. every close path γ with endpoint 0 is homotopic to a path that doesn't meet $z = 1$. Now $\widehat{\mathbb{C}} \smallsetminus \{1\} \cong \mathbb{C}$ and $\mathbb{C}$ is star-shaped. □

1.3 Unramified coverings

Definition 1.3.1. An **unramified covering** of a surface S is a continuous map

$$\pi : E \to S,$$

such that there exists a discrete space $D \neq \emptyset$ and for every $x \in S$ an open neighborhood U, so that the diagram of continuous maps

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "\pi"'] & & U \times D \arrow[dl, "p_1"] \\ & S & \end{tikzcd}$$

commutes. Here the horizontal arrow is a homeomorphism and p_1 is the projection onto the first coordinate. The space $U \times D$ is equipped with the product topology.

As D is discrete, the product $U \times D$ is homeomorphic with the disjoint union $\bigsqcup_{d \in D} U$ of copies of U.

A neighborhood U of x with this property is called a **trivializing neighborhood**.

The **degree** of the covering is the cardinality of the set$D$.

Examples 1.3.2.

- The trivial covering$p_1 : S \times D \to S$.
- The map $p : \mathbb{C}^\times \to \mathbb{C}^\times$ given by $p(z) = z^2$ is a non-trivial unramified covering of degree 2.

Definition 1.3.3. A **homomorphism of coverings** or a **deck transformation** from $E \to S$ to $F \to S$ is a continuous map $\psi : E \to F$ such that the diagram

$$\begin{tikzcd} E \arrow[rr, "\psi"] \arrow[dr] & & F \arrow[dl] \\ & S & \end{tikzcd}$$

commutes.

Lemma 1.3.4 (Lifting of paths)**.** *Let $\pi : E \to S$ be a covering. Let $\gamma : [0,1] \to S$ be a continuous map and let $x_0 = \gamma(0)$. Then for every $y \in \pi^{-1}(x_0)$ there is exactly one path $\tilde{\gamma}_y : [0,1] \to E$ with $\tilde{\gamma}_y(0) = y$ and $\pi \circ \tilde{\gamma}_y = \gamma$.*

Every such $\tilde{\gamma}_y$ is called a **lifting** *of γ. The map $y \mapsto \tilde{\gamma}_y(1)$ is a bijection from $\pi^{-1}(x_0)$ to $\pi^{-1}(x_1)$, where $x_1 = \gamma(1)$.*

If γ and τ are paths in S with $\gamma \simeq \tau$, then one has $\tilde{\gamma}_y \simeq \tilde{\tau}_y$.

Proof. Let $y \in \pi^{-1}(x_0)$. Let $U \subset S$ be a trivializing neighborhood of x_0, i.e., $\pi^{-1}(U) \cong U \times D$. Then there is a neighborhood U_y of y such that $\pi|_{U_y}$ is a homeomorphism of U_y to U. Let $t_0 > 0$ such that $\gamma([0, t_0)) \subset U$, then γ can on $[0, t_0)$ be lifted in a unique way to a path $\tilde{\gamma}$ with $\tilde{\gamma}(0) = y$. Let $t_1 > 0$ be the supremum of all $t_0 > 0$ such that $\gamma|_{[0,t_0)}$ has a unique lifting $\tilde{\gamma}$ with $\tilde{\gamma}(0) = y$. Let V be a trivializing neighborhood of $\gamma(t_1)$. In this neighborhood we can extend this lifting, if $t_1 < 1$, so by maximality of t_1 we get $t_1 = 1$.

The map $y \mapsto \tilde{\gamma}_y(1)$ is bijective, because the corresponding map for $\check{\gamma}$ is an inverse.

Let $\gamma \simeq \tau$ in S and let $h : [0,1] \times [0,1] \to x$ ba a homotopy with fixed ends. As above one sees that h has a unique lifting to a continuous map $\tilde{h} : [0,1] \times [0,1] \to E$ with $\pi \circ \tilde{h} = h$ and $\tilde{h}(0,0) = y$. Then $\tilde{h}$ is the desired homotopy. □

Proposition 1.3.5. *Let S be connected. Let $\pi : E \to S$ be a covering. Then every connected component C of E is a covering of S. Further C is open in E and path-connected.*

So every covering decomposes into disjoint connected coverings. Every connected covering of a path-connected space is path-connected.

Proof. Let $y_0 \in E$ and let $x_0 = \pi(y_0)$. Let $W(y_0)$ be the set of all points $z \in E$, which can be joined with y_0 by a path in E. This is the **path component** of y_0. Let $D = \pi^{-1}(x_0) \cap W(y_0)$. Let $x_1 \in S$ and let γ be a path from x_0 to x_1. For every $z \in D$ there is exactly one lifting $\tilde{\gamma}_z$ of γ to E, which starts in z. Then $\tilde{\gamma}_z(1)$ lies in $W(y_0)$ and the map $z \mapsto \tilde{\gamma}_z(1)$ is a bijection from D to $D' = \pi^{-1}(x_1) \cap W(y_0)$. So $W(y_0)$ is a covering. Further the set $W(y_0)$ is a union of sets of the form $U \times D_C$, where U is a trivializing neighborhood. This means that $W(y_0)$ is open and E decomposes into path components, which are open, so they coincide with the connected components. □

1.4 The universal covering

Theorem 1.4.1. *Let S be a connected surface.*

(a) *$x_0 \in S$. Let $\pi_E : E \to S$ and $\pi_F : F \to S$ connected coverings and let E be simply connected. Choose some fixed $e \in \pi_E^{-1}(x_0)$ and $f \in \pi_F^{-1}(x_0)$. Then there is exactly one homomorphism of coverings $\psi : E \to F$ with $\psi(e) = f$. The map ψ is surjective.*

(b) *In particular, if S has a simply connected covering, then it is unique up to isomorphism. We call it the* **universal covering** *and write it as $\tilde{S} \to S$.*

Proof. (a) Let $y \in E$ and let γ_y be a path in E from e to y. Define

$$\psi(y) = (\widetilde{\pi_E \circ \gamma_y})_f(1).$$

This means, we project γ_y to S first, then lift it to F and evaluate at 1. Since E is simply connected, γ_y is uniquely determined by y up to homotopy with fixed ends. So the projection $\pi_E \circ \gamma_y$ is uniquely determined up to homotopy and so the lift is unique up to homotopy. So the map ψ is well-defined. As π_E and π_F are local homeomorphisms, ψ is continuous. Commutativity of the diagram is clear by definition. Uniqueness is clear as well, since a given homomorphism of coverings from E to F must map the path γ_y to the unique lifting of $\pi_E \circ \gamma_y$.

(b) If E and F are simply connected coverings, and if e, f are as above, then there are uniquely determined homeomorphisms $\psi : E \to F$ and $\phi : F \to E$ with $\psi(e) = f$ and

$\phi(f) = e$. Then $\phi \circ \psi$ is the uniquely determined homeomorphism $E \to E$ that maps e to e, so $\phi \circ \psi = \mathrm{Id}$. In the same way one gets $\psi \circ \phi = \mathrm{Id}$. $\square$

Theorem 1.4.2. *Let S be a connected surface. Then S has a universal covering*

$$p : \tilde{S} \to S.$$

The fundament group $\Gamma = \pi_1(S, x_0)$ acts by homeomorphisms on $\tilde{S}$, in a way that $S \cong \Gamma\backslash\tilde{S}$. For every connected covering $E \to S$ there is a subgroup Σ of Γ, such that $E \cong \Sigma\backslash\tilde{S}$.

Proof. We construct $\tilde{S}$ as follows: Choose a base point $x_0 \in S$ and define $\tilde{S}$ as the set of all paths τ with starting point x_0 modulo homotopy with fixed ends. The projection $p : \tilde{S} \to S$ is

$$p([\tau]) = \tau(1).$$

The fundamental group $\Gamma = \pi_1(S, x_0)$ acts on $\tilde{S}$ by

$$[\gamma][\tau] = [\gamma.\tau]$$

for $[\gamma] \in \Gamma$ and $[\tau] \in \tilde{S}$. We give $\tilde{S}$ a topology as follows: Let $[\tau] \in \tilde{S}$. Let $x = \tau(1)$ and U a simply connected open neighborhood of x in S. For every $y \in U$ choose a path σ_y from x to y, which completely lies in U. Then σ_y is uniquely determined up to homotopy with fixed ends. Let

$$\tilde{U} = \left\{[\tau.\sigma_y] : y \in U\right\}$$

Then $p|_{\tilde{U}}$ is a bijection $\tilde{U} \to U$. On $\tilde{U}$ we instal the topology induced by this bijection. Finally, on $\tilde{S}$ we put the topology induced by all inclusion maps $\tilde{U} \hookrightarrow \tilde{S}$, where U runs through the set of all simply connected open subsets of S. We now show: If $\gamma \in \Gamma$ and $\gamma\tilde{U} \cap \tilde{U} \neq \emptyset$, then $\gamma = 1$. For this let $\gamma\tilde{U} \cap \tilde{U} \neq \emptyset$. Then there are $y, z \in U$ with $\gamma.\tau.\sigma_y \simeq \tau.\sigma_z$. Evaluating at 1, one sees $y = z$, so $\gamma.\tau.\sigma_y \simeq \tau.\sigma_y$. Therefore

$$x_0 \simeq \gamma.\tau.\sigma_y.\check{\sigma}_y.\check{\tau}.$$

We have $\sigma_y.\check{\sigma}_y \simeq x_0$ and so $\tau.\sigma_y.\check{\sigma}_y.\check{\tau} \simeq x_0$. This yields $\gamma \simeq x_0$, which means that γ represents the neutral element of Γ. Next we show

$$p^{-1}(U) = \bigsqcup_{\gamma \in \Gamma} \gamma\tilde{U} \;\cong\; \tilde{U} \times \Gamma \;\cong\; U \times \Gamma.$$

For this let $[\eta] \in p^{-1}(U)$, so $\eta(1) = y \in U$. Then $\gamma = [\eta.\check{\sigma}_y.\check{\tau}] \in \Gamma$ and one has $[\eta] = \gamma[\tau.\sigma_y] \in \gamma\tilde{U}$. Therefore p is a covering.

The space $\tilde{S}$ is path-connected, since for each $[\tau] \in \tilde{S}$ there is a path σ in $\tilde{S}$, which connects $[\tau]$ to the constant path:

$$\sigma(s) = [t \mapsto \tau((1-s)t)].$$

Let $\pi : E \to S$ be a connected covering. Then E is path-connected. Choose $f \in \pi^{-1}(x_0)$. Define $\eta : \tilde{S} \to E$ by

$$\eta([\tau]) = \tilde{\tau}_f(1),$$

where $\tilde{\tau}_f$ is the unique lift of the path τ to E with $\tilde{\tau}_f(0) = f$. Since $\pi \circ \tilde{\tau}_f = \tau$, the diagram

$$\begin{tikzcd} \tilde{S} \arrow[dr, "\eta"] \arrow[dd] & \\ & E \arrow[dl, "\pi"] \\ S & \end{tikzcd}$$

commutes. To see that the map η is surjective, let $e \in E$ and let τ be a path in E from f to e. Since the lift is unique, one has $e = \eta([\pi \circ \tau])$. Therefore η is a covering. Let

$$\Sigma = \left\{\lambda \in \tilde{S} : \eta(\lambda) = f\right\}$$

Then $\Sigma \subset \Gamma$ and η induces a homeomorphism $\Sigma\backslash\tilde{S} \to E$.

It remains to show that $\tilde{S}$ is simply connected. For this let σ be a closed path in $\tilde{S}$ with starting point y_0. We may assume that y_0 is the class of the constant path in S of value $x_0 \in S$. Then σ is the unique lifting of the path $p \circ \sigma$, which starts at y_0. Then the class of $p \circ \sigma$ is an element of $\tilde{S}$ which projects to x_0. Let τ_t be the path $s \mapsto p \circ \sigma(st)$. Then $t \mapsto [\tau_t]$ is a path which connects y_0 to $[p \circ \sigma]$. This path $T : t \mapsto [\tau_t]$ is a lift of $p \circ \sigma$, with $T(0) = y_0$, so by uniqueness $T = \sigma$. So $y_0 = \sigma(1) = T(1) = [\tau_1] = [p \circ \sigma]$. This means that $p \circ \sigma$ is homotopic to the constant path and this homotopy lifts to a homotopy of σ to the constant path in $\tilde{S}$. □

Definition 1.4.3. A group action of Γ on a set M is a **free action** or a **fixed-point free action**, if for every $m \in M$ and every $\gamma \in \Gamma$ one has

$$\gamma m = m \quad \Rightarrow \quad \gamma = 1.$$

So an action is free, if all stabilizer groups

$$\Gamma_m = \{\gamma \in \Gamma : \gamma m = m\}$$

are trivial.

Definition 1.4.4. Let Y be a topological space. A group action of Γ on S is a **discontinuous action**, if every point $y \in Y$ has an open neighborhood U such that

$$\gamma U \cap U \neq \emptyset \quad \to \quad \gamma = 1.$$

If an action is discontinuous, it is free.

Definition 1.4.5. A group action of Γ on a topological space S is a **continuous action**, if for every $\gamma \in \Gamma$ the map $S \to S$, $x \mapsto \gamma x$ is continuous.

Lemma 1.4.6. *Let G be a finite group which acts continuously and freely on a metric space. Then G acts discontinuously.*

Proof. **Assume**, G does not act diskontinuously. Then there exists a point $x \in S$ such that for every open set $U_n = B_{1/n}(x)$, $n \in \mathbb{N}$ there is a $g_n \in G$ with $g_n \neq 1$ and $g_n U_n \cap U_n \neq \emptyset$. Since G is finite, we can assume $g_n = g$ for some $g \in G$. This means that for every n there is $x_n, y_n \in U_n$ with $x_n = g y_n$. The sequences x_n and y_n both converge to x, so by continuity we get $gx = x$. **Contradiction!** □

Definition 1.4.7. A group Γ acts **transitively** on a set M, if M consists of a single G-orbit only, i.e., if

$$m, n \in M \quad \Rightarrow \quad \exists\, \gamma \in \Gamma \;:\; \gamma m = n.$$

Let S be connected and $\pi : E \to S$ a covering. A **deck transformation** is a bijection $d : E \to E$ such that the diagram

$$\begin{tikzcd} E \arrow[rr, "d"] \arrow[dr, "\pi"'] & & E \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

commutes. Let $\mathrm{Deck}(\pi)$ be the group of all deck transformations. One also calls $\mathrm{Deck}(\pi)$ the **Galois group** of the covering and writes it as $\mathrm{Gal}(\pi)$ or $\mathrm{Gal}(E/S)$.

Proposition 1.4.8. *Let S be connected and $\pi : E \to S$ be a covering.*

(a) *If E is connected and d a deck-transformation with $d(e) = e$ for some $e \in E$, then $d = \mathrm{Id}_E$. In particular we get: If π is the universal covering, then* $\mathrm{Deck}(\pi) \cong \pi_1(S)$.

(b) *If Y is simply connected and if a group Γ acts discontinuously on Y, then $Y = \tilde{S}$ with $S = \Gamma\backslash Y$ and $\Gamma \cong \pi_1(S)$.*

Proof. (a) Let $d(e) = e$ and let $f \in E$. Then there exists a path α from e to f in E. Let $\gamma = \pi \circ \alpha$. Then α is the unique lifting of γ to E with $\alpha(0) = e$, so $\alpha = \tilde{\gamma}_e$. On the other hand $d \circ \alpha$ is a lifting as well γ with $d \circ \alpha(0) = d(\alpha(0) = d(e) = e$, so we get $d \circ \alpha = \alpha$ and thus $d(f) = d \circ \alpha(1) = \alpha(1) = f$, which means, $d = \mathrm{Id}$.

If π is universal, then $\Gamma = \pi_1(S, x_0)$ acts discontinuously on $\tilde{S}$ by deck transformations, so $\Gamma \hookrightarrow \Gamma(\pi)$. The group Γ also acts transitively on the fibre $F = \pi^{-1}(x_0)$. So let d be adeck transformation and let $e \in F$. Then there exists $\gamma \in \Gamma$ with $d(e) = \gamma(e)$, so $\gamma^{-1}d(e) = e$, which means $\gamma^{-1}d = \mathrm{Id}$, or $\gamma = d$.

(b) Let Y be simply connected and let Γ acts discontinuously. We show that the projection $\pi : Y \to S := \Gamma\backslash Y$ is a covering. For this let $x \in \Gamma\backslash Y$, say $x = \Gamma y$. Let V be an open neighborhood of y with $\gamma V \cap V \neq \emptyset \;\Rightarrow\; \gamma = 1$. Then $U = \pi(V)$ is an open neighborhood of x and the diagram

$$\begin{tikzcd}[column sep=small]
\pi^{-1}(U) = \bigsqcup_{\gamma\in\Gamma} \gamma V \arrow[rr, "\cong"] \arrow[dr, "\pi"'] & & V \times \Gamma \cong U \times \Gamma \arrow[dl, "p_1"] \\
& U &
\end{tikzcd}$$

commutes. Therefore π is a covering. Since Y is simply connected, we get $Y \cong \tilde{S}$. $\square$

Definition 1.4.9. A covering $\pi : E \to S$ is called **normal**, if the canonical map

$$E/\operatorname{Gal}(E/S) \to S$$

is a homeomorphism.

Lemma 1.4.10. *Let $\pi : E \to S$ be a connected covering of a surface S. Then the following are equivalent.*

(a) *π is normal,*

(b) *for every $x \in S$ the Galois group $\operatorname{Gal}(\pi)$ acts transitively on the fibre $\pi^{-1}(x)$,*

(c) *Ther is a point $x \in S$, such that $\operatorname{Gal}(\pi)$ acts transitively on the fibre $\pi^{-1}(x)$.*

Proof. (a)⇒(b): Let $x \in S$. The fibre of $E/\operatorname{Gal}(\pi)$ is the set of $\operatorname{Gal}(\pi)$-orbits of $\pi^{-1}(x)$.

(b)⇒(c) is clear.

(c)⇒(a): The map $E/\operatorname{Gal}(\pi) \to S$ is a connected covering, which has trivial fibre and one place. Then it has trivial fibre everywhere. A covering with trivial fibre is a homeomorphism. □

Theorem 1.4.11 (Main Theorem of Galois-theory of coverings)**.** *Let S be a surface and let $\tilde{S}$ be the universal covering. The map*

$$H \mapsto \tilde{S}/H$$

is a bijection between the set of all conjugacy classes of subgroups H of $\Gamma = \pi_1(S)$ and the set of isomorphy classes of connected coverings $\tilde{S}/H \to \tilde{S}/\Gamma = S$ of S, so

$$\begin{aligned} &\{\textit{subgroups } H \textit{ of } \Gamma\}\,/\textit{conjugation} \\ &\xrightarrow{\cong} \{\textit{conn. coverings } E \to S\}/\textit{isomorphy} \end{aligned}$$

A covering is normal if and only if the corresponding subgroup H is a normal subgroup of Γ. So then the map specializes to

$$\left\{\textit{normal subgroups}\right\} \xrightarrow{\cong} \left\{\textit{normal coverings}\right\}/\textit{isomorphy}.$$

Proof. First we show well-definedness: Let the groups $G, H \subset \Gamma$ be conjugate, say

$$\psi h \psi^{-1} \in G.$$

Then ψ induces a homeomorphism $\phi : \tilde{S}/H \to \tilde{S}/G$ such that the diagram

$$\begin{tikzcd} \tilde{S} \arrow[r, "\psi"] \arrow[d, "p_H"'] & \tilde{S} \arrow[d, "p_G"] \\ \tilde{S}/H \arrow[r, "\phi"] & \tilde{S}/G \end{tikzcd}$$

commutes. This implies well-definedness.

We show injectivity of the map $H \mapsto \left(\tilde{S}/H \to S\right)$: let H, G be two subgroups of Γ giving isomorphic coverings. Then there is a homeomorphism $\phi : \tilde{S}/H \to \tilde{S}/G$ over S. Because of the uniqueness of the universal covering the map $\tilde{S} \to \tilde{S}/H \to \tilde{S}/G$ lifts to a homeomorphism $\psi : \tilde{S} \to \tilde{S}$ such that the diagram

$$\begin{tikzcd} \tilde{S} \arrow[r, "\psi"] \arrow[d, "p_H"'] & \tilde{S} \arrow[d, "p_G"] \\ \tilde{S}/H \arrow[r, "\phi"] & \tilde{S}/G \end{tikzcd}$$

commutes. If $h \in H$, so $p_H(hx) = p_H(x)$, then $p_G(\psi(hx)) = p_G(\psi(x))$, which means that

$$\psi h \psi^{-1} \in G.$$

This also works from G to H, so that $\psi H \psi^{-1} = G$. Thus we get injectivity.

Surjectivity is obtained from the universal property of the universal covering. □

1.5 Determining the fundamental group

Proposition 1.5.1. *The fundamental groups of* $\mathbb{C}^\times$, $\mathbb{T}$ *and* $\mathbb{E}^\times = \{z \in \mathbb{C} : 0 < |z| < 1\}$ *each are isomorphic to* $\mathbb{Z}$.

Proof. If γ is a closed path in $\mathbb{C}^\times$, then $\frac{\gamma(t)}{|\gamma(t)|}$ is a closed path in S^1. So we get isomorphisms $\pi_1(\mathbb{C}^\times) \cong \pi_1(\mathbb{T}) \cong \pi_1(\mathbb{E}^\times)$. Now $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$, so $\mathbb{R}$ is the universal covering and by Proposition 1.4.8 the fundamental group is $\mathbb{Z}$. □

Definition 1.5.2. Let G, H, L be groups and let $\phi : L \to H$, $\psi : L \to H$ be group homomorphisms. Then the **amalgamate product**

$$G *_L H$$

is by definition equal to the set of all finite tuples (words) of the form $(x_1, x_2, \dots, x_n)$ with $x_j \in G \sqcup H$ modulo the reduction rules:

(a) $(\dots, x, y, \dots) = (\dots, xy, \dots)$,
if x, y both lie in G or both lie in H, and

(b) $(\dots, x, 1, y, \dots) = (\dots, x, y, \dots)$
where 1 is the neutral element of G or H. Finally,

(c) $(\dots, \phi(x), \dots) = (\dots, \psi(x), \dots)$
for every $x \in L$.

In the special case $L = \{1\}$ one writes this group as $G * H$ and calls it the **free product** of G and H.

Remark 1.5.3. (i) The composition

$$(x_1, \dots, x_m)(y_1, \dots, y_m) = (x_1, \dots, x_m, y_1, \dots, y_n)$$

makes $G *_L H$ a group. The tuple (1) is the neutral element.

(ii) By the reduction rules every element of $G *_L H$ can be written in the form $(x_1, y_1, x_2, y_2, \dots, x_n, y_n)$ with $x_1, \dots, x_n \in G$ and $y_1, \dots, y_n \in H$, all non-trivial with the possible exception of x_1 or y_n. We call this the **standard form** of elements of $G *_L H$.

(iii) The map $x \mapsto (x)$ is a canonical group homomorphism $s_G : G \to G *_L H$. Same for H. These homomorphisms need not be injective. For example, if $G = L$ and $\phi = \mathrm{Id}$, as well as $H = \{1\}$, then $G *_L H = \{1\}$.

The group $G *_L H$ is generated by the images of these homomorphisms.

Examples 1.5.4. • $\mathbb{Z} * \mathbb{Z} = F_2$ the free group in two generators.
If α and β are generators of the free group, then $\phi : \mathbb{Z} * \mathbb{Z} \to F_2$, given in the standrad form by

$$\phi(j_1, k_1, \dots, j_n, k_n) = a^{k_1} b^{j_1} \cdots a^{j_n} b^{k_n}$$

is an isomorphism.

- If $L = H$ and $\psi = \mathrm{Id}$, then it does not matter what ϕ looks like, we always have $G *_H H \cong G$.

Proposition 1.5.5 (Universal property of the amalgamate product)**.** *There is a commutative diagram*

$$\begin{tikzcd}
 & G *_L H & \\
G \arrow[ur, "s_G"] & & H \arrow[ul, "s_H"'] \\
 & L \arrow[ul, "\phi"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

This has the universal property that for every commutative diagram of group homomorphisms

$$\begin{tikzcd}
 & Z & \\
G \arrow[ur, "\alpha"] & & H \arrow[ul, "\beta"'] \\
 & L \arrow[ul, "\phi"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

there is exactly one homomorphism $\eta : G *_L H \to Z$ such that the diagram

$$\begin{tikzcd}
 & Z & \\
 & G *_L H \arrow[u, "\eta"] & \\
G \arrow[uur, "\alpha"] \arrow[ur, "s_G"] & & H \arrow[uul, "\beta"'] \arrow[ul, "s_H"'] \\
 & L \arrow[ul, "\phi"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

commutes.

Proof. One defines $\eta(g_1, h_1, \ldots, g_n, h_n) = \alpha(g_1)\beta(h_1) \cdots \alpha(g_n)\beta(h_n)$. The well-definedness and the unversal property are easily seen. □

Theorem 1.5.6 (Seifert-van Kampen). *Let $S = U \cup V$ with U, V open, such that $U, V, U \cap V \neq \emptyset$ are path connected. Then*

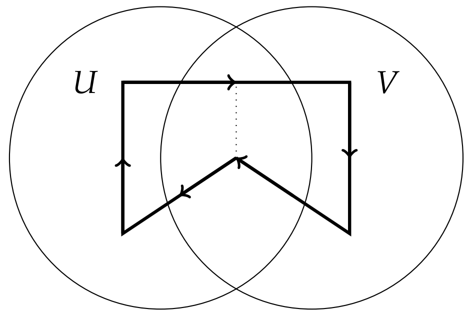
$$\pi_1(S) \cong \pi(U) *_{\pi_1(U \cap V)} \pi_1(V).$$

Proof. Choose a base point $a \in U \cap V$. The maps in the diagram

$$\begin{tikzcd}
 & \pi_1(S) & \\
\pi_1(U) \arrow[ur, "\alpha"] & & \pi_1(V) \arrow[ul, "\beta"'] \\
 & \pi_1(U \cap V) \arrow[ul, "\phi"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

are induced by the inclusions. By the universal property there is a homomorphism $\eta : \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \to \pi_1(S)$ such that α and β factorize over η. We want to show that η is an isomorphism.

Surjectivity: We have to show that every closed path γ in S can be written as composition of closed paths in U and V. The following picture gives the idea:



We cover the unit interval I with the connected components of $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$. As I is compact, finitely many suffice. So there is a decomposition $0 = t_0 < \cdots < t_n = 0$ such that, say, $\gamma([t_{2k}, t_{2k+1}]) \subset U$ and $\gamma([t_{2k+1}, t_{2k+2}]) \subset V$ for all k. We connect $\gamma(t_1)$ with a inside of $U \cap V$ and we get a closed path inside U. From a we go back to $\gamma(t_1)$, follow γ to $\gamma(t_2)$ and connect this point in $U \cap V$ with a. This is a closed path in V. Iteration yields the claim.

Injectivity: Let γ be a path, which is nullhomotopic in $U \cup V$. Let $H : I \times I$ be a homotopy to the trivial path. We cover the compact set $I \times I$ with finitely many connected components of $h^{-1}(U)$ and $h^{-1}(V)$. Then there is $n \in \mathbb{N}$ such that $h(Q_{k,l})$ lies completely in U or completely in V for any $0 \leq k, l \leq n-1$, where $Q_{k,l}$ is the square

$$Q_{k,l} = \left[\frac{k}{n}, \frac{k+1}{n}\right] \times \left[\frac{l}{n}, \frac{l+1}{n}\right].$$

Let x_{ij} be the vertices of these squares. and let η_{ij} be paths from a to $h(x_{ij})$, lying entirely in $U \cap V$ if $x_{ij} \in h^{-1}(U \cap V)$ or entirely in U or entirely in V otherwise. The path $\gamma(j/n, t)$ can be split into the paths $\eta_{j,i+1} \cdot h(j/n, t) \cdot \eta_{j,i}^{-1}$, which by the restriction of h entirely inside U or V are homotopic to $h((j+1)/n, t)$. The claim follows. □

Theorem 1.5.7. *Let $a_1, \dots, a_n$ be distinct points. Then*

$$\pi_1(\mathbb{C} \smallsetminus \{a_1, \dots, a_n\}) \;\cong\; F_n.$$

Proof. The space $\mathbb{C}$ can be covered by open sets $U_1, \dots, U_n$ such that every U_j contains exactly one of the points $a_1, \dots, a_n$, U_j is simply connected, and $(U_1 \cup \cdots \cup U_k) \cap U_{k+1}$ is simply connected. Let $V_j = U_j \smallsetminus \{a_1, \dots, a_n\}$. Then $\pi_1(V_j) \cong \mathbb{Z}$ and

$\pi_1(V_1 \cup \cdots \cup V_{k+1}) \cong \pi_1(V_1 \cup \cdots \cup V_k) * \pi_1(U_{k+1})$ and therefore, by induction

$$\pi_1(\mathbb{C} \setminus \{a_1, \dots, a_n\}) \cong \pi_1(V_1) * \cdots * \pi_1(V_n) \cong \mathbb{Z} * \cdots * \mathbb{Z} \cong F_n. \qquad \square$$

2 Classification of surfaces

2.1 Triangulations

Definition 2.1.1. Let Δ be the closed triangle in $\mathbb{C}$ with vertices $0, 1, i$. A **2-Simplex** of the surface S is an injective continuous map $\sigma : \Delta \to S$, which maps the interior $\mathring{\Delta}$ homeomorphically to an open subset of S. The images of $0, 1, i$ are called the **vertices** of the simplex. The images of the lines $[0, 1], [1, i], [i, 0]$ are called the **edges** of the simplex. A **triangulation** of S is a family $(\alpha_j : \Delta \to S)_{j \in J}$ of 2-simplices, such that every $x \in S$ either lies

- in the interior of exactly one simplex or
- on the edge of exactly two simplices or
- on the vertex of finitely many simplices.

Corollary 2.1.2. *Let $(\alpha_j)_j$ be a triangulation. Then*

(a) *Any two simplices intersect either in a common edge or in a common vertex or in the empty set.*

(b) *For every $a \in S$ the set of all simplices, which contain a is finite and their union is a neighborhood of a.*

(c) *If S is compact, any triangulation is finite.*

Proof. Clear. □

Theorem 2.1.3. *Every surface has a triangulation.*

Proof. (Idea) By separability there is a locally finite atlas $(U_i, \phi_i)_{i=1}^n$, $n \in \mathbb{N} \cup \{\infty\}$ with $\phi_i(U_i) = B_{1+\varepsilon}(0)$, $\varepsilon > 0$, such that the sets $V_i = \phi_i^{-1}(B_1(0))$ still cover all of S. By induction on n and possibly changing some radii, one sees that the boundaries ∂V_i form a finite union of disjoint arcs between the set of intersection points

$$\bigcup_{i<j} \partial V_i \cap \partial V_j$$

□

2.2 Simplicial complexes

Definition 2.2.1. Let E be a set. An abstract **simplicial complex** with vertex set E is a set $\mathcal{S} \subset \mathcal{P}(E)$ with the properties:

- Every $S \in \mathcal{S}$ is finite,
- $\bigcup_{M \in \mathcal{S}} M = E$,
- $A \subset B \in \mathcal{S} \;\Rightarrow\; A \in \mathcal{S}$.

The elements of $\mathcal{S}$ are called (abstract) **simplices**.

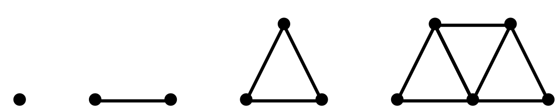
Definition 2.2.2. Fix $N \in \mathbb{N}$. A **geometric simplex** of dimension d is the convex hull of $d+1$ affine independent points in $\mathbb{R}^N$.

A **geometric simplicial complex** is a set $\mathcal{S}_{\text{geom}}$ of geometric simplices in $\mathbb{R}^N$ such that

- $A, B \in \mathcal{S} \;\Rightarrow\; A \cap B \in \mathcal{S}$,
- Let E be the set of all vertices of simplices in $\mathcal{S}_{\text{geom}}$. Then
$$\mathcal{S} = \left\{ E \cap S : S \in \mathcal{S}_{\text{geom}} \right\}$$
is an abstract simplicial complex with vertex set E and one has
$$\mathcal{S}_{\text{geom}} = \left\{ \text{conv}(S) : S \in \mathcal{S} \right\}.$$

In this case one says that $\mathcal{S}_{\text{geom}}$ is a **geometric realization** of $\mathcal{S}$.

Beispiele



The existence of a triangulation implies:

Theorem 2.2.3. *Every surface is homeomorphic to a 2-dimensional geometric simplicial complex.*

2.3 Classification

Definition 2.3.1 (Gluing)**.** Let S, Y, Z be topological spaces and let there be maps

$$\begin{tikzcd} S & & Y \\ & Z \arrow[ul, "\phi"] \arrow[ur, "\psi"'] & \end{tikzcd}$$

The **Gluing**

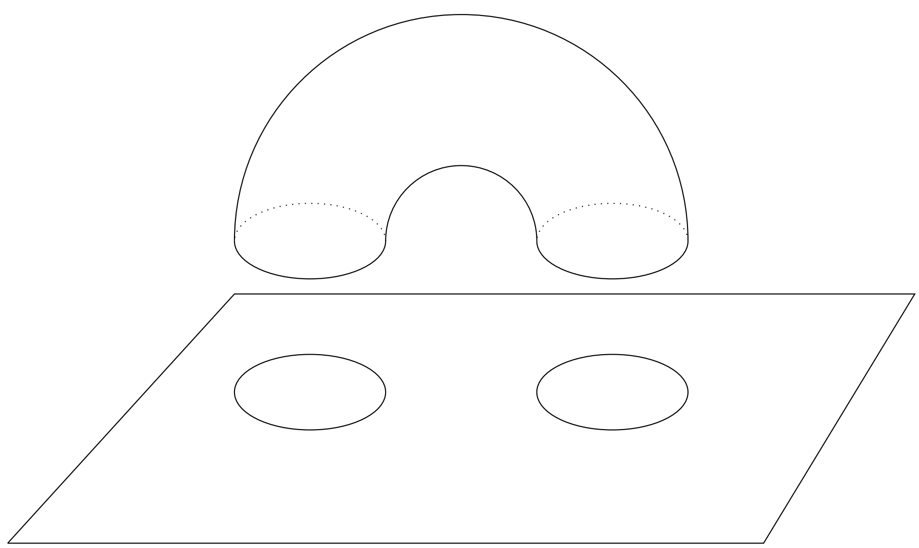
$$S \sqcup_Z Y$$

is defined as the set $S \sqcup Y / \sim$, where

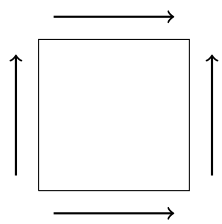
$$a \sim b \quad \Leftrightarrow \quad \left\{ \begin{array}{c} a = b \\ \text{or} \\ a = \phi(z), b = \psi(z) \text{ for some } z \in Z \end{array} \right\}$$

On $S \sqcup_Z Y$ one instals the topology induced by the two maps $S \to S \sqcup_Z Y$ and $Y \to S \sqcup_Z Y$.

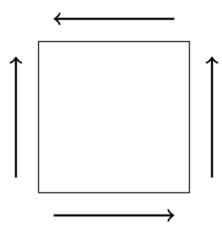
Definition 2.3.2. Let S be a surface. We attach a **handle** to S choosing two disjoint closed disks D_1, D_2, then cutting out their interiors and gluing in a cylinder $[0,1] \times S^1$.



Definition 2.3.3. The torus $\mathbb{R}^2/\mathbb{Z}^2$ can be described as the closed square $I \times I$, $I = [0,1]$ modulo identifying opposite boundary lines:



If one instead identifies like this:



one gets a non-orientable surface$K$, called the **Klein bottle**.

Another construction of the Klein bottle is this:

$$K \cong S^2/\sim,$$

where $\sim$ is the equivalence relation where each point is equivalent to its opposite.

Theorem 2.3.4. *Every compact surface is homeomorphic to exactly one of the following:*

(a) *for every $g = 0, 1, 2, \dots$ a sphere $S^2 \cong \widehat{\mathbb{C}}$ with g handles,*

(b) *for every $g = 0, 1, 2, \dots$ a Klein bottle g handles.*

The number g in the theorem is called the **genus** of the surface. The torus has genus 1, a pretzel has genus 3.

Proof. Note that case (a) gives all orientable surfaces, whereas (b) gives the non-orientable ones.

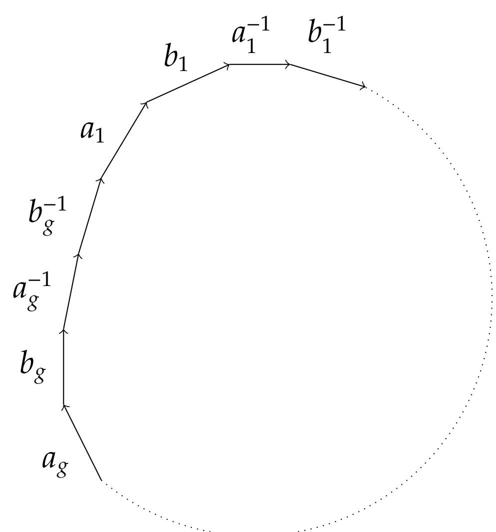
Let Z be a sphere with $2g$ handles, one can order these (up to homeomorphism) symmetrical with respect to $z \mapsto -z$, such that $S = Z/\pm 1$ is a KLein bottle with g handles.

Let S be a non-orientable compact surface. Then $S = \Gamma\backslash\tilde{S}$, where $\tilde{S}$ is the universal covering and Γ the group of deck transformations. The mapping

$$D : \gamma \mapsto \begin{cases} 1 & \gamma \text{ orientation-preserving,} \\ -1 & \gamma \text{ not orientation-preserving,} \end{cases}$$

is a gorup homomorphism and so the kernel $\Sigma = \ker(D)$ is a subgroup of index 2. Then $\Sigma\backslash\tilde{S}$ is a 2-fold covering, which is orientable. If you apply the theorem to $\Sigma\backslash\tilde{S}$ one gets the assertion for S as well. This means that it suffices to prove the theorem in the case of an orientable surface.

Lemma 2.3.5. *A connected, orientable, compact surface S is either homeomorphic to S^2, or there is exactly one $g = 1, 2, \dots$, such that S is homeomorphic to the following polygon modulo boundary identifications: One starts with a disk-shaped polygon with $4g$ boundary segments, which are lines and which are identified in the following way: every segment c is identified with c^{-1} in the opposite orientation:*

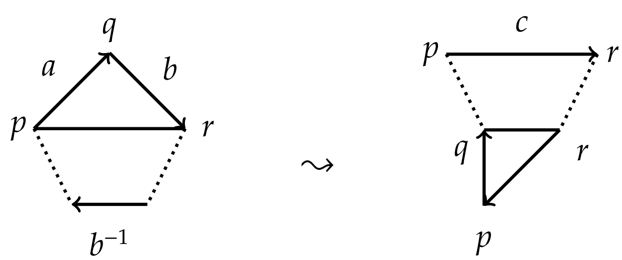


This induces the **canonical triangulation** *of S by joining an arbitrary point p in the interior of the n-gon by line segments with the vertices of the edges a_j, b_j.*

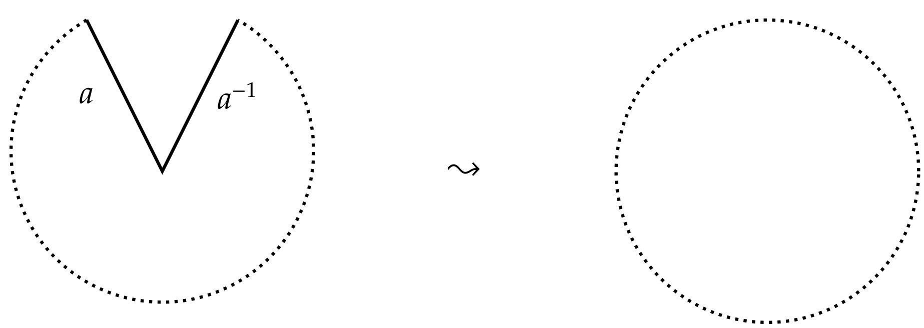
Proof. Choose a finite triangulation. Start with a triangle add neighboring triangles according to the following recipe: At every step one has a finite set of triangles that constitute a connected subset of $\mathbb{C}$ which interior maps injectively to S. Some of the boundary edges get identified by this map to S. If this is so for every boundary edge, the construction stops. Otherwise, choose an edge, which has not been identified with another and add the corresponding triangle to the set.

This construction stops and yields a closed polygon T, which maps surjective onto S under identifying boundary edges. As S is orientable, every edge gets identified with its inverse. Starting at an arbitrary vertex and denoting the edges by $a, b, c, \dots$ one gets a word of the form $abcdb^{-1}efa^{-1}\dots$ (example) in which every letter occurs once and once again as inverse. Next we reglue some triangles to get the desired form.

(a) If we have the situation that two consecutive edges belong to the same triangle, we remove this triangle and reglue it at b^{-1}:

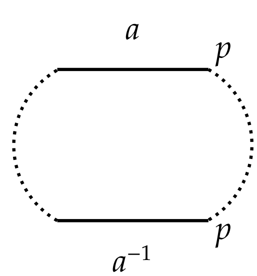


(b) By chnaging edge lengthes of the triangles , we cancel expressions of the form aa^{-1}:

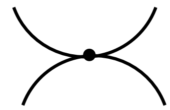


(c) Two edges a, b are said to be **chained**, if they occur in the order $a \dots b \dots a^{-1} \dots b^{-1}$.

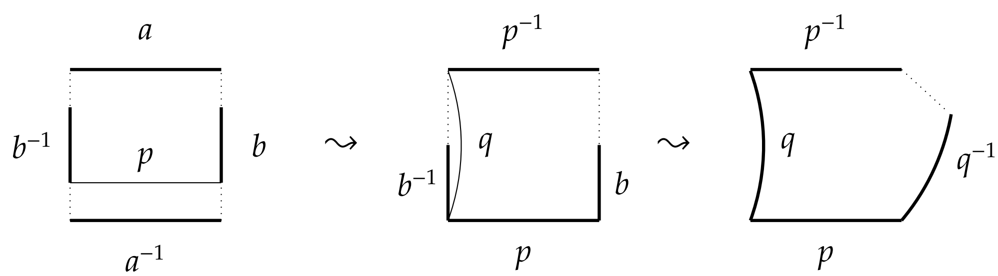
We show that for every edge a there is an edge b such that a and b are chained. If this was not so, then a would occur as follows:



On the right hand side there are only edges b, for which the edge b^{-1} is also on the right. The same is true for the left hand side. But then the polygonal line closes between the two occurrences of p and the ensuing topological space has two components, which are glued together at p, hence is not a surface!

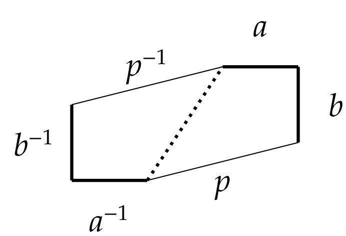


We finally have to show that the edges can be arranged in a way that chained edges come directly after one another. To achieve this, we reglue twice:



Here p and q are in general not single edges, but paths. In the first step we chop off everything below c and reglue it by gluing a and a^{-1}.

Now, in the vertex, where c and d meet, there are two consecutive edges, which are chained. In the next picture we cut along the dotted path and reglue along p to get the desired sequence $aba^{-1}b^{-1}$:



Repeating these operations with other chained pairs, we get the desired form. □

The theorem follows from this, since every sequence $aba^{-1}b^{-1}$ glues a handle to S. □

2.4 Fundamental group and Euler number

Theorem 2.4.1. *Let S be an orientable compact surface, i.e., a sphere with g handles. Then there are generators of the fundamental group $a_1, \dots, a_g$ and $b_1, \dots, b_g$ such that $\pi_1(S)$ is the group generated by $a_1, \dots, a_g, b_1, \dots, b_g$ with the only relation:*

$$[a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1.$$

Proof. Write S a a convex polygon P with boundary idientifications as in Lemma 2.3.5. Let p be a point in the interior of the polygon. Let $U = S \setminus \{p\}$ and let V be S minus the boundary of the polygon. Then $S = U \cup V$ and $U \cap V$ is homeomorphic to $\mathbb{C}^\times$, hence has fundamental group $\cong \mathbb{Z}$, generated by $\prod_{j=1}^{g}[a_j, b_j]$. The set V is contractible, so $\pi_1(V) = 1$. Centric dilation defines a homotopy, so an isomorphism

$$\pi_1(U) \cong \pi_1(\partial(U)) = \pi_1(4g\text{-gon}).$$

The boundary of U is a bouquet B_{2g} of $2g$ cicles and $\pi_1(B_{2g})$ is the free group in the generators $a_1, \dots, a_g, b_1, \dots, b_g$. By the Seifert-van Kampen Theorem we get

$$\pi_1(S) \cong \pi_1(B_{2g}) *_{\mathbb{Z}} \underbrace{\pi_1(V)}_{=1} \cong \pi(B_{2g}) / \prod_j [a_j, b_j]. \qquad \square$$

Theorem 2.4.2 (Euler number)**.** *Let S be a compact orientable surface of genus g. Fix a triangulation with T triangles, E edges and V vertices. Then one has*

$$V - E + T = 2 - 2g.$$

Proof. The following are easy to show

(i) The number $\chi = V - E + T$ remains the same when the triangulation is refined.

(ii) Any two triangulations have (after moving some edges and vertices) a common refinement.

(iii) The formula is correct for the canonical triangulation of Lemma 2.3.5

The theorem follows. $\square$

3 Riemann surfaces

3.1 Definition

Definition 3.1.1. An atlas $(U_i, h_i)_{i\in I}$ of a surface S is called a **holomorphic atlas**, if every **transition map** $\alpha = h_j \circ h_i^{-1}$:

$$\begin{tikzcd} & U_i \cap U_j \arrow[dl, "h_i"'] \arrow[dr, "h_j"] & \\ \mathbb{C} \arrow[rr, "\alpha"'] & & \mathbb{C} \end{tikzcd}$$

is holomorphic $h_i(U_i \cap U_j) \to h_j(U_i \cap U_j)$ ist. Since this then holds for any pair of indices, this map is biholomorphic.

Definition 3.1.2. Let $\mathcal{A} = (U_i, h_i)_{i\in I}$ be a holomorphic atlas. An arbitrary chart (U, h) is called **compatible** with $\mathcal{A}$, if every transition map $h_i \circ h^{-1}$ is biholomorphic. If this is the case, one can extend the atlas by the chart (U, h) and still get a holomorphic atlas.

Therefore, a given holomorphic atlas $\mathcal{A}$ is contained in exactly one maximal holomorphic atlas $\mathcal{A}_{\max}$, given by

$$\mathcal{A}_{\max} = \Big\{(U, h) : h \text{ is compatible with } \mathcal{A}\Big\}.$$

Definition 3.1.3. A maximal holomorphic atlas is called a **holomorphic structure** or **complex structure** on the surface S.

A tuple $(S, \mathcal{A})$ consisting of a surface S and a complex structure $\mathcal{A}$ is called a **Riemann surface**.

Examples 3.1.4.
- The Riemann number sphere is a Riemann surface with the charts $z \mapsto z$ and $z \mapsto \frac{1}{z}$.
- In the case of the torus the transition maps are translations, hence holomorphic. So the torus is a Riemann surface.

Examples 3.1.5.
- Let $S = Y = \mathbb{C}$ and $U = \mathbb{C}^\times$. Further let

$$\phi(z) = z, \qquad \psi(z) = \frac{1}{z}.$$

Then $S \sqcup_U Y = \widehat{\mathbb{C}}$ is the Riemann number sphere.

- Let $S = Y = \mathbb{C}$ and $U = \mathbb{C}^\times$. Further let

$$\phi(z) = z, \qquad \psi(z) = z.$$

  Then $S \sqcup_U Y$ is not a Hausdorff space. It is the space of Example 1.1.2 (d).

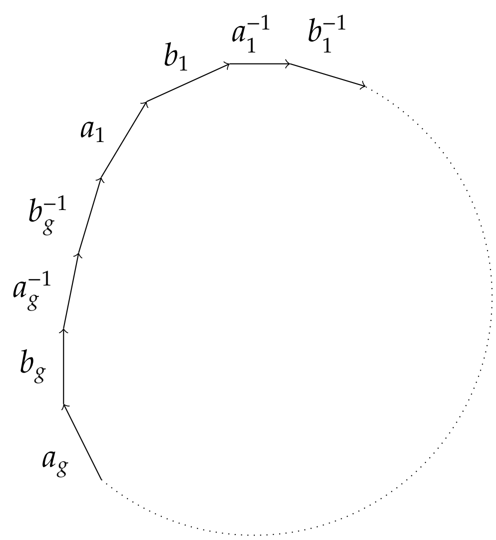
3.2 Existence

Theorem 3.2.1. (a) *Every Riemann surface is orientable.*

(b) *On every orientable compact surface there exists a complex structure.*

Proof. (a) Multiplication with i induces an $\mathbb{R}$-linear endomorphism $J : T_pS \to T_pS$ on each tangent space T_pS. One has $J^2 = -1$. If $v \in T_pS$ is non-zero, then the basis (v, Jv) of T_pS induces an orientation on T_pS and therefore the map J induces an orientation on S.

(b) On $S^2 \cong \widehat{\mathbb{C}}$ there is a complex structure. Any $S \neq S^2$ can be written (Lemma 2.3.5) as a polygon modulo boundary identifications:



We can assume all edges $a_j, b_j, a^{-1}, b_j^{-1}$ to be of equal length. Then one can assume that the identification maps are rotations followed by translations, i.e., they are of the form $z \mapsto az + b$ with $a \in \mathbb{C}$, $|a| = 1$ and $b \in C$. These maps exted to neighborhoods of the respective edge and are biholomorphic there. So they define holomorphic charts in neighborhoods of every boundary point. $\square$

3.3 Holomorphic maps

Definition 3.3.1. Let S, Y be Riemann surfaces. A map $F : S \to Y$ is called a **holomorphic map**, if for any two holomorphic charts $(U, \phi), (V, \psi)$, $U \subset S$ $V \subset Y$ the induced map $\psi \circ F \circ \phi^{-1}$

$$\mathbb{C} \xrightarrow{\phi^{-1}} S \xrightarrow{F} Y \xrightarrow{\psi} \mathbb{C}$$

is holomorphic, where defined, i.e., on $\phi\left(F^{-1}(V) \cap U\right)$.

The map F is called **biholomorphic**, or an **isomorphism** of Riemann surfaces, if F is homolomorphic and bijective.
(In this case the inverse map F^{-1} os automatically holomorphic, too.)

Definition 3.3.2. Let S be a Riemann surface. The set of all biholomorphic maps $F : S \to S$ is a group, the **automorphism group** $\mathrm{Aut}(S)$.

Definition 3.3.3. A holomorphic map $f : S \to \mathbb{C}$ is called a **holomorphic function** on S. The set of all holomorphic functions is denoted by

$$\mathcal{O}(S).$$

Proposition 3.3.4. (a) *The automorphism group of* $\mathbb{C}$ *is*

$$\mathrm{Aut}(\mathbb{C}) = \left\{z \mapsto az + b : a \in \mathbb{C}^\times,\ b \in \mathbb{C}\right\}.$$

(b) *The automorphism group of the* **unit disk**

$$\mathbb{E} := \{z \in \mathbb{C} : |z| < 1\}$$

is

$$\mathrm{Aut}(\mathbb{E}) = \left\{z \mapsto c\frac{z - a}{\bar{a}z - 1} : |c| = 1,\ a \in \mathbb{E}\right\}.$$

(c) *The automorphism group of the* **upper half plane**

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

is

$$\mathrm{Aut}(\mathbb{H}) = \left\{z \mapsto \frac{az + b}{cz + d} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}\right\}.$$

Proof. (a) Each map $\phi_{a,b}(z) = az + b$ is biholomorphic. Let $\phi : \mathbb{C} \to \mathbb{C}$ be an arbitrary bihilomorphic map. Let $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ be its power series. **Assume**, the power series is infinite. Then $\phi(z)$ takes every value with one possible exception infinitely often by the Theorem of Picard. Therefore, it cannot be bijective. So $\phi(z)$ is a polynomial. Let n be its degree. Ist $z \in \mathbb{C}$ is not a zero of the derivative of ϕ, then ϕ takes the value z at n different places. For ϕ to be bijective, we must have $n = 1$.

(b) follows from the Schwarz lemma and (c) follows after the application of the Cayley map. $\tau(z) = \frac{z-i}{z+i}$, which maps the upper half plane biholomorphically to the unit disk. □

Theorem 3.3.5 (Coverings)**.** *Let $F : S \to Y$ be a covering of surfaces. If Y is a Riemann surface, i.e., has a given complex structure, then there is exactly one complex structure on S, which makes F a holomorphic map.*

In particular, the universal covering becomes a Riemann surface in a unique way.

Proof. Let (V_i, ϕ_i) be a holomorphic atlas of S. After decomposing the V_i, one can assume that the pre-image $F^{-1}(V_i)$ is a disjoint union of connected open sets $\bigsqcup_j U_{i,j}$, where for each j the map $F|_{U_{i,j}}$ is a homeomorphism onto V_i. Then $(U_{i,j}, \phi_i \circ F)$ is a holomorphic atlas on S, which makes F holomorphic. The uniqueness is clear, as F is a lokal homeomorphism. □

Definition 3.3.6. Let $F : S \to Y$ be a holomorphic map between Riemann surfaces. Let $a \in S$ and let (U, ϕ), (V, ψ) charts centered around a and $F(a)$. After post-composing with a translation in $\mathbb{C}$ we can assume $\psi(F(a)) = 0$. The **order** of F in the point a is defined as

$$\operatorname{ord}_a(F) = \min\{n : b_n \neq 0\}, \text{ where } \psi \circ F \circ \phi^{-1}(z) = \sum_{n \geq 1} b_n z^n.$$

The order does not depend on the choice of charts, since a change of charts only menas per- and post-composing with biholomorphic maps, which does not change the vanishing order of a holomorphic function.

Note that

$$\operatorname{ord}_a(F) = \infty \quad \Leftrightarrow \quad F \text{ is constant in a neighborhood of } a.$$

Proposition 3.3.7 (Local model of holomorphic maps)**.** *Let $F : S \to Y$ be holomorphic and $a \in S$ with $n = \operatorname{ord}_a(F) < \infty$. Then there are charts (U, ϕ), centered around a and (V, ψ), centered around $F(a)$, such that*

$$\psi \circ F \circ \phi^{-1}(z) = z^n.$$

Proof. First let ϕ, ψ be any centered charts, such that $\psi \circ F \circ \phi^{-1}(z) = z^n \sum_{j=0}^{\infty} a_j z^j$, $a_0 \neq 0$. In a small disk around zero the function $h(z) = \sum_{j=0}^{\infty} a_j z^j$ has no zero and therefore has a holomorphic n-th root $g(z)$. The function $zg(z)$ has non-vanishing derivative in $z = 0$, so it is biholomorphic in a small disk. Let η be the inverse map. Then we have
$\psi F \phi^{-1}(z) = (zg(z))^n$ Set $w = zg(z)$, then one has $\psi \circ F \circ \phi^{-1}(\eta(w)) = w^n$. The charts ψ and $\eta^{-1} \circ \phi$ now do the job. □

Corollary 3.3.8 (Open Mapping Theorem)**.** *Let $F : S \to Y$ and let S be connected. Then the image $F(S)$ is open in Y.*

Proof. Let $y \in Y$ be in the image, so $y = F(a)$ with $a \in S$. If F was constant in a neighborhood of a, then the connected component of $F^{-1}(y)$, which contains a, would be closed on one hand, open by the identity theirem on the other, therefore it 'd be equal S, as S is connected. Therfore F is not constant in any neighborhood of a. This means that we can apply the proposition. Since the map z^n maps any zer-neighborhood to a zero-neighborhood, the image $F(S)$ contains a neighborhood of y. □

Definition 3.3.9. A subset $A \subset S$ is called **analytic**, if A is closed and every point in A possesses a neighborhood U, such that $A \cap U$ is the zero-set of a holomorphic function $U \to \mathbb{C}$.

A set $A \subset S$ is called **locally-finite**, if every point of S has a neighborhood U, such that $A \cap U$ is finite.

Corollary 3.3.10. *For a subset A of a Riemann surface S the following are equivalent:*

(a) *A is locally-finite,*

(b) *A is closed and discrete,*

(c) *A has no accumulation point in S.*

Proof. (a)⇒(b): Let $x \in S \setminus A$. Then there is a neighborhood U of x such that $A \cap U$ is finite. As S is a Hausdorff space, A is locally finite and $x \notin A$, there is an open neighborhood V of x with $V \cap A = \emptyset$. So the set $S' = S \setminus A$ contains a neighborhood around each of its membergs, hence is open, so A is closed. Discreteness of A is clear as S is a Hausdorff space.

(b)⇒(c): Assume that x_0 is an accumulation point of A. Then $x_0 \in A$, as A is closed. Since A is discrete, there is a neighborhood U of x_0 with $A \cap U = \{x_0\}$. But then x_0 cannot be an accumulation point of A.

(c)⇒(a) A having no accumulation point means that ever $x \in S$ has a neighborhood U with $U \cap A$ being finite. □

Theorem 3.3.11. *Let S be a connected Riemann surface and $A \subset S$ be an analytic set. Then $A = S$ or A is locally finite in S.*

Proof. Assume that A has an accumulation point a_0. Then there is an open neighborhood U of a_0 and a holomorphic function f on U, such that $A \cap U$ is the zero set of f ist. Then a_0 is an accumulation point of zeros of f, therefore f is identically zero in a neighborhood of. This means that A has inner points. By the same argument each boundary point of $\mathring{A}$ is an inner point itself, so $A = S$. □

3.4 Ramified coverings

Definition 3.4.1. A holomorphic map $F : S \to Y$ between Riemann surfaces is called **proper**, if for compact set $K \subset Y$ the pre-image $F^{-1}(K) \subset S$ is compact, too.

A holomorphic map $F : S \to Y$ is called **finite**, if F is proper and for every point $y \in Y$ the fibre $F^{-1}(y)$ is finite.

A holomorphic map $F : S \to Y$ is called a **(ramified) covering**, if for every $y \in Y$ there are charts (V, ϕ) around y and (U_i, ψ_i) around $x_i \in F^{-1}(y)$, such that

$$F^{-1}(V) = \bigsqcup_{i \in I} U_i,$$

further $F(U_i) = V$ and F in these charts is of the form

$$\phi \circ F \circ \psi_i^{-1} : z \mapsto z^{n_i}.$$

Such charts are called **standard charts**.

Convention: In topology, the word *covering* means unramified covering. In complex analysis it also means possibly ramified coverings.

From this point onward, we switch to the tradition of complex analysis. So

- *covering* now means possibly ramified covering and
- *unramified covering* means unramified covering.

Lemma 3.4.2. *Every proper map $F : S \to Y$ between surfaces is closed.*

Proof. Let $A \subset S$ be closed and $b \in Y \setminus F(A)$. Let U be a neighborhood of b with compact closure $\overline{U}$. Then $F^{-1}(\overline{U}) \cap A$ is compact. Then the image $F(F^{-1}(\overline{U}) \cap A) = \overline{U} \cap F(A)$ is compact. This means that $U \setminus (\overline{U} \cap F(A))$ is a neighborhood of b, which does not intersect $F(A)$, so $Y \setminus F(A)$ is open, hence $F(A)$ is closed. □

Theorem 3.4.3. *Let $F : S \to Y$ be a holomorphic map and $S \neq \emptyset$.*

(a) *F is finite if and only if F is proper and non-constant.*

(b) *If F is finite, then F is a covering.*

(c) *If F is a covering, Y is connected and if for one point $y \in Y$ the index set of the definition $I = I_y$ is finite, then F is finite and in particular proper.*

In particular, every non-constant holomiorphic map between connected compact Riemann surfaces is a finite covering.

Proof. (a) Let F be finite. Then every fibre $F^{-1}(y)$ is finite, so in particular, different from S. So F is proper and non-constant. For the converse, if F is proper, then every fibre $F^{-1}(y)$ is compact. The fibre is also analytic and different from S, hence locally-finite and by compactness, finite.

(b) Let $y \in Y$. As F is proper, Y connected and $S \neq \emptyset$, it follows that F is surjective:

Proof. The image $F(S)$ is open by the open mapping theorem. It is also closed by Lemma 3.4.2. Since Y is connected, we get $F(S) = Y$. □

So let $F^{-1}(y) = \{x_1, \dots, x_d\}$. We choose charts (V, ψ) around y and (U_i, ϕ_i) around x_i and after decreasing the U_i we can assume that the U_i are disjoint and $F(U_i) \subset V$. By Proposition 3.3.7 for every i there is a chart $(\tilde{U}_i, \tilde{\phi}_i)$ around x_i with $\tilde{U}_i \subset U_i$ such that

$$\psi \circ F \circ \tilde{\phi}_i^{-1} : z \mapsto z^{n_i}$$

with $n_i = \mathrm{ord}_{x_i}(F)$. The claim follows.

(c) We show first, that $F^{-1}(y)$ is finite for every $y \in Y$. For this we define

$$\deg(F, \cdot) : Y \to \mathbb{N} \cup \{\infty\}, \quad y \mapsto \sum_{x \in f^{-1}(y)} \mathrm{ord}_x(F).$$

Since f is a covering, this map is locally constant. It is finite in one point, hence globally finite, as Y is connected. It remains to show properness. For this let $K \subset Y$ be compact. For every $y \in Y$ there are neighborhoods V_y and $U_{y,i}$ as in the definition of a covering. As Y is locally compact, there are open neighborhoods V'_y such that $\overline{V'_y}$ is compact and contained in V_y. Since K is compact, finitely many $V'_1, \dots, V'_n$ suffice to cover K. It is enoufg, to show that each $f^{-1}(\overline{V'_i})$ is compact. This follows from the fact that the map $z \mapsto z^n$ is proper. □

Definition 3.4.4. The map $\deg(F, \cdot)$ is called the **degree** of the holomorphic map $F : S \to Y$. Every point $x \in S$ with $\mathrm{ord}_x(F) > 1$ is called a **ramification point** of F.

If S and Y are connected and compact and F is non-constant, the degree map is constant. In this case we denote this constant by $\deg(F)$ and call it the **degree of** F.

Lemma 3.4.5. *Let $F : S \to Y$ be a covering. For $x \in S$ the following are equivalent:*

(a) *x is a ramification point of F,*

(b) *There are local charts ϕ, ψ around x and $F(x)$ with $\phi(x) = 0$, such that for the function $g = \psi^{-1} \circ F \circ \phi$ one has $g'(0) = 0$.*

(c) *For all local charts ϕ, ψ around x and $F(x)$ with $\phi(x) = 0$ one has $g'(0) = 0$, where $g = \psi^{-1} \circ F \circ \phi$.*

Proof. This follows from the fact that for an open neighborhood $U \subset \mathbb{C}$ of zero and a holomorphic, non-constant $g : U \to \mathbb{C}$ the point $z = 0$ is a ramification point if and only if $g(z) - g(0) = z^n h(z)$ for some $n \geq 2$. This is equivalent to $g'(0) = 0$. □

Lemma 3.4.6. *For a covering, the set of ramification points is locally-finite.*

Proof. Let $F : S \to Y$ be a covering. Let x be a ramification point. Then there is a neighborhood in which F acts like $z \mapsto z^n$. This map has zeor for its only ramification point. □

Definition 3.4.7. Let $F : S \to Y$ be a covering of Riemann surfaces.

(a) The group $\mathrm{Gal}(S/Y) = \left\{\sigma \in \mathrm{Aut}(S) : F \circ \sigma = F\right\}$ is called the **Galois group** of F.

(b) Let $A \subset S$ be the set of ramification points of F. Then F is called **normal**, if the unramified covering $S \setminus A \to Y \setminus F(A)$ is normal.

Lemma 3.4.8. *Let $F : \mathbb{C} \to \mathbb{C}$ be the covering $z \mapsto z^n$. Then* $\mathrm{Gal}(F) \cong \mathbb{Z}/n\mathbb{Z}$.

Proof. Let $\tau(z) = e^{2\pi i/n}z$. We claim that $\mathrm{Gal}(F)$ is generated by the element τ. So let $\sigma : \mathbb{C} \to \mathbb{C}$ be biholomorphic with $\sigma(z)^n = F(\sigma(z)) = F(z) = z^n$. Then $\sigma(z) = az + b$ for suitable $a, b \in \mathbb{C}$, $a \neq 0$. It follows $b = 0$ and $a^n = 1$. □

Proposition 3.4.9. *Let $F : S \to Y$ be a normal finite covering, where Y is connected. Let $y \in Y$. Then the group* $\mathrm{Gal}(F)$ *acts transitively on the fibre $F^{-1}(y)$. In particular one has* $\mathrm{ord}_x(F) = \mathrm{ord}_{x'}(F)$ *for all $x, x' \in F^{-1}(y)$.*

Proof. Let $A \subset S$ be the set of ramification points and $S' = S \setminus A$, as well as $Y' = Y \setminus F(A)$. Then $F' = F|_{S'}$ is an unramified covering. So the Galois group acts transittove;y on $F^{-1}(y)$, $y \in Y'$ by Lemma 1.4.10. As Y' is dense in Y, the claim follows generally. □

3.5 Riemann-Hurwitz Theorem

Theorem 3.5.1 (Riemann-Hurwitz Formula)**.** *Let $F : S \to Y$ be a non-constant holomorphic map between two compact, connected Riemann surfaces. Then*

$$2g(S) - 2 = \deg(F)\,(2g(Y) - 2) + \sum_{x \in S} (\mathrm{ord}_x(F) - 1)\,.$$

Proof. We choose a triangulation of Y, such that every triangle lies in a standard chart and that the images of the ramification points are vertices of the triangulation. Then the pre-images of the vertices, edges and triangles define a triangulation of S. Let $V_S, V_Y, E_S, E_Y, T_S, T_Y$ be the sets of the vertices, edges and triangles.

Then $|E_S| = |E_Y| \deg(F)$ and $|T_S| = |T_Y| \deg(F)$ and one has

$$|V_S| = |V_Y| \deg(F) + \sum_{x \in S} (1 - \mathrm{ord}_x(F)).$$

By the Euler formula $2 - 2g = V - E + T$ of Theorem 2.4.2 we get

$$\begin{aligned} 2 - 2g(S) &= |V_S| - |E_S| + |T_S| \\ &= (|V_Y| - |E_Y| + |T_Y|)\deg(F) + \sum_{x\in S}(1 - \operatorname{ord}_x(F)) \\ &= (2 - 2g(Y))\deg(F) + \sum_{x\in S}(1 - \operatorname{ord}_x(F)). \end{aligned}$$

Multiplying by (-1) we get the claim. □

3.6 Meromorphic functions

Let $U \subset \mathbb{C}$ be a domain. A **meromorphic function** on U is a holomorphic function $f : U \setminus P \to \mathbb{C}$, where P is a closed discrete subset of U and every $p \in P$ is a pole of f. This means that in a neighborhood of p one has

$$f(z) = \sum_{n=-N}^{\infty} c_n (z-p)^n, \quad z \neq p.$$

Definition 3.6.1. Let S be a Riemann surface. A **meromorphic function** on S is a function $f : S \setminus P \to \mathbb{C}$, where $P \subset S$ is a closed discrete subset and for every holomorphic chart (U, ϕ) the funkction $f \circ \phi^{-1}$ is meromorphic on $\phi(U) \subset \mathbb{C}$. The set $\mathcal{M}(S)$ of all meromorphic functions on S forms a field with the point-wise operations.

Proposition 3.6.2. *Let f be a meromorphic function on S Define $f(a) = \infty$ for every pole of f. In this way we get a holomorphic map of Riemann surfaces*

$$f : S \to \widehat{\mathbb{C}}.$$

Conversely, let $f : S \to \widehat{\mathbb{C}}$ be a non-constant holomorphic map. Then f, restrictes to $S \setminus f^{-1}(\infty)$ is a meromorphic function.

3.7 Differential forms

Definition 3.7.1. Let S be a Riemann surface and $p \in S$. A **point derivation** in p is a linear map $D : C^\infty(S) \to \mathbb{C}$ with

$$D(fg) = D(f)g(p) + f(p)D(g).$$

this equation is called the **Leibniz rule**. The $\mathbb{C}$-vector space of point derivations in p is denoted by T_pS and is called the **tangent space** at the point p.

Lemma 3.7.2. *The tangent space at p has dimension 2. For a local chart z with $z(p) = z_0$, the two maps*

$$\frac{\partial}{\partial x} : f \mapsto \frac{\partial f}{\partial x}(z_0)$$

and

$$\frac{\partial}{\partial y} : f \mapsto \frac{\partial f}{\partial y}(z_0)$$

are a basis of T_p.

Proof. Analysis 3. □

Definition 3.7.3. A map $V : S \to \bigsqcup_{p \in S} T_p$ with $V_p = V(p) \in T_p$ is called **smooth vector field**, if for every $f \in C^\infty(S)$ the function

$$Vf(p) = V_p(f)$$

is smooth.

In local coordinates a smooth vector field can be written in the form $V = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$, where f and g are smooth functions.

Definition 3.7.4. Let T_p^* be the dual space of T_p, it is called the **cotangent space**. A (smooth) **differential form** or **1-Form** is a map $\omega : S \to \bigsqcup_{p \in S} T_p^*$ such that $\omega(p) \in T_p^*$ for every p and such that for every smooth vector field S the map

$$\begin{aligned} \omega(V) : S &\to \mathbb{C}, \\ p &\mapsto \omega(p)(V_p) \end{aligned}$$

is smooth.

In local coordinates z every differential form ω can be written in the form $\omega = \alpha dx + \beta dy$, where (dx, dy) is the basis dual to $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ and α, β are smooth functions.

So $V = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$ implies

$$\omega(V)(z) = \alpha(z)f(z) + \beta(z)g(z).$$

Definition 3.7.5. (Exterior derivative) Let $u : S \to \mathbb{R}$ or $u : S \to \mathbb{C}$. Locally in a chart define

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy.$$

This notation is konsistent for the coordinate functions $x, y : U \to \mathbb{R}$, that make up the chart, i.e., $dx = dx$ and $dy = dy$. In Analysis 3 one shows that this construction is independent of the chart. This means that to every smooth function u one gets a smooth differential form du.

The functions $z : U \to \mathbb{C}$ and $\overline{z} : U \to \mathbb{C}$ give differential forms. One gets

$$dz = dx + idy, \qquad d\overline{z} = dx - idy.$$

A given form $\omega = \alpha dx + \beta dy$ can also be written as

$$\omega = udz + vd\overline{z}$$

with

$$u = \frac{1}{2}(\alpha - i\beta), \qquad v = \frac{1}{2}(\alpha + i\beta).$$

The form ω is called $(1,0)$**-form**, if in every holomorphic chart it can be written as

$$\omega = udz.$$

It is called a $(0,1)$**-form**, if it can likewise be written as $\omega = vd\overline{z}$.

Lemma 3.7.6. *ω is a (1,0)-form, if and only if for every $p \in S$ the $\mathbb{R}$-linear map $\omega(p) : T_p \to \mathbb{C}$ is $\mathbb{C}$-linear.*

ω is a (0,1)-Form, if and only if for every $p \in S$ the $\mathbb{R}$-linear map $\omega(p) : T_p \to \mathbb{C}$ is anti-$\mathbb{C}$-linear, i.e., if for every $\lambda \in \mathbb{C}$ and $v \in T_p$ one has

$$\omega(p)(\lambda v) = \overline{\lambda}\omega(p)(v).$$

Every differential form is the sum of a uniquely determined (1,0) and a unique (0,1)-form. One write $\Omega^1(S)$ for the space of 1-forms, $\Omega^{1,0}(S)$ for the space of (1,0)-forms and $\Omega^{0,1}(S)$ for the space of (0,1)-forms. One has a direct decomposition into complex vector spaces

$$\Omega^1(S) = \Omega^{0,1}(S) \oplus \Omega^{0,1}(S).$$

Proof. The form dz is $\mathbb{C}$-linear and the form $d\bar{z}$ is anti-linear. Every $\mathbb{R}$-linear map $T : \mathbb{C} \to \mathbb{C}$ can be written in a unique way as $T = A + B$, where A is complext-linear and B is anti-linear. $\square$

Definition 3.7.7. The exterior differential $d : C^\infty(S) \to \Omega(S)$ can in local coordinates be written as

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy = \frac{\partial f}{\partial z}\, dz + \frac{\partial f}{\partial \bar{z}}\, d\bar{z},$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

We write $d' : C^\infty(S) \to \Omega^{1,0}(S)$ and $d'' : C^\infty(S) \to \Omega^{0,1}(S)$ for the projections,so

$$d'f = \frac{\partial f}{\partial z}\, dz, \qquad d''f = \frac{\partial f}{\partial \bar{z}}\, d\bar{z}.$$

Definition 3.7.8. Since $dz \wedge dz = 0 = d\bar{z} \wedge d\bar{z}$, we conclude that every 2-form η can, in local coordinates, be written as $\eta = u\, dz \wedge d\bar{z}$. We write the components of the exterior differential $d = d' + d''$ as in the **Hodge diamond**

$$\begin{tikzcd}
 & \Omega^2 & \\
\Omega^{1,0} \arrow[ur, "d''"] & & \Omega^{0,1} \arrow[ul, "d'"'] \\
 & C^\infty \arrow[ul, "d'"] \arrow[ur, "d''"'] &
\end{tikzcd}$$

Definition 3.7.9. A complex-valued differential form ω is called a **holomorphic form**, if in every holomorphic chart $\omega = udz$, where additionally u is required to be a holomorphic function. The $\mathbb{C}$-vector space of all holomorphic differential forms is written as $\Omega_{\text{hol}}(S)$.

A point $p \in S$ is called a **zero** of order n of a holomorphic differential form ω, if one can write ω around p in the form $\omega = \alpha dz$ with $\operatorname{ord}_p \alpha = n$.

Lemma 3.7.10. *The map $d : \mathcal{O}(S) \to \Omega_{\text{hol}}(S)$ is* **derivation**, *i.e., it is $\mathbb{C}$-linear and one has the* **Leibniz rule**:

$$d(fg) = fdg + gdf.$$

In local coordinates one has

$$df = \frac{\partial f}{\partial z} dz.$$

Proof. Let $D \in T_p$. One has

$$\begin{aligned} d_p(fg)(D) &= D(fg) = f(p)D(g) + g(p)D(f) \\ &= f(p)d_p(g)(D) + g(p)d_p(f)(D). \end{aligned}$$

The first formula follows from this. For the second, let $D \in T_p$ and z be a local coordinate with $z(p) = 0$. Then there is exactly one $\lambda \in \mathbb{C}$ with $D = \lambda \frac{\partial}{\partial z}$. So

$$df(D) = D(f) = \lambda \frac{\partial f}{\partial z} = \lambda \frac{\partial f}{\partial z} \cdot 1 = \frac{\partial f}{\partial z}\, dz \left(\lambda \frac{\partial}{\partial z} \right) = \frac{\partial f}{\partial z}\, dz\,(D)\,. \qquad \square$$

There are meromorphic differential forms, too:

Definition 3.7.11. Let S be a Riemann surface. A meromorphic differential form ω on S is by definition a holomorphic differential form on $S \smallsetminus P$, where P is a closed discrete subset of S, such that locally,

$$\omega = f dz$$

with a meromorphic function f.

Proposition 3.7.12. *Let S be a connected Riemann surface.*

(a) *Let $f \in \mathcal{M}(S)$. Then df is a meromorphic differential form.*

(b) *If ω is a meromorphic differential form and $f \in \mathcal{M}(S)$, then $f\omega$ is a meromorphic differential form. In this way the set $\Omega_{\text{mer}}(S)$ of all meromorphic differential forms is a vector space over the field $\mathcal{M}(S)$. The dimension of this space is at most 1.*

Proof. (a) In any local coordinate z we have $df = \frac{\partial f}{\partial z} dz$ and $\frac{\partial f}{\partial z}$ is meromorphic, again.

(b) The only non-trivial assertion is that about the dimension. If $\Omega_{\text{mer}}(S) = 0$, we have nothing to show. Otherwise, there is a meromorphic differential form $\omega \neq 0$ on S. Let η be another meromorphic differential form. In local coordinates we have $\eta = f dz$ and $\omega = g dz$, such that locally one has $\eta = (f/g)\omega$. This means that there is an open covering $(U_i)_{i \in I}$ of S and $h_i \in \mathcal{M}(U_i)$, such that on U_i one has $\eta = h_i \omega$. Then h_i and h_j agree on $U_i \cap U_j$ and the h_i define a meromorphic function $h \in \mathcal{M}(S)$ with $\eta = h\omega$. $\square$

Remark 3.7.13. One can show that on each Riemann surface there exists a non-constant meromorphic function. We don't present the proof as it requires more complex analysis than we are willing to invest here. For compact surfaces, however, we will give a proof later.

If f is a non-constant meromorphic function, then $\omega = df$ is a non-vanishing meromorphic differential form and for connected S the space $\Omega_{\text{mer}}(S)$ is indeed a one-dimensional $\mathcal{M}(S)$ vector space.

Definition 3.7.14. A complex-valued **2-Form** is a map η, that to each point $p \in S$ attaches an alternating bilinear form $\eta(p) : T_p \times T_p \to \mathbb{C}$.

For a 1-form $\omega \in \Omega^1(S)$, locally $\omega = \alpha dx + \beta dy$ the form

$$d\omega = \frac{\partial \alpha}{\partial y} dy \wedge dx + \frac{\partial \beta}{\partial x} dx \wedge dy = \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}\right) dx \wedge dy$$

is a well-defined 2-form, as is shown in Analysis 3.

Lemma 3.7.15. *We write*

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \qquad \text{und} \qquad \frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}.$$

If f is a smooth function, then

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \overline{z}} d\overline{z} =: d'f + d''f.$$

The function f is holomorphic if and only if $\frac{\partial f}{\partial \overline{z}} = 0$ and this is equivalent to df being a (1,0)-form.

Let $\omega = u dz + v d\overline{z}$ be a 1-form. Then

$$d\omega = \left(\frac{\partial v}{\partial z} - \frac{\partial u}{\partial \overline{z}}\right) dz \wedge d\overline{z}.$$

A (1,0)-form ω is holomorphic if and only if $d\omega = 0$.

Proof. The equation $\frac{\partial f}{\partial \overline{z}} = 0$ is equivalent to the Cauchy-Riemann equations. The rest is computation. □

Definition 3.7.16. A 1-form ω on S is called **closed**, if $d\omega = 0$. It is called **exact**, if there exists a smooth function f, such that $\omega = df$.

It follows that a (1,0)-form ω is exact if and only if it is the derivative of a holomorphic function. In this case ω is holomorphic.

Definition 3.7.17. Let $\omega = \alpha_1 dx + \beta_1 dy$ and $\eta = \alpha_2 dx + \beta_2 dy$ be 1-forms. Then

$$\omega \wedge \eta = (\alpha_1\beta_2 - \alpha_2\beta_1)\, dx \wedge dy$$

is a 2-form, called the **exterior product** of ω and η.

3.8 Integration of differential forms

Let $\gamma : [0,1] \to S$ be a smooth path and ω a 1-form. Then γ defines a piecewise smooth oriented submanifold and the integral $\int_\gamma \omega$ is defined as:

$$\int_\gamma \omega = \int_0^1 \omega(\gamma'(t))\, dt.$$

Here one views $\gamma'(t)$ as a tangent vector in $\gamma(t)$ and evaluates ω in this vector.

The integral of a 2-form over a surface S is defined by a partition of unity:

$$\int_S \omega = \sum_{i \in I} \int_{U_i} u_i \omega,$$

where the U_i are chart sets. In a local chart $z = x + iy$ one can write ω as $f(z)\, dx \wedge dy$ and define the integral as $\int_U f(z)\, dx\, dy$.

Lemma 3.8.1 (Local computation of path intgerals)**.** *Let ω be a 1-form on S, which in a chart z can be written as $\omega = \alpha dx + \beta dy$. Let $\gamma : [0,1] \to U$ be a smoot path U. Then*

$$\int_\gamma \omega = \int_0^1 \left(\alpha(\gamma(t)) \frac{\partial x(\gamma(t))}{\partial t} + \beta(\gamma(t)) \frac{\partial y(\gamma(t))}{\partial t} \right) dt.$$

Proof. A direct computation in local coordinates. □

Theorem 3.8.2 (Stokes)**.** *Let ω be a 1-form on the Riemann surface S and let the domain $D \subset S$ be relatively compact with piecewise smooth boundary. Then*

$$\int_{\partial D} \omega = \int_D d\omega.$$

Proof. Analysis 3. □

Corollary 3.8.3. *A 1-form ω is closed if and only if for any two homotopic paths γ, τ one has*

$$\int_\gamma \omega = \int_\tau \omega.$$

Proof. "⇐" Suppose that for any two homotopic paths the integrals are equal. Then for any disk $D \subset S$ the integral $\int_{\partial D} \omega$ vanishes, as the boundary is homotopic to a constant path. By Stokes's Theorem one gets $\int_D d\omega = 0$ for every disk and therefore $d\omega = 0$.

"⇒" Suppose ω is closed. If γ, τ are two homotopic paths, that doe not cross and are such that the image of a homotopy which moves γ into τ, covers the interior of the domain which is bounded by γ and τ, then the equality of the integrals follows from Stokes's Theorem. By dividing paths and homotopies into parts one gets the claim in general. □

Proposition 3.8.4. *Let S be a simply connected Riemann surface and ω a closed complex-valued smooth 1-form. Then ω possesses a primitive, i.e., there is a smooth function $f : S \to \mathbb{C}$ with $\omega = df$.*

Proof. Fix a point p_0 in S and define

$$f(p) = \int_{p_0}^{p} \omega,$$

where the integral is taken over any path connecting p_0 to p. This gives a well-defined function f, since any two paths connecting p_0 to p are homotopic, hence give the same integral. A local computation as in complex analysis, shows that $df = \omega$. □

Lemma 3.8.5. *Any two primitives for a given ω differ by a locally-constant function.*

Locally, every holomorphic form has a primitive.

Proof. In a local coordinate z we have $\omega = f dz$. A local primitive is a holomorphic function F with $F' = f$. Locally, such a function exists and is unique up to a constant. □

By the example $S = \mathbb{C}^\times$ we know that not every holomorphic differential form has a primitive.

Recall: Let $S = U \subset \mathbb{C}$ be open and $\gamma : [0, 1] \to S$ continuous differentiable and set

$$\int_\gamma f(z)\, dz = \int_0^1 f(\gamma(t))\, \gamma'(t)\, dt.$$

If f has a primitive F, then

$$\int_\gamma f(z)\, dz = F(\gamma(1)) - F(\gamma(0)).$$

3.9 Analytic continuation

Definition 3.9.1. Let S be a Riemann surface and $x \in S$. Consider the set M of all pairs (U, f), where $U \subset S$ is a connected open neighborhood of x and $f : U \to \mathbb{C}$ is a holomorphic function. Call two pairs (U, f) and $V, g)$ **equivalent**, if

$$f \equiv g \quad \text{on} \quad C(x, U \cap V),$$

where $C(x, U \cap V)$ is the connected component of x in $U \cap V$. A **function germ** or **germ** around x is an equivalence class $[U, f]$. The set of all germs around x is denoted by $\mathcal{O}_x(S)$ or $\mathcal{O}_x$.

Definition 3.9.2. Let $\gamma : [0, 1] \to S$ be a path. Let $f \in \mathcal{O}_{\gamma(0)}$ be a germ. A germ $g \in \mathcal{O}_{\gamma(1)}$ is called an **analytic continuation** of f along γ, if there exists a sequence $0 = t_0 < t_1 < \cdots < t_n = 1$, as well as open sets $U_i \subset S$ with $\gamma([t_i, t_{i+1}] \subset U_i$ and holomorphic functions $f_i : U_i \to \mathbb{C}$, such that

- $f_0 = f$, $f_n = g$ and
- f_i and f_{i+1} coincide on $U_i \cap U_{i+1}$.

By the identity theorem, an analytic continuation is unique, if it exists.

Examples 3.9.3.

- An analytic continuation does not always exist. For instance let $S = \mathbb{C}$, $\gamma(t) = t$ and $f(z) = \frac{1}{z-1}$. Then there is no analytic continuation.
- An analytic continuation depends on the path, not only the endpoints. Let for instance $f(z) = \log z$ be the standard-branch of the holomorphic logarithm in $U = \{\mathrm{Re}(z) > 0\} \subset \mathbb{C} = S$. Then let $m \in \mathbb{N}$ and $\gamma(t) = e^{2\pi i k t}$. Then the analytic continuation along the closed path γ equals $g(z) = 2\pi i k + \log z$.

Theorem 3.9.4. *Let $\omega \in \Omega_{\mathrm{hol}}(S)$ be a holomorphic differential form. Let $\gamma : [0, 1] \to S$ be a path and let $F \in \mathcal{O}_{\gamma(0)}$ be a germ of a primitive of ω at $\gamma(0)$. Then there is an analytic continuation of F along γ to $\gamma(1)$.*

Proof. For every point $t \in [0, 1]$ there is a connected open neighborhood $U_t \subset S$ on which ω has a primitive. The pre-image of U_t in $[0, 1]$ is open, so it contains an open interval $I(t) = (t - \varepsilon_t, t + \varepsilon_t)$ around t. As $[0, 1]$ is compact, finitely many of these sets $I(t_0), I(t_1), \dots, I(t_n)$ will cover the unit interval. Here we can assume

$0 = t_0 < t_1 < \cdots < t_n = 1$. Let $f_0 = F$ be a primitive of ω on $U_0 = U_{t_0}$. On U_1 there is exactly one primitive f_1 of ω, which coincides with f_0 on $U_0 \cap U_1$. On U_2 there is exactly one f_2 coinciding with f_1 and so on up to f_n. □

Corollary 3.9.5. *Let $\omega \in \Omega_{\mathrm{hol}}(S)$ satisfy $\int_\gamma \omega = 0$ for every closed path γ. Then ω has exactly one primitive up to constants.*

Theorem 3.9.6 (Analytic continuation and homotopy)**.** *Let S be a Riemann surface, α, β homotopic paths with endpoints p, q and let $f \in \mathcal{O}_p$. Assume that the analytyc continuation of f exists along each path $\gamma_s(\cdot) = h(s, \cdot)$, where h is a homotopy. Then the analytic continuations of f along α coincides with the continuation along β.*

Proof. Let $s \in [0, 1]$. As the analytic continuation along γ_s exists, there are open sets $U_0, \dots, U_n$ on which there are holomorphic functions successively extending f. In particular we have $\gamma_s([0, 1]) \subset U = U_0 \cup \cdots \cup U_n$. A simple compactness argument shows that there is $\varepsilon = \varepsilon(s) > 0$, such that for every $s' \in (s - \varepsilon, s + \varepsilon)$ the image of $\gamma_{s'}$ lies in U, too. Then the U_j give an analytic continuation along $\gamma_{s'}$, too and so that analytic continuations along all $s' \in (s - \varepsilon, s + \varepsilon)$ coincide.

As the unit interval is compact, there are $s_1 < \cdots < s_m$, such that the intevals $(s_j - \varepsilon(s_j), s_j + \varepsilon(s_j))$ cover all of $[0, 1]$. So the continuation along γ_0 coincides with the continuation along γ_{s_1} and this with the one along γ_{s_2} and so on until γ_1. □

Definition 3.9.7. Let $\omega \in \Omega_{\mathrm{mer}}(S)$ be a meromorphic differential form on S and in a local coordinate z with $z(p) = 0$ let $\omega = f dz$. Then

$$\mathrm{Res}_p \omega = \mathrm{Res}_0 f$$

is called the **residue** of ω in p.

Lemma 3.9.8. *The residue is well-defined, i.e., it does not depend on the choice of the local coordinate.*

Proof. In a given coordinate the residue can be described by the integral over a path encircling the point p. □

3.10 Harmonic forms

Definition 3.10.1. For a smooth 1-form $\omega \in \Omega(S) = \Omega^1(S)$ on a Riemann surface S, the complex conjugate $\overline{\omega}$ is a smooth 1-form again, since complex conjugation is $\mathbb{R}$-linear. We say that ω is **real**, if $\omega = \overline{\omega}$. The **real part** of a form ω is defined by

$$\mathrm{Re}(\omega) = \frac{1}{2}(\omega + \overline{\omega}).$$

A given form ω is uniquely decomposed as

$$\omega = \omega^{1,0} + \omega^{0,1} \in \Omega^{1,0} \oplus \Omega^{0,1} = \Omega.$$

We define the **Hodge star operator** as

$$*\omega = i\left(\overline{\omega^{1,0}} - \overline{\omega^{0,1}}\right).$$

Lemma 3.10.2. *The Hodge $*$-operator is an $\mathbb{R}$-linear map $\Omega \to \Omega$ with the following properties*

(a) $*\Omega^{0,1} = \Omega^{1,0}$ *and vice versa,*

(b) $**\omega = -\omega, \quad \overline{*\omega} = *\overline{\omega},$

(c) $*d'f = id''\overline{f}, \qquad *d''f = -id'\overline{f},$

(d) $d*df = 2id'd''\overline{f}.$

Proof. A computation. □

Definition 3.10.3. A 1-form $\omega \in \Omega^1(S)$ on a Riemann surface is called **harmonic**, if

$$d\omega = d*\omega = 0.$$

A smooth function f is called harmonic, if df is, i.e., of $d*df = 0$, or, equivalently, $d'd''f = 0$.

Proposition 3.10.4. *For a 1-form $\omega \in \Omega^1(S)$ the following are equivalent:*

(a) ω *is harmonic,*

(b) $d'\omega = d''\omega = 0,$

(c) $\omega = \omega_1 + \omega_2$, *where* $\omega_1 \in \Omega_{\mathrm{hol}}(S)$ *and* $\omega_2 \in \overline{\Omega_{\mathrm{hol}}(S)}$,

(d) *for every $p \in S$ there exists an open neighborhood U and a harmonic function f on U such that $\omega = df$.*

Proof. The equivalence of (a), (b) and (c) follows from Lemma 3.10.2.

(a)⇒(d): since a harmonic form ω is closed, the Poincaré Lemma implies that there exists a neighbourhood U of p, such that $\omega = df$ for same function f. Since $0 = d * \omega = d * df$, it follows that f is harmonic. The converse is trivial. □

Definition 3.10.5. The complex vector space of all harmonic 1-forms on a surface S will be denoted as $\mathrm{Harm}^1(S)$. We have

$$\mathrm{Harm}^1(S) = \Omega_{\mathrm{hol}}(S) \oplus \overline{\Omega_{\mathrm{hol}}(S)}.$$

Theorem 3.10.6. *Every real harmonic 1-form $\sigma \in \mathrm{Harm}^1(S)$ is the real part of exactly one holomorphic 1-form $\omega \in \Omega_{\mathrm{hol}}(S)$.*

Proof. Suppose $\sigma = \omega_1 + \overline{\omega}_2$ with $\omega_1, \omega_2 \in \Omega_{\mathrm{hol}}(S)$. Since $\omega_1 + \overline{\omega}_2 = \sigma = \overline{\sigma} = \overline{\omega}_1 + \omega_2$ we get $\omega_1 = \omega_2$ and hence $\sigma = \mathrm{Re}(2\omega_1)$. To prove uniqueness, assume $\omega \in \Omega_{\mathrm{hol}}(S)$ with $\mathrm{Re}(\omega) = 0$. Since locally we have $\omega = df$ for a holomorphic function f it follows that f has constant real part. Then f is constant and $\omega = 0$. □

For later use, we note the Lemma of Dolbeault:

Lemma 3.10.7 (Dolbeault's lemma)**.** *Let $U \subset \mathbb{C}$ be open and star-shaped. For every $f \in C^\infty(U)$ there exists $g \in C^\infty(U)$, such that*

$$\frac{\partial g}{\partial \bar{z}} = f.$$

Proof. Consider the 1-form $\omega = f d\bar{z}$. By the Poincaré Lemma there exists $g \in C^\infty(U)$ such that $\frac{\partial g}{\partial z} dz + \frac{\partial g}{\partial \bar{z}} d\bar{z} = dg = \omega$. The claim follows. □

Corollary 3.10.8. (a) *For a Riemann surface, the sequence of sheaves*

$$0 \to \mathcal{O} \to C^\infty \xrightarrow{d''} \Omega^{0,1} \to 0$$

is exact. This follows from Dolbeault's lemma 3.10.7.

(b) *For a Riemann surface S one has*

$$H^1(S,\mathcal{O}) \cong \Omega^{0,1}(S)/d''C^\infty(S)$$

and

$$H^1(S,\Omega_{\mathrm{hol}}) \cong \Omega^2(S)/d\Omega^{1,0}(S).$$

Proof. □

* * *

4 Sheaves

4.1 Presheaves

Definition 4.1.1. Let S be a topological space. A **resheaf** $\mathcal{F}$ is a mapping

$$U \mapsto \mathcal{F}(U),$$

attaching to an open set $U \subset S$ an abelian group $\mathcal{F}(U)$, together with group homomorphisms, the so called **restriction homomorphisms**,

$$\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V),$$

for all open sets $V \subset U$, such that the following axioms are satisfied

(i) $\mathcal{F}(\emptyset) = 0$,

(ii) $\rho_U^U = \mathrm{Id}$ for every U,

(iii) $\rho_W^V \circ \rho_V^U = \rho_W^U$, whenever $W \subset V \subset U$.

We write $s|_V$ instead of $\rho_V^U(s)$. The elements of $\mathcal{F}(U)$ are called **sections** of $\mathcal{F}$ on U.

Definition 4.1.2. A **morphism of presheaves** $\phi : \mathcal{F} \to \mathcal{G}$ is given by group homomorphisms $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for all open $U \subset S$, which commute with the restriction homomorphisms.

A **sheaf** is a presheaf which additionally satisfies two principles:

(i) (uniqueness): If $s \in \mathcal{F}(U)$ and $U = \bigcup_{i \in I} U_i$ is an open covering of the open set $U \subset S$, and if $s|_{U_i} = 0$ for every $i \in I$, then $s = 0$.

(ii) (existence) If $U = \bigcup_{i \in I}$ is an open covering and if for every $i \in I$ there is given an $s_i \in \mathcal{F}(U_i)$ with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for every pair $i, j \in I$, then there is $s \in \mathcal{F}(U)$ with $s_i = s|_{U_i}$ for every $i \in I$.

Definition 4.1.3. Morphismen of sheaves are defined to be the morphisms of the presheaves.

Examples 4.1.4.
- Let S be a Riemann surface. Then $U \mapsto \mathcal{O}(U)$ and $U \mapsto \mathcal{M}(U)$ are sheaves on S with addition as group law. The map $U \mapsto \mathcal{O}(U)^\times$ is a sheaf, where $\mathcal{O}(U)^\times$ is the multiplicative group of nowhere vanishing holomorphic functions.
- Let S be a Riemann surface, then the map $U \mapsto \Omega_{\mathrm{hol}}(U)$ is a sheaf.

- Let S be a topological space. Then the map $U \mapsto C(U, \mathbb{R})$ is a sheaf.
- Let S be a topological space and A an abelian group. The map $U \mapsto A$ isnot generally a sheaf, since the axioms imply that for two disjoint open sets $U, V \subset S$ one has $\mathcal{F}(U \cup V) = \mathcal{F}(U) \oplus \mathcal{F}(V)$.

  The so-called **constant sheaf** $\mathcal{K}_A$ with group A maps any open set U to the group $\mathcal{K}_A(U)$ of all locally-constant maps $\sigma : U \to A$.
- Let S be a Riemann surface. The **structure sheaf** is the sheaf $\mathcal{O}$, which attaches to any open set U the additive group $\mathcal{O}(U)$ of holomorphic functions on U. This is indeed a **sheaf of rings**.

Remark 4.1.5. Let $\mathcal{F}$ be a sheaf on the topological space S. Then

$$\mathcal{F}(\emptyset) = 0.$$

Proof. Let $s \in \mathcal{F}(\emptyset)$ and let $I = \emptyset$. Then we have the empty covering

$$\emptyset = \bigcup_{i \in I} U_i$$

and for every $i \in I$ (there are none) we have that $s|_{U_i} = 0$ (we could claim whatever, here). Hence by uniqueness, we have $s = 0$. □

Definition 4.1.6. Let S be a topological space, $\mathcal{F}$ a presheaf of abelian groups, $p \in S$ a point. The **stalk** of $\mathcal{F}$ in p is defined to be

$$\mathcal{F}_p = \varinjlim_{U \ni p} \mathcal{F}(U),$$

where U runs through the set of all open neighborhoods of x. An element of $\mathcal{F}_p$ therefore is a germ in p.

This means that

$$\mathcal{F}_p = \left\{(s, U) : U \text{ is open}, x \in U,\ s \in \mathcal{F}(U)\right\} \Big/ \sim$$

where $\sim$ is the equivalence relation given as follows: $(s, U) \sim (t, V)$ if and only if there exists $W \subset U \cap V$ open with $x \in W$ and

$$s|_W = t|_W.$$

Lemma 4.1.7. *Let $\mathcal{F}$ be a sheaf with $\mathcal{F}_p = 0$ for every $p \in S$. Then $\mathcal{F} = 0$.*

Proof. Let $s \in \mathcal{F}(U)$ for some open $U \subset S$ and let $p \in U$. Since the germ s_p vanishes, there is an open set $U_p \subset U$ with $s|_{U_p} = 0$. The sets $(U_p)_{p\in U}$ form an open covering of U, so that by the local uniqueness, we have $s = 0$. □

4.2 Complexes

Definition 4.2.1. A sequence of homomorphisms of abelian groups

$$\dots \xrightarrow{d_{-2}} G_{-1} \xrightarrow{d_{-1}} G_0 \xrightarrow{d_0} G_1 \xrightarrow{d_1} \dots$$

is called a **complex**, if $d_j d_{j-1} = 0$ for every $j \in \mathbb{Z}$. This is equivalent to

$$\mathrm{Im}(d_{j-1}) \subset \ker(d_j)$$

for every j. A complex is called **exact**, if $\mathrm{Im}(d_j) = \ker(d_{j+1})$ holds for all j. In this case one also speaks of an **exact sequence**.

Definition 4.2.2. For a complex $A = (A_i, d_i)_{i\in\mathbb{Z}}$ the quotient group

$$H^j(A) = \ker(d_j)/\,\mathrm{Im}(d_{j-1})$$

is called the j-th **cohomology group** of the complex.

Definition 4.2.3. A **homomorphism of complexes** $\phi : A_\bullet \to B_\bullet$ is a family of group homomorphisms $\phi_p : A_p \to B_p$ such that for every $p \in \mathbb{Z}$ the diagram

$$\begin{tikzcd} A_p \arrow[r, "d"] \arrow[d, "\phi_p"'] & A_{p+1} \arrow[d, "\phi_{p+1}"] \\ B_p \arrow[r, "d"] & B_{p+1} \end{tikzcd}$$

commutes.

A homomorphism of complexes will map the kernel of d to the kernel of d and likewise for the image, so it induces a homomorphism of the cohomology

$$\phi_* : H^p(A) \to H^p(B).$$

Definition 4.2.4. A homomorphism of complexes $\phi : A \to B$ is called **null-homotopic**, if there are group homomorphisms $h_p : A_p \to B_{p-1}$ such that

$$\phi_p = dh_p + h_{p+1}d.$$

We picture this situation by the (non-commutative!) diagram

$$\begin{tikzcd} A_{p-1} \arrow[r] \arrow[d, "\phi_{p-1}"'] & A_p \arrow[r] \arrow[dl, "h_p"'] \arrow[d, "\phi_p"'] & A_{p+1} \arrow[dl, "h_{p+1}"'] \arrow[d, "\phi_{p+1}"'] \\ B_{p-1} \arrow[r] & B_p \arrow[r] & B_{p+1} \end{tikzcd}$$

Lemma 4.2.5. *Let $\phi : A \to B$ be a morphism of complexes. If ϕ is null-homotopic, then ϕ_* is the zero map,*

$$\phi_* = 0.$$

Proof. Assume $\phi_p = dh_p + h_{p+1}d$. Let $s \in A_p$ with $ds = 0$. Then one has

$$\phi_p(s) = dh_p(s) + h_{p+1}\underbrace{ds}_{=0} = d(h_p(s)),$$

so $\phi(s)$ lies in the image of d and therefore is zero in the cohomology. □

4.3 Cech-cohomology

Definition 4.3.1. Let $\underline{U} = (U_i)_{i\in I}$ be a covering of S. A **Cech-cochain** of a sheaf $\mathcal{F}$ is an element of the group

$$C^p(\underline{U}, \mathcal{F}) = \prod_{(i_0,\dots,i_p)\in I^{p+1}} \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_p}).$$

We view such an element as a map s mapping a tuple $(i_0, \dots, i_p) \in I^{p+1}$ to an element $s(i_0, \dots, i_p)$ of the group $\mathcal{F}(U_{i_0} \cap \dots \cap U_{i_p})$.

Definition 4.3.2. The **coboundary-operator** $\check{d} : C^p(\underline{U}, \mathcal{F}) \to C^{p+1}(\underline{U}, \mathcal{F})$ is defined by

$$\check{d}(s)(i_0, \dots, i_{p+1}) = \sum_{j=0}^{p+1} (-1)^j s(i_0, \dots, \widehat{i_j}, \dots, i_{p+1}).$$

Lemma 4.3.3. *One has $\check{d}^2 = 0$, so*

$$0 \to C^0(\underline{U}, \mathcal{F}) \xrightarrow{\check{d}} C^1(\underline{U}, \mathcal{F}) \xrightarrow{\check{d}} \dots$$

is a complex.

Proof. We compute

$$\begin{aligned}
\check{d}^2 s(i_0,\dots,i_{p+2}) &= \sum_{j=0}^{p+2} (-1)^j \check{d}s(i_0,\dots,\widehat{i_j},\dots,i_{p+2}) \\
&= \sum_{j=0}^{p+2} (-1)^j \sum_{k=0}^{j-1} (-1)^k s(i_0,\dots,\widehat{i_k},\dots,\widehat{i_j},\dots,i_{p+2}) \\
&\quad + \sum_{j=0}^{p+2} (-1)^j \sum_{k=j+1}^{p+2} (-1)^{k-1} s(i_0,\dots,\widehat{i_j},\dots,\widehat{i_k},\dots,i_{p+2}) \\
&= \sum_{j<k} (-1)^{j+k} s(i_0,\dots,\widehat{i_k},\dots,\widehat{i_j},\dots,i_{p+2}) \\
&\quad - \sum_{k<j} (-1)^{j+k} s(i_0,\dots,\widehat{i_j},\dots,\widehat{i_k},\dots,i_{p+2}) = 0.
\end{aligned}$$

□

Definition 4.3.4. A given $s \in C^p(\underline{U},\mathcal{F})$ is called **cocycle**, if $\check{d}(s) = 0$. The group of all p-cocycles is denoted by $Z^p(\underline{U},\mathcal{F})$. The image $B^p(\underline{U},\mathcal{F}) := d(C^{p-1}(\underline{U},\mathcal{F}))$ is a subgroup of $Z^p(\underline{U},\mathcal{F})$, the elements of which we call p-**coboundaries**.

Definition 4.3.5. The group

$$H^p(\underline{U},\mathcal{F}) := Z^p(\underline{U},\mathcal{F})/B^p(\underline{U},\mathcal{F})$$

is called the p-th **cohomology group** of $\mathcal{F}$ with respect to $\underline{U}$.

By definition we have that

$$H^0(\underline{U},\mathcal{F}) = Z^0(\underline{U},\mathcal{F}) = \mathcal{F}(S)$$

is the group of global sections of $\mathcal{F}$.

Definition 4.3.6. A covering $\underline{V}$ is called a **refinement** of a covering $\underline{U}$, if for every $V \in \underline{V}$ there is a $U \in \underline{U}$ with $V \subset U$.

In this case we write $\underline{V} \geq \underline{U}$.

Let $\underline{V} \geq \underline{U}$ and $\underline{U} = (U_i)_{i\in I}$, as well as $\underline{V} = (V_j)_{j\in J}$. We then get a map $\tau : J \to I$ with $V_j \subset U_{\tau(j)}$. The restriction induces a comparison map

$$\tau^c : C^p(\underline{U},\mathcal{F}) \to C^p(\underline{V},\mathcal{F}),$$

given by

$$\tau^c(s)(j_0,\dots,j_p) = s(\tau(j_0),\dots,\tau(j_q))\Big|_{V_{j_0}\cap\dots\cap V_{j_q}}.$$

Lemma 4.3.7. *The map τ^c is a morphism of complexes and thus induces comparison maps*

$$v_{\underline{U}}^{\underline{V}} := \tau^* : H^p(\underline{U}, \mathcal{F}) \to H^p(\underline{V}, \mathcal{F}).$$

These do not depend on the choice of τ. For $p = 0$ this map is bijective and for $p = 1$ it is injective.

Proof. It is easily verified that τ^c is a group morphism and that $\tau^c d = d\tau^c$.

Bijectivity at $p = 0$ is clear as both sides coincide with the group $\mathcal{F}(S)$.

Now let $\gamma : J \to I$ be another map with the property, that $V_j \subset U_{\gamma(j)}$ for every $j \in J$. We have to show that $\tau^* = \gamma^*$. To show this, we construct a homotopy. Define $h_q : C^q(\underline{U}, \mathcal{F}) \to C^{q-1}(\underline{V}, \mathcal{F})$ by

$$h_q(f)(j_0, \ldots, j_{q-1}) = \sum_{k=0}^{q-1} (-1)^k f(\tau(j_0), \ldots, \tau(j_k), \gamma(j_k), \ldots, \gamma(j_{q-1}))\Big|_{V_{j_0} \cap \cdots \cap V_{j_{q-1}}}.$$

Next we show that $\tau_q^c - \gamma_q^c = h_{q+1}\check{d} + \check{d}h_q$, so that $\tau_q^c - \gamma_q^c$ is null-homotopic. For this we compute

$$\begin{aligned}
&(h_{q+1}\check{d} + \check{d}h_q)(s)(j_0, \ldots, j_q) \\
&= \sum_{k=0}^{q} (-1)^k \check{d}s\Big(\tau(j_0), \ldots, \tau(j_k), \gamma(j_k), \ldots, \gamma(j_q)\Big)\Big|_{V_{j_0} \cap \cdots \cap V_{j_q}} \\
&\quad + \sum_{k=0}^{p} (-1)^k h_q s(j_0, \ldots, \widehat{j_k}, \ldots, j_q) \\
&= \sum_{k=0}^{q} (-1)^k \sum_{\nu=0}^{k} (-1)^\nu s\Big(\tau(j_0), \ldots, \widehat{\tau(j_\nu)}, \ldots \tau(j_k), \gamma(j_k), \ldots, \gamma(j_q)\Big)\Big|_{V_{j_0} \cap \cdots \cap V_{j_q}} \\
&\quad + \sum_{k=0}^{q} (-1)^k \sum_{\nu=k}^{q} (-1)^{\nu+1} s\Big(\tau(j_0), \ldots, \tau(j_k), \gamma(j_k), \ldots, \widehat{\gamma(j_\nu)}, \ldots, \gamma(j_q)\Big)\Big|_{V_{j_0} \cap \cdots \cap V_{j_q}} \\
&\quad + \sum_{k=0}^{p} (-1)^k \sum_{\nu=0}^{k-1} (-1)^\nu s(\tau(j_0), \ldots, \tau(j_\nu), \gamma(j_\nu), \ldots, \widehat{\gamma(j_k)}, \ldots, \gamma(j_q))\Big|_{V_{j_0} \cap \cdots \cap V_{j_q}} \\
&\quad + \sum_{k=0}^{p} (-1)^k \sum_{\nu=k}^{q} (-1)^{\nu+1} s(\tau(j_0), \ldots, \widehat{\tau(j_k)}, \ldots, \tau(j_\nu), \gamma(j_\nu), \ldots, \gamma(j_q))\Big|_{V_{j_0} \cap \cdots \cap V_{j_q}}
\end{aligned}$$

The first and the last row cancel. The two middle rows cancel up to the first and the last summand. These are $\tau^c - \gamma^c$. By Lemma 4.2.5 it follows $\tau^* = \gamma^*$.

We finally show injectivity at $p = 1$. For this let $s \in Z^1(\underline{U}, \mathcal{F})$ with the property, that $\tau^c(s) = \check{d}t$ for some $t \in C^0(\underline{V}, \mathcal{F})$, i.e., for all $(j_0, j_1) \in J^2$ we have $s(\tau(j_0), \tau(j_1)) = t(j_1) - t(j_0)$ on $V_{j_0} \cap V_{j_1}$. The cocycle property of s says that on $U_{i_0} \cap U_{i_1} \cap U_{i_2}$ one has $s(i_1, i_2) - s(i_0, i_2) + s(i_0, i_1) = 0$ for every $(i_0, i_1, i_2) \in I^3$. Let $i \in I$ and $j_0, j_1 \in J$, then on $U_i \cap V_{j_0} \cap V_{j_1}$ one has

$$t(j_1) - t(j_0) = s(\tau(j_0), \tau(j_1)) = s(\tau(j_0), i) - s(\tau(j_1), i),$$

or

$$s(\tau(j_0), i) + t(j_0) = s(\tau(j_1), i) + t(j_1).$$

This means that one can glue these sums to get a section $h_i \in \mathcal{F}(U_i)$, which satisfies $h_i = s(\tau(j), i) + t(j)$ on $U_i \cap V_j$. These h_i hence give an element $h \in C^0(\underline{U}, \mathcal{F})$ and we claim that $\check{d}(-h) = s$. On $U_{i_0} \cap U_{i_1} \cap V_{j_0}$ one has

$$\begin{aligned} -\check{d}h(i_0, i_1) &= h(i_0) - h(i_1) \\ &= s(\tau(j_0), i_1) - s(\tau(j_0), i_0) \\ &= s(i_0, i_1). \end{aligned}$$

□

Definition 4.3.8. For any two coverings $(U_i)_{i\in I}$ and $V_j)_{j\in J}$ there is a common refinement $(U_i \cap V_j)_{(i,j)}$. So the set of all coverings forms a directed set. On the union $\bigsqcup_{\underline{U}} H^p(\underline{U}, \mathcal{F})$ one defines an equivalence relation by

$$s \sim v^{\underline{U}}_{\underline{V}}(s)$$

if $s \in H^p(\underline{U}, \mathcal{F})$ and $\underline{V}$ is finer than $\underline{U}$. The quotient becomes a group with the addition of H^p and is called the **direct limit**. It is written as

$$\varinjlim_{\underline{U}} H^p(\underline{U}, \mathcal{F}).$$

Definition 4.3.9. The **Cech-cohomology group** of the sheaf $\mathcal{F}$ is defined by

$$H^p(S, \mathcal{F}) = \varinjlim_{\underline{U}} H^p(\underline{U}, \mathcal{F}).$$

In the case when $\mathcal{F}$ is a sheaf of $\mathbb{C}$-vector spaces, the cohomology groups are $\mathbb{C}$-vector spaces, too and we write

$$h^p(S, \mathcal{F}) = \dim^p_{\mathcal{H}}(S, \mathcal{F}).$$

Proposition 4.3.10. *Let S be a Riemann surface and let C^∞ be the sheaf of all infinitely differentiable functions $f : U \to \mathbb{C}$. Then*

$$H^1(S, C^\infty) = 0.$$

Proof. By Analysis 3 there is a partition of unity, i.e., for every locally finite covering $(U_i)_{i\in I}$ there is a family $(u_i)_{i\in I}$ of smooth functions $u_i : S \to [0,1]$ with

- $\operatorname{supp}(u_i) \subset U_i$,
- for every $x \in S$ there is a neighborhood U, such that $u_i|_U = 0$ for all but finitely many $i \in I$,
- $\sum_{i\in I} u_i = 1$.

Let $s \in Z^1(\underline{U}, C^\infty)$, so $\check{d}s = 0$, which means that

$$0 = s(j,i) - s(k,i) + s(k,j)$$

on $U_k \cap U_j \cap U_i$. The function $u_i s(i,j) \in C^\infty(U_i \cap U_j)$ has support in U_i. The extension by zero to U_j still is infinitely differentiable, so $u_i s(i,j) \in C^\infty(U_j)$. For every $j \in I$ we set $g_j = \sum_{k\in I} u_k s(k,j) \in C^\infty(U_j)$. Then $g \in C^0(\underline{U}, C^\infty)$ and

$$\check{d}g(i,j) = g(j) - g(i) = \sum_{k\in I} u_k(s(k,j) - s(k,i)) = -\sum_{k\in I} u_k s(j,i) = -s(j,i),$$

so $\check{d}(-g) = s$ and therefore the cohomology vanishes. □

Proposition 4.3.11. *Let Ω^1 be the sheaf of all $\mathbb{C}$-valued, smooth 1-differential forms on th Riemann surface S. Then*

$$H^1(S, \Omega^1) = 0.$$

Proof. Up to trivial reformulation the same proof as in the last proposition applies. □

Theorem 4.3.12. *The first cohomology groups with constant coefficients $\mathbb{Z}$ and $\mathbb{C}$ vanish on a simply connected surface S, i.e.,*

$$H^1(S, \mathbb{C}) = H^1(S, \mathbb{Z}) = 0.$$

Proof. Let $s \in Z^1(\underline{U}, \mathbb{C})$ be given. Since locally constant functions are infinitely differentiable we can view s as an element of $Z^1(\underline{U}, C^\infty)$. By Proposition 4.3.10 there is $g \in C^0(\underline{U}, C^\infty)$ with $\check{d}(g) = s$. Every g_i is a smooth function on U_i. The derivatives $dg_i \in \Omega^1(U_i)$ coincide on the intersections, as

$$dg_i|_{U_i \cap U_j} - dg_j|_{U_i \cap U_j} = ds(i,j) = 0.$$

Therefore there is a globally defined differential form $\omega \in \Omega^1(S)$ with $\omega|_{U_i} = dg_i$. The form ω is exact and so it is closed. Since S is simply connected, by Proposition 3.8.4 there is a primitive to ω, so a smooth function f with $df = \omega$. The function $g_i - f$ has derivatie zero on U_i, so it is locally constant and so $(g_i - f)_{i \in I} \in C^0(\underline{U}, \mathbb{C})$. The coboundary of this is $\check{d}(g_i - f) = \check{d}(g) = s$, so s yields the trivial cohomology class.

For the case of integer cohomology let $b \in Z^1(\underline{U}, \mathbb{Z})$ be given. As $\mathbb{Z} \subset \mathbb{C}$, by the first part there is $g \in C^0(\underline{U}, \mathbb{C})$ with $\check{d}g = b$, so $b(i,j) = g(j) - g(i)$. This means that g_j is locally constant on U_j and $g_i - g_j$ is integer valued on $U_i \cap U_j$. Write $e(z) = e^{2\pi i z}$. The locally constant functions $e(g_j)$ coincide on the intersections, so $e(g_i) = e(g_j)$ on $U_i \cap U_j$, such that there exists a locally constant function E on S with $E|_{U_i} = e(g_i)$. Since S is connected, the function E is constant. Let $f \in C$ such that $e(f) = E$. Then $g_i - f$ is integer-valued, as $e(g_i - f) = 1$. So $(g_i - f)_i \in C^0(\underline{U}, \mathbb{Z})$ and $\check{d}(g_i - f) = b$. $\square$

Theorem 4.3.13 (Leray)**.** *Let $\mathcal{F}$ be a sheaf and $\underline{U} = (U_i)_{i \in I}$ an open covering with the property $H^1(U_i, \mathcal{F}) = 0$ for every $i \in I$. Then the natural map*

$$H^1(\underline{U}, \mathcal{F}) \to H^1(S, \mathcal{F})$$

is an isomorphism.

Such a covering is called a **Leray-covering**.

Proof. It is enough to show that for every finer covering $\underline{V} = (V_\nu)_{\nu \in J}$ the comparison map $H^1(\underline{U}, \mathcal{F}) \to H^1(\underline{V}, \mathcal{F})$ is an isomorphism. So let $\tau : J \to I$ be a map with $V_\nu \subset U_{\tau(\nu)}$ for every $\nu \in J$. Let $f \in Z^1(\underline{V}, \mathcal{F})$ be a cocycle. We have to show that there is a cocycle $F \in Z^1(\underline{U}, \mathcal{F})$, such that

$$\tau^c F - f$$

lies in $B^1(\underline{V}, \mathcal{F})$. Let $i \in I$. Then $\underline{V}|_{U_i}$ is an open covering of U_i. By assumption we have $H^1(U_i, \mathcal{F}) = 0$ and since the map $H^1(\underline{V}|_{U_i}, \mathcal{F}) \to H^1(U_i, \mathcal{F})$ is injective by Lemma 4.3.7, we get $H^1(\underline{V}|_{U_i}, \mathcal{F}) = 0$. This means that for every $\nu \in J$ there is a $g(i,\nu) \in \mathcal{F}(U_i \cap V_\nu,$

such that

$$f(\nu,\mu) = g(i,\nu) - g(i,\mu) \quad \text{on} \quad U_i \cap V_\nu \cap V_\mu.$$

On the intersection $U_i \cap U_j \cap V_\nu \cap V_\mu$ one has

$$g(j,\nu) - g(i,\nu) = g(j,\mu) - g(i,\mu).$$

So there exists an $F(i,j) \in \mathcal{F}(U_i \cap U_j)$ such that

$$F(i,j) = g(j,\nu) - g(i,\nu) \quad \text{on} \quad U_i \cap U_j \cap V_\nu.$$

This F satisfies the cocycle relation and lies in $Z^1(\underline{U}, \mathcal{F})$. Let $h_\nu = g(\tau(\nu),\nu)|_{V_\nu} \in \mathcal{F}(V_\nu)$. On $V_\nu \cap V_\mu$ one has

$$\begin{aligned} F(\tau(\nu),\tau(\mu)) - f(\nu,\mu) &= (g(\tau(\mu),\nu) - g(\tau(\nu),\nu)) - (g(\tau(\mu),\nu) - g(\tau(\mu),\mu)) \\ &= g(\tau(\mu),\mu) - g(\tau(nu),\nu) = h_\mu - h_\nu. \end{aligned}$$

□

Example 4.3.14. We show

$$H^1(\mathbb{C}^\times, \mathbb{Z}) = \mathbb{Z}.$$

Proof. Let $U_1 = \mathbb{C}^\times \smallsetminus \mathbb{R}_-$ and $U_2 = \mathbb{C}^\times \smallsetminus \mathbb{R}_+$, where $\mathbb{R}_+$ and $\mathbb{R}_-$ is the positive, resp. negative real axis. Since U_i is star-shaped, we have $H^1(U_i, \mathbb{Z}) = 0$ by Theorem 4.3.12. By Leray's Theorem we therefore get $H^1(\mathbb{C}^\times, \mathbb{Z}) = H^1(\underline{U}, \mathbb{Z})$. Let $a \in Z^1(\underline{U}, \mathbb{Z})$ be a cocycle, so $a(j,k) - a(i,k) + a(i,j) = 0$. Then with $i = j = k$, we get $a(i,i) = 0$ and with $k = i$, that $a(i,j) = -a(j,i)$. Therefore a is completely determined by $a(1,2)$ and so $Z^1(\underline{U}, \mathbb{Z}) \cong \mathbb{Z}(U_1 \cap U_2) \cong \mathbb{Z} \times \mathbb{Z}$, as $U_1 \cap U_2$ decomposes in two components. As U_i is connected, one has $\mathbb{Z}(U_i) = \mathbb{Z}$ and so $C^0(\underline{U}, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$. The coboundary operator is, via these isomorphisms, given by

$$\begin{aligned} &\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, \\ &(b_1, b_2) \mapsto (b_2 - b_1, b_1 - b_2). \end{aligned}$$

The claim follows. □

Analogously one gets $H^1(\mathbb{C}^\times, \mathbb{C}) \cong \mathbb{C}$.

4.4 The long cohomology sequence

Definition 4.4.1. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on S. A **sheaf homomorphism** ϕ consists of a group homomorphism $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for every open $U \subset S$, which are compatible with the restriction homomorphisms, so for all $U \subset V$ ooen in S, the diagram

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\rho_U^V"] \arrow[d, "\phi(V)"'] & \mathcal{F}(U) \arrow[d, "\phi(U)"] \\ \mathcal{G}(V) \arrow[r, "\rho_U^V"'] & \mathcal{G}(U) \end{tikzcd}$$

commutes.

Examples 4.4.2.
- The inclusion of the constant sheaf $\mathbb{C}$ into the sheaf C^∞ is a sheaf homomorphism.
- The exterior derivative $d : C^\infty \to \Omega$ is a sheaf homomorphism.
- Let $\mathcal{O}$ be the structure sheaf and let $\mathcal{O}^\times$ be the sheaf defined by

$$\mathcal{O}^\times(U) = \left\{ f \in \mathcal{O}(U) : f \text{ has no zero} \right\}$$

  Then the map $f \mapsto \mathrm{e}(f) = e^{2\pi i f(z)}$ is a sheaf homomorphism $\mathcal{O} \to \mathcal{O}^\times$.

Definition 4.4.3. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a sheaf homomorphism. Then the kernel $\ker(\phi)$ is a sheaf defined by

$$\ker(\phi)(U) := \ker(\phi(U)).$$

Example 4.4.4. Let $\mathrm{e} : \mathcal{O} \to \mathcal{O}^\times$ be the exponential map. Then the image $\mathrm{Im}(\mathrm{e})(U) = \mathrm{Im}(\mathrm{e}(U))$ defines a pre-sheaf, but generally not a sheaf. As an example let $S = \mathbb{C}^\times$, let $U_1 = \mathbb{C} \smallsetminus (-\infty, 0]$ and $U_2 = \mathbb{C} \smallsetminus [0, +\infty)$. As the U_i are simply connected, the map $\mathrm{e}(U_i)$ is surjective, so, for instance the map $f(z) = z$ lies in the image of each $e(U_i)$. These functions glue to give $f(z) = z$, but thie map is not in the image of the exponential map, as on $\mathbb{C}^\times$ there is no holomorphic logarithm.

Proposition 4.4.5. *Let $\mathcal{F}$ be a presheaf. Then there exists a sheaf $\mathcal{F}^+$ and a presheaf homomorphism $\theta : \mathcal{F} \to \mathcal{F}^+$ with the property that every presheaf homomorphism $\phi : \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf, factors uniquely over θ. So for each ϕ there is a uniquely determined sheaf homomorphism ψ such that the diagram*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\theta"] \arrow[dr, "\phi"'] & \mathcal{F}^+ \arrow[d, dotted, "\exists!\ \psi"] \\ & \mathcal{G} \end{tikzcd}$$

commutes. The pair $(\mathcal{F}^+, \theta)$ is unique up to isomorphy. One calls $\mathcal{F}^+$ the **sheafification** *of*

$\mathcal{F}$. So for every sheaf $\mathcal{G}$ one has

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}^+, \mathcal{G}).$$

The homomorphism θ induces isomorphisms of all stalks

$$\theta_x : \mathcal{F}_x \xrightarrow{\cong} \mathcal{F}_x^+.$$

Proof. We construct the sheaf $\mathcal{F}^+$ as followst. For an open set $U \subset S$ let $\mathcal{F}^+(U)$ be the set of all maps s from U to the disjoint union of stalks $\bigsqcup_{x \in U} \mathcal{F}_x$ such that

- for every $x \in U$ one has $s(x) \in \mathcal{F}_x$ and
- for every $x \in U$ there is an open neighborhood $V \subset U$ and a $t \in \mathcal{F}(V)$, so that for every $y \in V$ we have $t(y) = s(y)$.

The properties of $\mathcal{F}^+$ are immediate. The uniqueness follows formally from the universal property. For $x \in S$ the map θ induces an isomorphism of the stalk $\mathcal{F}_x$ to the stalk $\mathcal{F}_x^+$. In the case when $\mathcal{F}$ already is a sheaf, the map θ is an isomorphism. □

Definition 4.4.6. We define the **image sheaf** of a sheaf homomorphism $\phi : \mathcal{F} \to \mathcal{G}$ as the sheafification of the presheaf $U \mapsto \mathrm{Im}\, \phi_U$ and we write this sheaf as $\mathrm{Im}(\phi)$.

Definition 4.4.7. A sequence of sheaf homomorphisms

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

is called **exact**, if $gf = 0$ and the induced homomorphism $\mathrm{Im}(f) \to \mathrm{Ker}(g)$ is an isomorphism of sheaves.

Proposition 4.4.8. *A sequence of sheaf homomorphisms*

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

is exact, if and only if for every $x \in S$ the induced si=equence of stalks

$$\mathcal{F}_x \xrightarrow{f_x} \mathcal{G}_x \xrightarrow{g_x} \mathcal{H}_x$$

is an exact sequence of groups.

Proof. Let $\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$ be exact and let $x \in S$. Then $(gf)_x = g_x f_x$, so $g_x f_x = 0$ and therefore $\mathrm{Im}(f_x) \subset \ker(g_x)$. Let $\alpha \in \ker(g_x)$ and let (α_U, U) be a representative of α, i..e., U is an open neighborhood of x and $\alpha_U \in \mathcal{G}(U)$ with $g(U)\alpha_U = 0$. By definition of the image sheaf there is an open neighborhood $V \subset U$ of x, such that the restriction of α to V is of the form $f(V)\beta_V$ for some $\beta_V \in \mathcal{F}(U)$. Then $f_x(\beta) = \alpha$.

The converse direction is proven similarly. □

Example 4.4.9. The inclusion and the exponential map $\mathrm{e}(f(z)) = \exp(2\pi i f(z))$ yield an exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_S \to \mathcal{O}_S^\times \to 0.$$

Definition 4.4.10. A sheaf homomorphism $\alpha : \mathcal{F} \to \mathcal{G}$ yields a group homomorphism $C^p(\underline{U}, \mathcal{F}) \to \mathbb{C}^p(\underline{U}, \mathcal{G})$ for every p and every covering $\underline{U}$. The coboundary operator $\check{d}$ by definition is a linear combination of restriction maps. As these commute with α, we get $\check{d}\alpha = \alpha\check{d}$ and α maps kernel and image $\check{d}$ to kernel and image of $\check{d}$. Hence we get induced maps $\alpha_p : H^p(\underline{U}, \mathcal{F}) \to H^p(\underline{U}, \mathcal{G})$ and so

$$\alpha_p : H^p(S, \mathcal{F}) \to H^p(S, \mathcal{G}).$$

Note that for two sheaf homomorphisms α and β which can be composed, we have

$$(\beta \circ \alpha)_p = \beta_p \circ \alpha_p.$$

Lemma 4.4.11. *Let $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ be an exact sequence of sheaves over S. Then for every open $U \subset S$ the sequence*

$$0 \to \mathcal{F}(U) \xrightarrow{\alpha_U} \mathcal{G}(U) \xrightarrow{\beta_U} \mathcal{H}(U)$$

is exact. But for each $h \in \mathcal{H}(U)$ there is a covering $(V_i)_{i \in I}$ of U such that $h|_{V_i}$ lies in the image of β_{V_i} for every i. Consequently, for each covering $\underline{U}$ of S the sequence

$$0 \to C^p(\underline{U}, \mathcal{F}) \xrightarrow{\alpha_{\underline{U}}} C^p(\underline{U}, \mathcal{G}) \xrightarrow{\beta_{\underline{U}}} C^p(\underline{U}, \mathcal{H})$$

is exact. But for each $h \in C^p(\underline{U}, \mathcal{H})$ there exists a refinement $\underline{V}$ of $\underline{U}$ such that the image of h in $C^p(\underline{V}, \mathcal{H})$ lies in the image of $\beta_{\underline{V}}$.

Proof. For $s \in \mathcal{F}(U)$ with $\alpha(s) = 0$ we have $s_x = 0$ in every $x \in U$ and therefore $s = 0$ by the local uniqueness. If $t \in \mathcal{G}(U)$ satisfies $\beta(t) = 0$, then for every $x \in U$ one has $t_x = \alpha_x(s_x)$ for some $s_x \in \mathcal{F}_x$. So there is an open neighborhood V of x and $s_V \in \mathcal{F}(V)$ with $t_V = \alpha(V)s_V$. Since α is locally injective, the different s_V are compatible and together define a section s over U. This section satisfies $\alpha(U)s = t$.

For the second assertion repeat the same argument with β instead of α. This works up to the point where the different s_V should be compatible. Now they are not, but the V

form a covering which satisfies the claim.

The assertions about the cochain sets C^p follow by applying the first part to the intersections $U_{i_0} \cap \ldots, \cap U_{i_p}$. $\square$

Theorem 4.4.12 (The long exact cohomology sequence)**.** *Let*

$$0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$$

be an exact sequence of sheaves. Then there are morphisms, called **connection morphisms**

$$\delta_p : H^p(S, \mathcal{H}) \to H^{p+1}(S, \mathcal{F}),$$

such that the sequence

$$0 \to H^0(S, \mathcal{F}) \xrightarrow{\alpha_0} H^0(S, \mathcal{G}) \xrightarrow{\beta_0} H^0(S, \mathcal{H}) \xrightarrow{\delta_0} H^1(S, \mathcal{F}) \xrightarrow{\alpha_1} H^1(S, \mathcal{G}) \to \ldots$$
$$\ldots H^p(S, \mathcal{F}) \xrightarrow{\alpha_p} H^p(S, \mathcal{G}) \xrightarrow{\beta_p} H^p(S, \mathcal{H}) \xrightarrow{\delta_p} H^{p+1}(S, \mathcal{F}) \xrightarrow{\alpha_{p+1}} H^{p+1}(S, \mathcal{G}) \to \ldots$$

is exact.

Proof. Let $h \in H^p(S, \mathcal{H})$ and choose a representative $h \in Z^p(\underline{U}, \mathcal{H})$. We can choose the covering $\underline{U}$ so fine that surjectivity of the stalk maps β_x guarantees the existence of $g \in C^p(\underline{U}, \mathcal{G})$ with $\beta(g) = h$. In order to construct an element of $H^{p+1}(S, \mathcal{F})$ from this, we apply the coboundary operator $\check{d}(g)$. Because of $\check{d}^2 = 0$ we have $\check{d}(g) \in Z^{p+1}(\underline{U}, \mathcal{G})$. By $\beta\check{d} = \check{d}\beta$ one has $\beta(\check{d}(g)) = 0$, since h is a cocycle. By Lemma 4.4.11, applied to all $p + 2$-folds intersections of the sets U_i, there is $f \in C^{p+1}(\underline{U}, \mathcal{F})$ with $\alpha(f) = \check{d}(g)$. As $0 = \check{d}^2(g) = \check{d}\alpha(f) = \alpha(\check{d}(f))$ and by the injectivity in Lemma 4.4.11, applied to all $(p + 3)$-fold intersections, we get $d(f) = 0$. Therefore $f \in Z^{p+1}(\underline{U}, \mathcal{F})$ and we take its class in $H^{p+1}(S, \mathcal{F})$ to define $\delta(g)$.

We write this as $h \to g \to f$ with

$$\beta(g) = h \quad \text{and} \quad \alpha(f) = \check{d}g.$$

and then we have $\delta([h]) = [f]$. So we get $\delta(h)$ when in the diagram

$$\begin{array}{ccc} & & C^p(\mathcal{G}) \xrightarrow{\beta} C^p(\mathcal{H}) \\ & & \big\downarrow \check{d} \\ C^{p+1}(\mathcal{F}) & \xrightarrow{\alpha} & C^{p+1}(\mathcal{G}) \end{array}$$

we first take a preimage under β, then apply $\check{d}$ and again take a preimage, this time under α.

It remains to check, that δ does not depend on the choice of pre-images g and f. We don't have to worry about f here, as α is injective. Let $\tilde{g} = (\tilde{g}_{i_o \dots i_p})$ on a covering $\underline{\tilde{U}}$, consisting of open sets $\tilde{U}_i$, be another pre-image. We change to a covering $\underline{V}$, which is finer than $\underline{U}$ and $\underline{\tilde{U}}$ and we consider g and $\tilde{g}$ as elements of $C^p(\underline{V}, \mathcal{F})$. We have to show that the image $\delta(h) \in H^{p+1}(\underline{V}, \mathcal{F})$ does not depend on whether one has used g to construct f or if one has used $\tilde{g}$ to construct $\tilde{f}$. By construction we have $\beta(g - \tilde{g}) = 0$. By exactness of the sheaf sequences, every point $x \in S$ has a neighborhood $W(x)$, on which there exists an element u with $\alpha(u) = (g - \tilde{g})|_U$. By choosing finer coverings we can assume that $\underline{V}$ contains these $W(x)$, so that there is an element $u \in C^p(\underline{V}, \mathcal{F})$ with $\alpha(u) = g - \tilde{g}$. Then we have

$$\alpha(h - \tilde{h}) = \check{d}(g - \tilde{g}) = \check{d}(\alpha(u)) = \alpha(\check{d}(u))$$

and by injectivity of α this difference is a coboundary, the class of h in $H^p(S, \mathcal{F})$ is well-defined. By construction, δ is a group homomorphism.

We now show the exactness of the sequence. For the exactness at $H^p(S, \mathcal{G})$ we first observe that $0 = \beta \circ \alpha$ implies $0 = \beta_p \circ \alpha_p$ and so $\mathrm{Im}(\alpha_p) \subset \ker(\beta_p)$. For the converse inclusion let $[g] \in \ker(\beta_p)$. So $g \in C^p(\underline{U}, \mathcal{G})$ with $\check{d}g = 0$ and $\beta(g) = \check{d}h$ for some $h \in C^{p-1}(\underline{U}, \mathcal{H})$. By refining the covering $\underline{U}$ we find some $\tilde{g} \in C^{p-1}(S, \mathcal{G})$ with $h = \beta(\tilde{g})$. The class of g is the same as the class of $g' = g - \check{d}\tilde{g}$ and we have

$$\beta(g') = \beta(g) - \check{d}\beta(\tilde{g}) = \check{d}h - \check{d}h = 0.$$

By further refining the covering there is $f \in C^p(\underline{U}, \mathcal{F})$ with $\alpha(f) = g'$. Now $\alpha(\check{d}f) = \check{d}g' = 0$ and as α is injective, $\check{d}f = 0$ and it follows that $\alpha_p([f]) = [g]$.

For the inclusion $\mathrm{Im}(\beta) \subset \ker(\delta)$ it suffices to reconsider the proof of well-definedness of δ.

Recall the construction of δ as $h \to g \to F$ with

$$\beta(g) = h \quad \text{and} \quad \alpha(f) = \check{d}g.$$

and then we have $\delta([h]) = [f]$. In order to show $\ker(\delta) \subset \mathrm{Im}(\beta)$ let $[h] \in \ker(\delta)$. Then there is $u \in C^p(\underline{U}, \mathcal{F})$ with $\check{d}(u) = f$.

We change the chosen pre-image $g \in C^p(\underline{U}, \mathcal{F})$ by $-\alpha(u)$, i.e., $g' = g - \alpha(u)$. Then $\check{d}(g') = \check{d}(g) = \check{d}(\alpha(u)) = \check{d}(g) - \alpha(f) = 0$, so g' is, a cocycle and $\beta(g') = \beta(g) - \beta(\alpha(u)) = \beta(g) = h$.

The inclusion $\mathrm{Im}(\delta) \subset \ker(\alpha)$ is clear from the construction of δ. For the converse inclusion let $f \in \ker(\alpha)$ be given, so $\alpha(f) = \check{d}(g)$ for some cochain $g \in C^{p-1}(\underline{U}, \mathcal{F})$. Set $h = \beta(g)$. Then $\check{d}(h) = \beta(\check{d}(g)) = \beta(\alpha(f)) = 0$, and so $h \in Z^{p-1}(\underline{U}\mathcal{H})$, which implies $\delta(h) = f$. □

Corollary 4.4.13. *Let*

$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \to 0$$

be an exact sequence of sheaves on a topological space X with $H^1(\mathcal{G}, X) = 0$. Then

$$H^1(\mathcal{F}, X) = \mathcal{H}(X)/g\mathcal{H}(X).$$

Proof. Clear by the exact sequence. □

Examples 4.4.14.

- Let

  $$\mathcal{Z} = \ker(\Omega^2 \xrightarrow{d} \Omega^2)$$

  be the sheaf of closed forms. The sequence

  $$0 \to \mathbb{C} \to \mathbb{C}^\infty \to \mathcal{Z} \to 0$$

  is exact by the Poincaré Lemma.

- The sequence

  $$0 \to \mathbb{C} \to \mathcal{O} \to \Omega_{\mathrm{hol}} \to 0$$

  is exact, since locally, every holomorphic function has a primitive.

4.5 The Cohomology of the structure sheaf, first examples

It is clear that
$$\dim_{\mathbb{C}} \mathcal{O}(\mathbb{P}_1) = 1$$
and
$$\dim_{\mathbb{C}} \left\{ f \in \mathcal{M}(\mathbb{P}^1) : \begin{array}{c} \text{holomorphic outside zero} \\ \text{at most simple pole at zero} \end{array} \right\} = 2.$$
Generally, these dimensions are connected to the cohomology of the structure sheaf $\mathcal{O}$.

Theorem 4.5.1. (a) *Let $S = \left\{ z \in \mathbb{C} : |z| < R \right\}$ for some $0 < R \le \infty$. Then $H^1(S, \mathcal{O}) = 0$.*

(b) *One has $H^1(\mathbb{P}^1, \mathcal{O}) = 0$.*

We later will show that for a compact connected Riemann surface S we have
$$\dim H^1(S, \mathcal{O}) = g(S)$$
the genus.

Proposition 4.5.2. *Let $S = \left\{ x \in \mathbb{C} : |z| < R \right\}$, $0 < R \le \infty$ and $g \in C^\infty(S)$. Then there is $f \in C^\infty(S)$ such that*
$$\frac{\partial f}{\partial \bar{z}} = g.$$

Proof. Let $0 < R_0 < R_1 < \cdots < R$ be a sequence with limit R and set
$$S_n = \left\{ z \in \mathbb{C} : |z| < R_n \right\}.$$

Let $\psi_n \in C^\infty(S)$ with $\operatorname{supp}(\psi_n) \subset S_{n+1}$ and $\psi_n|_{S_n} \equiv 1$. Since $\psi_n g$ has compact support, there is $f_n \in C^\infty(S)$ with
$$\bar{\partial} f_n = \psi_n g,$$
where we have written $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$. We change the sequence f_n inductively to a sequence $\tilde{f}_n$ such that

(i) $\bar{\partial} \tilde{f}_n = g$ on S_n and

(ii) $\left\| \tilde{f}_{n+1} - \tilde{f}_n \right\|_{S_{n-1}} \le \frac{1}{2^n}$.

Here $\left\| f \right\|_K = \sup_{x \in K} |f(x)|$ is the supremum norm. For this set $\tilde{f}_1 = f_1$. Let $f_1, \dots, f_n$ be constructed already. Then we have

$$\overline{\partial}(f_{n+1} - \tilde{f}_n) = 0 \quad \text{auf} \quad S_n,$$

and so $f_{n+1} - \tilde{f}_n$ is holomorphic in S_n. Therefore there is a polynomial P such that

$$\left\| f_{n+1} - \tilde{f}_n - P \right\|_{S_{n-1}} \leq \frac{1}{2^n}.$$

We set $\tilde{f}_{n+1} = f_{n+1} - P$. Then (ii) holds. On S_{n+1} one has

$$\overline{\partial} \tilde{f}_{n+1} = \overline{\partial} f_{n+1} - \overline{\partial} P = \overline{\partial} f_{n+1} = \psi_{n+1} g = g,$$

so (i) holds, too. Set

$$f(z) = \lim_n \tilde{f}_n(z).$$

This is a continuous function on S. On S_n one has

$$f = \tilde{f}_n + \sum_{k=n}^{\infty} \left(\tilde{f}_{k+1} - \tilde{f}_k \right).$$

For $k \geq n$ the functions $\tilde{f}_{k+1} - \tilde{f}_k$ are holomorphic on S_n, since there one has $\overline{\partial}\left(\tilde{f}_{k+1} - \tilde{f}_k\right) = 0$. The sum $\sum_{k=n}^{\infty} \left(\tilde{f}_{k+1} - \tilde{f}_k\right)$ converges uniformly on S_n and therefore is holomorphic. So f is infinitely differentiable on S_n and so on all if S. Further one has $\overline{\partial} f = \overline{\partial} \tilde{f}_n = g$ on S_n and so this equation holds on all of S. □

Proof of Theorem 4.5.1, part (a). $H^1(S, \mathcal{O}) = 0$
Let $\underline{U} = (U_i)_{i \in I}$ be an open covering of S and let $f \in Z^1(\underline{U}, \mathcal{O})$ be a cocycle. as $Z^1(\underline{U}, \mathcal{O}) \subset Z^1(\underline{U}, \mathcal{C}^\infty)$ and $H^1(S, \mathcal{C}^\infty) = 0$, there is $g \in C^0(\underline{U}, \mathcal{C}^\infty)$ such that

$$f(i,j) = g_j - g_i \quad \text{on} \quad U_i \cap U_j.$$

Since $\overline{\partial} f(i,j) = 0$, one has $\overline{\partial} g_i = \overline{\partial} g_j$ on $U_i \cap U_j$ and so there is $G \in \mathcal{C}^\infty(S)$ such that $G|_{U_i} = \overline{\partial} g_i$. By Proposition 4.5.2 there is $h \in \mathcal{C}^\infty(S)$ with $\overline{\partial} h = G$. Set

$$s_i = g_i - h.$$

Then s_i is holomorphic on U_i, since $\overline{\partial} s_i = \overline{\partial} g_i - \overline{\partial} h = \overline{\partial} g_i - G = 0$ and therefore $s \in C^0(\underline{U}, \mathcal{O})$. On $U_i \cap U_j$ we also have

$$s_j - s_i = g_j - g_i = f(i,j),$$

so $\check{d} s = f$. □

Proof of Theorem 4.5.1, part (b). $H^1(\mathbb{P}^1, \mathcal{O}) = 0$
Let $U_1 = \mathbb{P}^1 \setminus \{\infty\}$ and $U_2 = \mathbb{P}^1 \setminus \{0\}$. Since both theses sets are $\cong \mathbb{C}$, we get $H^1(U_i, \mathcal{O}) = 0$ and by Leray's Theorem we get $H^1(\mathbb{P}^1, \mathcal{O}) = H^1(\underline{U}, \mathcal{O})$. Let $f \in Z^1(\underline{U}, \mathcal{O})$. We must find holomorphic functions $f_i \in \mathcal{O}(U_i)$, such that

$$f(1,2) = f_2 - f_1 \quad \text{on} \quad U_1 \cap U_2 = \mathbb{C}^\times.$$

For this let

$$f(1,2)(z) = f_{1,2}(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

be the Laurent series of $f_{1,2}$. The functions $f_1(z) = -\sum_{n=0}^{\infty} c_n z^n$ and $f_2(z) = \sum_{n=-\infty}^{-1} c_n z^n$ will do the job. □

Corollary 4.5.3 (Dolbeault's Theorem)**.** *For a Riemann surface S one has*

$$H^1(S, \mathcal{O}) \cong \Omega^{0,1}(S)/d''C^\infty(S).$$

Proof. By the Dolbeault Lemma 3.10.7 one gets the exactness of

$$0 \to \mathcal{O} \to C^\infty \to \Omega^{0,1} \to 0.$$

The sheaf C^∞ is fine, hence $H^1(S, \mathbb{C}^\infty) = 0$, so Corollary 4.4.13 yields the claim. □

4.6 A finiteness theorem

In this section we show that for two open subsets U, V of a Riemann surface such that $\overline{U}$ is compact and $\overline{U} \subset V$, the restriction map

$$H^1(V, \mathcal{O}) \to H^1(U, \mathcal{O})$$

has finite-dimensional image.

Let $D \subset \mathbb{C}$ be open. For a function $f \in \mathcal{O}(D)$ let the L^2**-Norm** be defined by

$$\left\| f \right\|_{L^2(D)} := \left(\int_D |f(x+iy)|^2 \, dx \, dy \right)^{\frac{1}{2}}.$$

We call f **square-integrable**, if $\left\| f \right\|_{L^2(D)} < \infty$. Let $L^2(D, \mathcal{O})$ be the vector space of all square-integrable functions on D.

Proposition 4.6.1. *The space $L^2(D, \mathcal{O})$ is a Hilbert space with the inner product*

$$\langle f, g \rangle = \int_D f(z)\overline{g(z)}\, dx\, dy.$$

Proof. We have to show completeness, in other words, that $L^2(D, \mathcal{O})$ is a closed subspace of the Hilbert space $L^2(D)$. Let $a \in D$ and $R > 0$ such that $B_R(a) \subset D$. Let $\delta_a(f) = f(a)$ be the evaluation functional. By Cauchy's integral formula we get

$$\begin{aligned} |\delta_a(f)| = |f(a)| &= \frac{1}{2\pi(R-r)} \left| \int_r^R \int_{\partial D_s} \frac{f(z)}{z-a}\, dz\, ds \right| \\ &= \frac{1}{2\pi(R-r)} \left| \int_r^R \int_0^{2\pi} f(a + se^{i\theta}) i\, d\theta\, ds \right| \\ &\leq C \int_{D_{r,R}} |f(z)|\, dx\, dy \underbrace{\leq}_{\text{Cauchy-Schwarz}} C' \left\|f\right\|_{L^2(D_{r,R})} \leq C' \left\|f\right\|_{L^2(D)} \end{aligned}$$

Here $a \in B_r(a_0)$ and therefore the linear functional δ_a is bounded with a bound that does not depend on a (locally). This implies that if a sequence $f_n \in L^2(D, \mathcal{O})$ converges $f_n \to f$ in $L^2(D)$, then f_n converges locally uniformly and by a theorem of Weierstraß, f is holomorphic, so $L^2(D, \mathcal{O})$ is a closed subspace of the Hilbert space $L^2(D)$. □

Definition 4.6.2. For subsets A, B of a topological space S we write

$$A \Subset B,$$

if the closure $\overline{A}$ of A is compact and $\overline{A} \subset B$.

Definition 4.6.3. Let S be a Riemann surface and let (U_i^*, z_i), $i = 1, \ldots, n$ be charts such that every $z_i(U_i^*)$ is a disk. We do not insist this family to be a covering. Let $U_i \subset U_i^*$ be open subsets and set $\underline{U} = (U_i)1 \leq i \leq n$.

(i) For $f \in C^0(\underline{U}, \mathcal{O})$ let

$$\left\|f\right\|^2_{L^2(\underline{U})} := \sum_i \left\|f_i\right\|^2_{L^2(U_i)}.$$

(ii) For $g \in C^1(\underline{U}, \mathcal{O})$ let

$$\left\|g\right\|^2_{L^2(\underline{U})} := \sum_{i,j} \left\|g_{ij}\right\|^2_{L^2(U_i \cap U_j)}.$$

The norms of f_i and g_{ij} are computed with the help of the charts z_i, i.e.,

$$\begin{aligned} \left\|f_i\right\|_{L^2(U_i)} &:= \left\|f_i \circ z_i^{-1}\right\|_{L^2(z_i(U_i))}, \\ \left\|g_{ij}\right\|_{L^2(U_i \cap U_j)} &:= \left\|g_{ij} \circ z_i^{-1}\right\|_{L^2(z_i(U_i \cap U_j))}. \end{aligned}$$

Let $C^q_{L^2}(\underline{U}, \mathcal{O}) \subset C^q(\underline{U}, \mathcal{O})$ be the space of cochains of finite norm. These are Hilbert spaces. The cocycles in $C^1_{L^2}(\underline{U}, \mathcal{O}$ form a closed subspace, which we denote by $Z^1_{L^2}(\underline{U}, \mathcal{O})$.

Lemma 4.6.4. *Let $V_i \Subset U_i$ be relatively compact open subsets and $\underline{V} = (V_i)_{1 \le i \le n}$. We write this as $\underline{V} \Subset \underline{U}$. For every cochain $f \in C^q(\underline{U}, \mathcal{O}$ we have $\left\|f\right\|_{L^2(\underline{V})} < \infty$. Then for every $\varepsilon > 0$ there is a closed subspace $A \subset Z^1_{L^2}(\underline{U}, \mathcal{O})$ of finite codimension, such that*

$$\left\|f\right\|_{L^2(\underline{V})} \le \varepsilon \left\|f\right\|_{L^2(\underline{U})} \quad \textit{for every} \quad f \in A.$$

Proof. As the families of charts are finite, it suffices to show that for open subsets $C \Subset D$ of $\mathbb{C}$ and every $\varepsilon > 0$ there is a closed linear subspace $A \subset L^2(D, \mathcal{O})$ of finite codimension, such that

$$\left\|f\right\|_{L^2(C)} \le \varepsilon \left\|f\right\|_{L^2(D)} \quad \text{for every} \quad f \in A.$$

The set $\overline{C}$ is compact and in D. So there is $r > 0$ and finitely many points $a_1, \dots, a_k \in D$ such that

(i) $B_r(a_j) \subset D$ for $j = 1, \dots, k$,

(ii) $C \subset \bigcup_{j=1}^k B_{r/2}(a_j)$.

Choose n so large, that $2^{-n-1}k \le \varepsilon$. Let A be the set of all functions $f \in L^2(D, \mathcal{O})$, which at each point a_j vanish at least to order n. Then A is a closed subspace of $L^2(D, \mathcal{O})$ of co-dimension $\le kn$.

Let $f \in A$ and let

$$f(z) = \sum_{\nu=n}^{\infty} c_\nu (z - a_j)^\nu$$

be the Taylor-series of f around a_j. Applying polar coordinates, one sees from the corresponding fact on Fourier-series, that

$$(a - a_j)^\nu \quad \perp \quad (z - a_j)^\mu$$

if $\nu \neq \mu$ in $L^2(B_r(a_j))$. Therefore, for $\rho \leq r$ we have

$$\|f\|^2_{L^2(B_\rho(a_j))} = \sum_{\nu=n}^{\infty} \frac{\pi\rho^{\nu+1}}{\nu+1}|c_\nu|^2,$$

such that

$$\|f\|_{L^2(B_{r/2}(a_j))} \leq 2^{-n-1}\|f\|_{L^2(B_r(a_j))} \leq 2^{-n-1}\|f\|_{L^2(D)}.$$

We get

$$\|f\|_{L^2(C)} \leq \sum_{j=1}^{k} \|f\|_{L^2(B_{r/2}(a_j))} \leq k2^{-n-1}\|f\|_{L^2(D)} \leq \varepsilon\|f\|_{L^2(D)}. \qquad \square$$

Lemma 4.6.5. *Let S be a Riemann surface and $\underline{U}^*$ a finite family of charts as in Definition 4.6.3. Let there be further families with $\underline{W} \Subset \underline{V} \Subset \underline{U} \Subset \underline{U}^*$. Then there is a constant $C > 0$ such that for every $\xi \in Z^1_{L^2}(\underline{V}, \mathcal{O})$ there are $w \in Z^1_{L^2}(\underline{U}, \mathcal{O})$ and $\eta \in C^0_{L^2}(\underline{W}, \mathcal{O})$ such that*

$$w = \xi + \check{d}\eta \quad \textit{on} \quad \underline{W}$$

and

$$\max\left(\|w\|_{L^2(\underline{U})}, \|\eta\|_{L^2(\underline{W})}\right) \leq C\,\|\xi\|_{L^2(\underline{V})}.$$

Proof. (a) Let $\xi \in Z^1_{L^2}(\underline{V}, \mathcal{O})$ be given. First we ignore the assertions on norms and construct $\zeta \in Z^1_{L^2}(\underline{U}, \mathcal{O})$ and $\eta \in C^0_{L^2}(\underline{W}, \mathcal{O})$ such that $\zeta = \xi + \check{\eta}$ holds on $\underline{W}$. By Proposition 4.3.10 we have $H^1(S, \mathcal{C}^\infty) = 0$, so there exists a cochain $g \in C^0(\underline{V}, \mathcal{C}^\infty)$ such that

$$\xi_{ij} = g_j - g_i \quad \text{on} \quad V_i \cap V_j.$$

As $d''\xi_{ij} = 0$ we get $d''g_i = d''g_j$ on $V_i \cap V_j$ and so there is a differential form $\omega \in \Omega^{0,1}(|\underline{V}|)$ with $\omega|_{V_i} = d''g_i$. Here $|\underline{V}| = \bigcup_{i=1}^n V_i$ is the **support** of $\underline{V}$. Since $|\underline{W}| \Subset |\underline{V}|$, there is a function $\psi \in \mathcal{C}^\infty(S)$ with

$$\text{supp}(\psi) \subset |\underline{V}| \quad \text{and} \quad \psi|_{|\underline{W}|} = 1.$$

So one can view $\psi\omega$ as an element of $\Omega^{0,1}(|\underline{U}^*|)$. By Proposition 4.5.2 there are functions $h_i \in \mathcal{C}^\infty(U_i^*)$ such that

$$d''h_i = \psi\omega \quad \text{on} \quad U_i^*.$$

In particular one has $d''h_i = d''h_j$ on $U_i^* \cap U_j^*$. Hence

$$\zeta_{ij} := h_j - h_i \in \mathcal{O}(U_i^* \cap U_j^*).$$

As $\underline{U} \Subset \underline{U}^*$, one has $\zeta \in Z^1_{L^2}(\underline{U}, \mathcal{O})$. On W_i we have $d'' h_i = \psi\omega = \omega = d'' g_i$, so the function $h_i - g_i$ is holomorphic on W_i.

Since $h_i - g_i$ also is bounded on W_i, one has

$$\eta_i = (h_i - g_i)|_{\underline{W}} \in C^0_{L^2}(\underline{W}, \mathcal{O}).$$

On $W_i \cap W_j$ one has $\zeta_{ij} - \xi_{ij} = (h_j - g_j) - (h_i - g_i)$ and therefore

$$\zeta - \xi = \check{d}\eta \quad \text{on} \quad \underline{W}.$$

(b) To get the assertions on norms, consider the Hilbert space

$$H := Z^1_{L^2}(\underline{U}, \mathcal{O}) \times Z^1_{L^2}(\underline{V}, \mathcal{O}) \times C^0_{L^2}(\underline{W}, \mathcal{O})$$

with the norm

$$\left\|(\zeta, \xi, \eta)\right\|^2_H := \|\zeta\|^2_{L^2(\underline{U}} + \|\xi\|^2_{L^2(\underline{V})} + \left\|\eta\right\|^2_{L^2(\underline{W})}.$$

Let $L \subset H$ be the subspace

$$L := \left\{(\zeta, \xi, \eta) \in H : \xi = \zeta + \check{d}\eta \text{ auf } \underline{W}\right\}.$$

Since L is closed in H, it is a Hilbert space itself. By part (a) the continuous linear map

$$\pi : L \to Z^1_{L^2}(\underline{V}, \mathcal{O}), \quad (\zeta, \xi, \eta) \mapsto \xi$$

is surjective. By the open mapping theorem, π is open. So there is a constant $C > 0$, such that for every $\xi \in Z^1_{L^2}(\underline{V}, \mathcal{O})$ there is an $x = (\zeta, \xi, \eta) \in L$ with $\pi(x) = \xi$ and $\|x\|_H \le C\,\|\xi\|_{L^2(\underline{V})}$. The claim follows. □

Lemma 4.6.6. *Under the conditions of Lemma 4.6.5 there is a finite-dimensional linear subspace $L \subset Z^1(\underline{U}, \mathcal{O})$ with the following property: for every $\xi \in Z^1(\underline{U}, \mathcal{O})$ there are elements $\sigma \in L$ and $\eta \in C^0(\underline{W}, \mathcal{O})$ such that*

$$\sigma = \xi + \check{d}\eta \quad \text{on} \quad \underline{W}.$$

The lemma says that the restriction map

$$H^1(\underline{U}, \mathcal{O}) \to H^1(\underline{W}, \mathcal{O})$$

has finite-dimensional image.

Proof of the lemma. Let C be the constant of Lemma 4.6.5 and set $\varepsilon = \frac{1}{2C}$. By Lemma

4.6.4 there is a closed linear subspace $A \subset Z^1_{L^2}(\underline{U}, \mathcal{O})$ of finite codimension such that

$$\|\xi\|_{L^2(\underline{V})} \leq \varepsilon \, \|\xi\|_{L^2(\underline{U})} \quad \text{for every} \quad \xi \in A.$$

Let L be the orthogonal complement of A in $Z^1_{L^2}(\underline{U}, \mathcal{O})$, so $A \oplus L = Z^1_{L^2}(\underline{U}, \mathcal{O})$. Let $\xi \in Z^1_{L^2}(\underline{U}, \mathcal{O})$ be arbitrary. Because of $\underline{V} \Subset \underline{U}$ one has

$$M := \|\xi\|_{L^2(\underline{V})} < \infty.$$

By Lemma 4.6.5 there are $\zeta_0 \in Z^1_{L^2}(\underline{U}, \mathcal{O})$ and $\eta_0 \in C^0_{L^2}(\underline{W}, \mathcal{O})$ such that

$$\zeta_0 = \xi + \check{d}\eta_0 \quad \text{on} \quad \underline{W}$$

and $\|\zeta_0\|_{L^2(\underline{U})} \leq CM$, as well as $\left\|\eta_0\right\|_{L^2(\underline{W})} \leq CM$. We write the orthogonal decomposition as

$$\zeta_0 = \xi_0 + \sigma_0, \quad \xi_0 \in A, \quad \sigma_0 \in L.$$

We inductively construct elements

$$\zeta_\nu \in Z^1_{L^2}(\underline{U}, \mathcal{O}), \quad \eta_\nu \in C^0_{L^2}(\underline{W}, \mathcal{O}), \quad \xi_\nu \in A, \quad \sigma_\nu \in L$$

with the following propertues:

(i) $\zeta_\nu = \xi_{\nu-1} + \check{d}\eta_\nu \quad \text{on} \quad \underline{W}$,

(ii) $\zeta_\nu = \xi_\nu + \sigma_\nu$,

(iii) $\|\zeta_\nu\|_{L^2(\underline{U})} \leq 2^{-\nu} CM$. $\qquad M = \|\xi\|_{L^2(\underline{V})}$

The induction step $\nu \to \nu + 1$ is done as follows: As $\zeta_\nu = \xi_\nu + \sigma_\nu$ is an orthogonal decomposition, one has

$$\|\xi_\nu\|_{L^2(\underline{U})} \leq \|\zeta_\nu\|_{L^2(\underline{U})} \leq 2^{-\nu} CM.$$

So that

$$\|\xi_\nu\|_{L^2(\underline{V})} \leq \varepsilon \, \|\xi_\nu\|_{L^2(\underline{U})} \leq 2^{-\nu} \varepsilon CM \leq 2^{-\nu-1} M.$$

By Lemma 4.6.5 there are elements $\zeta_{\nu+1} \in Z^1_{L^2}(\underline{U}, \mathcal{O})$ and $\eta_{\nu+1} \in C^0_{L^2}(\underline{W}, \mathcal{O})$, such that

$$\zeta_{\nu+1} = \xi_\nu + \check{d}\eta_{\nu+1} \quad \text{on} \quad \underline{W}$$

and

$$\max\left(\|\zeta_{\nu+1}\|_{L^2(\underline{U})}, \left\|\eta_{\nu+1}\right\|_{L^2(\underline{W})}\right) \leq 2^{-\nu-1} CM.$$

Now there is an orthogonal decomposition $\zeta_{\nu+1} = \xi_{\nu+1} + \sigma_{\nu+1}$, where $\xi_{\nu+1} \in A$ and

$\sigma_{\nu+1} \in L$ and then the induction step is complete.

The equations (i) and (ii) imply

$$\xi_k + \sum_{\nu=0}^{k} \sigma_\nu = \xi + \check{d}\left(\sum_{\nu=0}^{k} \eta_\nu\right) \quad \text{on} \quad \underline{W}. \tag{*}$$

By (ii) and (iii) one gets

$$\max\left(\|\xi_\nu\|_{\mathbb{L}^2(\underline{U})}, \|\sigma_\nu\|_{L^2(\underline{U})}, \left\|\eta_\nu\right\|_{L^2(\underline{W})}\right) \leq 2^{-\nu}CM.$$

So $\lim_{k\to\infty} \xi_k = 0$ and the series

$$\sigma := \sum_{\nu=0}^{\infty} \sigma_\nu \in L$$
$$\eta := \sum_{\nu=0}^{\infty} \eta_\nu \in C^0_{L^2}(\underline{W}, \mathcal{O})$$

converge. By (*) it follows that $\sigma = \xi + \check{d}\eta$ on $\underline{W}$. □

Definition 4.6.7. Let S be a topological space, $Y \subset S$ open and $\mathcal{F}$ a sheaf on S. For everyopen covering $\underline{U} = (U_i)_{i\in I}$ of S the restriction $\underline{U}|_Y$ is an open covering of Y and the restriction map $Z^1(\underline{U}, \mathcal{F}) \to Z^1(\underline{U}|_Y, \mathcal{F})$ induces a homomorphism $H^1(\underline{U}, \mathcal{F}) \to H^1(\underline{U}|_Y, \mathcal{F})$. In this way one gets the restriction homomorphism on cohomology,

$$H^1(S, \mathcal{F}) \to H^1(Y, \mathcal{F}).$$

Theorem 4.6.8. *Let S be a Riemann surface and $U \Subset V \subset S$ open subsets. Then the restriction homomorphism*

$$H^1(V, \mathcal{O}) \to H^1(U, \mathcal{O})$$

has finite-dimensional image.

Proof. There are finite families of charts $\underline{W} \Subset \underline{V} \Subset \underline{U} \Subset \underline{U}^*$ such that

(i) $U \subset \bigcup_{i=1}^n W_i =: Y' \Subset Y'' := \bigcup_{i=1}^n U_i \subset V$,

(ii) the sets $z_i(U_i^*)$, $z_i(U_i)$ and $z_i(W_i)$ are disks in $\mathbb{C}$.

By Lemma 4.6.6 the restriction homomorphism $H^1(\underline{U}, \mathcal{O}) \to H^1(\underline{W}, \mathcal{O})$ has finite-dimensional image. By Theorem 4.5.1 we have $H^1(U_i, \mathcal{O}) = 0 = H^2(W_i, \mathcal{O})$ and by the Theorem of Leray 4.3.13 one has $H^1(\underline{U}, \mathcal{O}) = H^1(Y'', \mathcal{O})$ and $H^1(\underline{W}, \mathcal{O}) = H^1(Y', \mathcal{O})$. As the restriction homomorphism $H^1(V, \mathcal{O}) \to H^1(U, \mathcal{O})$ can be factorized:

$$H^1(V, \mathcal{O}) \to H^1(Y'', \mathcal{O}) \to H^1(Y', \mathcal{O}) \to H^1(U, \mathcal{O}),$$

the claim follows. □

Corollary 4.6.9. *For a compact Riemann surface S the space $H^1(S, \mathcal{O})$ is finite-dimensional.*

Proof. Apply the theorem to the case $Y_1 = Y_2 = S$. □

Definition 4.6.10. The **cohomological genus** of a Riemann surface S is by definition

$$g_{\mathrm{coh}}(S) := \dim H^1(S, \mathcal{O}_S).$$

Later we will see that the cohomological genus equals the topological genus.

Theorem 4.6.11. *Let S be a Riemann surface and $U \Subset S$ an open subset. Then for every $p \in U$ there exists a meromorphic function $f \in \mathcal{M}(U)$ which has a pole at p and is holomorphic on $U \setminus \{p\}$.*

Proof. By Theorem 4.6.8,

$$k = \dim \operatorname{im}\left(H^1(S, \mathcal{O}) \to H^1(U, \mathcal{O})\right) < \infty.$$

Let (U_1, z) be a chart around p with $z(p) = 0$. Let $U_2 = S \setminus \{p\}$. Then $\underline{U} = \{U_1, U_2\}$ is an open covering of S. For $j \in \mathbb{N}$ the function z^{-j} is holomorphic on $U_1 \cap U_2$ and thus is an element of $Z^1(\underline{U}, \mathcal{O}_S)$. As $H^1(\underline{U}, \mathcal{O}) \to H^1(S, \mathcal{O})$ is injective, the elements $z^{-1}, \dots, z^{-(k+1)}$ in $H^1(\underline{U}, \mathcal{O})$ are linearly dependent. This means that there are $(f_1, f_2) \in C^0(\underline{U}, \mathcal{O})$ and complex numbers $c_1, \dots, c_{g_{\mathrm{coh}}+1}$, such that

$$f_2 - f_1 = \sum_{j=1}^{g_{\mathrm{coh}}+1} c_j z^{-j} \quad \text{on} \quad U_1 \cap U_2,$$

or $f_2 = f_1 + \sum_{j=1}^{g_{\mathrm{coh}}+1} c_j z^{-j}$. So the function, defined by f_2 on U_2 has a pole of order $\leq k + 1$ in p and is holomorphic otherwise. □

Corollary 4.6.12. *Let $U \Subset S$ be open. Then there exists a holomorphic function $f : U \to \mathbb{C}$ which is not constant on any connected component of U.*

Proof. Choose a domain D with $U \Subset D \Subset S$ and a point $p \in D \setminus U$. Now apply Theorem 4.6.11 to p and D. □

Theorem 4.6.13. *Let S be a non-compact Riemann surface and $U \Subset V \subset X$ open sets. Then*

$$L = \operatorname{Im}\left(H^1(V,\mathcal{O}) \to H^1(U,\mathcal{O})\right) = 0.$$

Proof. By Theorem 4.6.8 this is a finite-dimensional space. Choose classes $\xi_1, \ldots, \xi_n \in H^1(V,\mathcal{O})$, such that their restrictions to U span L. By Corollary 4.6.12 there is $f \in \mathcal{O}(V)$ which is non-constant on every connected component of V. Note that $H^1(V,\mathcal{O})$ is a $\mathcal{O}(V)$ module. By the choice of the ξ_ν there are constants $c_{\nu,\mu} \in \mathbb{C}$ such that

$$f\xi_\nu = \sum_{\mu=1}^{n} c_{\nu,\mu}\xi_\mu \tag{$*$}$$

for $\nu = 1, \ldots, n$. Set $F = \det(f\delta_{\nu,\mu} - c_{\nu,\mu})_{\nu,\mu}$. Then F is a holomorphic function on V which is non-zero on any connected component of V. By $(*)$ it follows that

$$F\xi_\nu|_U = 0 \tag{$**$}$$

for $\nu = 1, \ldots, n$. An arbitrary cohomology class $\zeta \in H^1(V,\mathcal{O})$ can be represented by a cocycle $(f_{i,j}) \in Z^1(\underline{U},\mathcal{O})$, where $\underline{U} = (U_i)_{i\in I}$ is an open covering of V such that each zero of F is contained in at most one U_i. So for $i \neq j$ one has $F|_{U_i\cap U_j} \in \mathcal{O}^\times(U_i \cap U_j)$. Hence there exists a cocycle $(g_{i,j}) \in Z^1(\underline{U},\mathcal{O})$ such that $f_{i,j} = Fg_{i,j}$. Let $\xi \in H^1(V,\mathcal{O})$ be the cohomology class of $(g_{i,j})$. Then $\zeta = F\xi$. Hence from $(**)$ one gets $\zeta|_U = F\xi|_U = 0$. □

Corollary 4.6.14. *Let S be a non-compact Riemann surface and $U \Subset V \subset S$ open sets. Then for every differential form $\omega \in \Omega^{0,1}(V)$ there exists a function $f \in C^\infty(U)$ such that $d''f = \omega|_U$.*

Proof. By Dolbeault's lemma 3.10.7, the problem has local solutions. So there exists an open covering $\underline{U} = (U_i)_{i\in I}$ of V and functions $f_i \in C^\infty(U_i)$ such that $d''f_i = \omega|_{U_i}$. The differences $f_i - f_j$ are holomorphic on $U_i \cap U_j$ and thus define a cocycle in $Z^1(\underline{U},\mathcal{O})$. By Theorem 4.6.13 this cocycle is cohomologous to zero on U, and so there exist holomorphic functions $g_i \in \mathcal{O}(U_i \cap U)$ such that

$$f_i - f_j = g_i - g_j \qquad \text{on} \quad U_i \cap U_j \cap U.$$

This implies that there exists a function $f \in C^\infty(U)$ such that

$$f = f_i - g_i$$

on $U_i \cap U$ for every $i \in I$. But then the function f satisfies $d'' f = \omega|_U$. □

5 The Theorem of Riemann-Roch

5.1 Divisors

Definition 5.1.1. Let S be a Riemann surface. A **divisor** on S is a map

$$D : S \to \mathbb{Z},$$

which is **locally-finite** in the following sense: for every $p \in S$ there is an open neighborhood U, such that $D(u) = 0$ for all but finitely many $u \in U$. One writes a divisor as formal sum

$$D = \sum_{p\in S} D(p)p.$$

One can add divisors, they form an abelian group $\mathrm{Div}(S)$, the **divisor group** of S.

Examples 5.1.2.

- Let $f \in \mathcal{M}(S)$ be a meromorphic function. Then

$$(f) = \mathrm{div}(f) = \sum_{p\in S} \mathrm{ord}_p(f)p$$

is the divisor of f. Every divisor of this form is called a **principal divisor**.

- Let ω be a meromorphic differential form on S. If in local coordinates around the point p the form writes as $\omega = f dz$, then we set

$$\mathrm{ord}_p(\omega) = \mathrm{ord}_p(f).$$

This number does not depend on the choice of the coordinates. The **divisor of** ω is defined as

$$\mathrm{div}(\omega) = \sum_{p\in S} \mathrm{ord}_p(\omega)p.$$

Lemma 5.1.3. *For a compact Riemann surface S every divisor is finite, i.e., a finite formal sum. So we have*

$$\mathrm{Div}(S) = \bigoplus_{p\in S} \mathbb{Z}p.$$

Proof. Let D be a divisor. For every point $p \in S$ there is an open neighborhood U_p, on which D is finite. As S is compact, it can be covered by finitely many U_p, so D is finite. □

Definition 5.1.4. Two divisors $D, E \in \mathrm{Div}(S)$ are called **equivalent**, if $D - E$ is a principal divisor.

A divisor of the form $\operatorname{div}(\omega)$ with a meromorphic differential form ω is called a **canonical divisor**.

Lemma 5.1.5. *Any two canonical divisors are equivalent.*

Proof. Let ω, η be meromorphic differential forms. Then one has $\eta = f\omega$ for some $f \in \mathcal{M}(S)$. This means

$$\operatorname{div}(\eta) = \operatorname{div}(f) + \operatorname{div}(\omega). \qquad \square$$

Definition 5.1.6. Let S be a compact Riemann surface. The **degree** of a divisors D is defined by

$$\deg(D) = \sum_{p \in S} D(p).$$

Example 5.1.7. Let S be a compact Riemann surface of genus 1. Then $S = \Lambda\backslash\mathbb{C}$ for some lattice $\Lambda = \mathbb{Z}a \oplus \mathbb{Z}b \subset \mathbb{C}$. Integrating over the boundary of a fundamental mesh, one sees that for every pricipal divisor $D = \operatorname{div}(f)$ one has $\deg(D) = 0$. Hence not every divisor is principal.

Proposition 5.1.8. *Let S be a compact Riemann surface.*

(a) *For a meromorphic function $f \neq 0$ on S one has*

$$\deg(\operatorname{div}(f)) = 0.$$

(b) *Let $\omega \neq 0$ be a meromorphic differential form on S and $g = g(S)$ the (topological) genus. Then one has*

$$\deg(\operatorname{div}(\omega)) = 2g - 2.$$

Proof. (a) If f is constant, the claim is clear. Otherwise, the function f defines a ramified covering $f : S \to \mathbb{P}^1_{\mathbb{C}}$. Since the degree map given by f is constant on $\mathbb{P}^1$, it takes the same value at 0 and at ∞. We have $\deg(\operatorname{div}(f)) = \deg(f, 0) - \deg(f, \infty) = 0$.

(b) Any two forms are equivalent. So it suffices to show the claim for one particular form. First we consider the case $S = \mathbb{P}^1$. In this case we take $\omega = dz$ in one of the two charts. In the other chart $\tilde{z} = \frac{1}{z}$ we then have $\omega = d(\frac{1}{\tilde{z}}) = -\frac{1}{\tilde{z}^2} d\tilde{z}$ and so $\deg \operatorname{div}(\omega)) = -2$, as was to be shown.

For arbitrary S let $f : S \to \mathbb{P}^1_{\mathbb{C}}$ be a holomorphic non-constant map. After applying some element of $\operatorname{Aut}(\mathbb{P}^1_{\mathbb{C}})$ we can assume that f is unramified over ∞ and we set $\omega = df$. Then locally we have $\omega = f' dz$, so, if $f(p) \neq \infty$, then $\operatorname{ord}_p(\omega) = \operatorname{ord}_p(f) - 1$. In the case $f(p) = \infty$, the unramifiedness implies that f has the form $f(t) = c_1 t^{-1} + c_0 + \dots$ with $c_0 \neq 0$ in a chart (U, t) around p with $t(p) = 0$. So the vanishing order of the derivative at $t = 0$ equals -2 and so $\mathrm{ord}_p(\omega) = 2$. As there are $\deg(f)$ many such points, we get

$$\deg(\mathrm{div}(\omega)) = \sum_{p \in S} (\mathrm{ord}_p(f) - 1) - 2\deg(f) = 2g - 2$$

according to the Riemann-Hurwitz formula of Theorem 3.5.1. □

Definition 5.1.9. For a divisor D on S let

$$\mathcal{L}(D) := \left\{0\right\} \cup \left\{f : S \to \mathbb{P}^1_{\mathbb{C}} : \text{ holomorphic with } \mathrm{div}(f) + D \geq 0\right\}$$

and

$$\ell(D) = \dim_{\mathbb{C}} \mathcal{L}(D).$$

So for instance if $D = p$, then $f \in \mathcal{L}(D)$ may have a simple pole at p, but must be holomorphic otherwise.

Examples 5.1.10.

- For $D = 0$ we have $\mathcal{L}(D) = \mathcal{O}(S)$. In particular for S compact, the space $\mathcal{L}(D)$ with $D = 0$ only consists of the constant functions.
- For $C \leq D$, one has $\mathcal{L}(C) \subset \mathcal{L}(D)$.
- If $\deg(D) < 0$, then $\mathcal{L}(D) = 0$.

Theorem 5.1.11 (Riemann-Roch)**.** *Let S be a compact Riemann surface of genus g, let $\omega \neq 0$ a meromorphic differential form and $K = \mathrm{div}(\omega)$. Then for every divisor D on S we have $\ell(D) < \infty$ and*

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g.$$

In particular if $\deg(D) > \deg(K) = 2g - 2$*, then*

$$\ell(D) = \deg(D) + 1 - g.$$

Recall the cohomological genus $g_{\mathrm{coh}} = \dim H^1(S, \mathcal{O}_S)$. Later we will show that $g_{\mathrm{coh}} = g$, the topological genus.

The proof of the theorem will take the rest of the chapter.

Corollary 5.1.12. *In the case $D = 0$ the theorem says:*

$$\ell(K) = g$$

Corollary 5.1.13. *Let D be a divisor on the compact Riemann surface S.*

(a) *If* $\deg(D) = 0$*, then* $0 \le \ell(D) \le 1$*.*

(b) *If* $\deg(D) = \deg(K) = 2g - 2$*, then* $g - 1 \le \ell(D) \le g$*.*

Proof. (a) Suppose that $\deg(D) = 0$ and let $f \in \mathcal{L}(D)$, i.e., $\operatorname{div}(f) + D \ge 0$. Since $\deg(\operatorname{div}(f)) = 0 = \deg(D)$ it follows that $\operatorname{div}(f) = -D$. So any other $g \in \mathcal{L}(D)$ has the same poles and zeros as f and so f/g is holomorphic, i.e. constant. So $\mathcal{L}(D)$ can at most be one-dimensional.

(b) If $\deg(D) = \deg(K)$, then $\deg(K - D) = 0$, so (a) and the Riemann-Roch Theorem imply the claim. □

5.2 The cohomological version of Riemann-Roch

Definition 5.2.1. For a divisor D let the sheaf $\mathcal{O}_S(D)$ be defined by

$$\mathcal{O}_S(D)(U) = \left\{ f : U \to \mathbb{P}^1_{\mathbb{C}} \text{ holomorphic with } \operatorname{div}(f) \ge -D|_U \right\} \cup \{0\}.$$

The axioms of a sheaf are easily verified.

Definition 5.2.2. A divisor D is called **effective**, if $D(p) \ge 0$ for every $p \in S$.

Definition 5.2.3 (skyscraper sheaf)**.** Let $E = \sum_{p \in S} E(p)p$ be an effective divisor on S. Then

$$\mathbb{C}_E(U) = \bigoplus_{p \in U} \mathbb{C}^{E(p)}$$

defines a sheaf $\mathbb{C}_E$, called the **skyscraper sheaf** of E.

Let $f \in \mathcal{O}_S(E)(U)$ and $p \in U$ arbitrary. Then f has a power series expansion around p of the form $f = \sum_{i \ge -E(p)} c_i z^i$. Mapping f to its principal part $\sum_{i=-E(p)}^{-1} c_i z^i$, and this to the vector $(c_{-1}, c_{-2}, \dots, c_{-E(p)})$, one gets a sheaf homomorphism $\mathcal{O}_S(E) \to \mathbb{C}_E$. It is easy to see that the sequence $0 \to \mathcal{O}_S \to \mathcal{O}_S(E) \to \mathbb{C}_E \to 0$ is exact. More generally for every divisor D the sequence

$$0 \to \mathcal{O}_S(D) \to \mathcal{O}_S(D + E) \to \mathbb{C}_E \to 0$$

is exact, where the arrow to $\mathbb{C}_E$ maps to the principal part of f, translated by the coefficient of D.

Lemma 5.2.4. *Let E be an effective divisor. For a sheaf $\mathcal{F}$ of complex vector spaces write $h^k(S,\mathcal{F})$ for $\dim H^k(S,\mathcal{F})$. Then*

(a) *One has*
$$h^0(S,\mathbb{C}_E) = \deg(E) \quad \textit{und} \quad h^1(S,\mathbb{C}_E) = 0.$$

(b) *For an arbitrary divisor D, the map induced by the inclusion $\mathcal{O}(D) \subset \mathcal{O}(D+E)$,*
$$H^1(S,\mathcal{O}(D)) \to H^1(S,\mathcal{O}(D+E))$$
is surjective.

Proof. (a) The first assertion is trivial. For the second let $\underline{U}$ be a covering of S, which is such that every point x with $a_x \neq 0$ is in contained in only one of the U_i. Then any intersection of atleast two of the U_i will contain no such point. So if $f \in Z^1(\underline{U},\mathbb{C}_D)$ is a cocycle, then by definition $f = 0$.

(b) The exact sequemce $0 \to \mathcal{O}_S(D) \to \mathcal{O}_S(D+E) \to \mathbb{C}_E \to 0$ induces the exact cohomology sequence
$$\cdots \to H^1(S,\mathcal{O}(D)) \to H^1(S,\mathcal{O}(D+E)) \to H^1(S,\mathbb{C}_E) = 0. \qquad \square$$

Recall $g_{\mathrm{coh}} = h^1(S,\mathcal{O})$.

Theorem 5.2.5 (Riemann-Roch, cohomological version)**.** *Let D be a divisor on the compact Riemann surface S. Then $H^0(S,\mathcal{O}(D))$ and $H^1(S,\mathcal{O}(D))$ are finite-dimensional $\mathbb{C}$-vector spaces and one has*
$$h^0(S,\mathcal{O}(D)) - h^1(S,\mathcal{O}(D)) = \deg(D) + 1 - g_{\mathrm{coh}}.$$

Proof. If $D = 0$, then $h^0(S,\mathcal{O}(D)) = 1$ and the claim follows from the definition of g_{coh}. We set $D' = D + [p]$ and then we show that the claim for D implies the claim for D' and vice versa. For this we consider the exact sequence
$$0 \to \mathcal{O}_S(D) \to \mathcal{O}_S(D') \to \mathbb{C}_{[p]} \to 0$$

and the ensuing exact cohomology sequence

$$0 \to H^0(S,\mathcal{O}(D)) \xrightarrow{\alpha_0} H^0(S,\mathcal{O}(D')) \xrightarrow{\beta_0} \mathbb{C} \xrightarrow{\delta_0} H^1(S,\mathcal{O}(D)) \xrightarrow{\alpha_1} H^1(S,\mathcal{O}(D')) \to 0.$$

By exactness, the dimensions of the spaces added up with alternating signs, give zero, so $0 = h^0(S,\mathcal{O}(D)) - h^0(S,\mathcal{O}(D')) + 1 - h^1(S,\mathcal{O}(D)) + h^1(S,\mathcal{O}(D'))$ or

$$h^0(S,\mathcal{O}(D) - h^1(S,\mathcal{O}(D)) = h^0(S,\mathcal{O}(D')) - h^1(S,\mathcal{O}(D)) - 1.$$

As $1 = \deg(D') - \deg(D)$, the claim follows. □

Corollary 5.2.6. *For an arbitrary divisor D on a compact Riemann surface we have*

$$h^0(S,\mathcal{O}(D)) \geq \deg(D) + 1 - g_{\mathrm{coh}}.$$

5.3 Serre-Duality

Definition 5.3.1. For a divisor D consider the sheaf $\Omega(D)$ given by

$$\Omega(D)(U) = \left\{\omega \in \Omega_{\mathrm{mer}}(U) : \mathrm{div}(\omega) + D \geq 0\right\}.$$

Theorem 5.3.2 (Serre-Duality)**.** *For a compact Riemann surface S one has*

$$h^0(\mathcal{O}(K - D)) = h^1(S,\mathcal{O}(D)).$$

Note that Serre-Duality together with the cohomological Riemann-Roch implies the classical Riemann-Roch with g replaced by g_{coh}.

Lemma 5.3.3. *Let K be a canonical divisor. One has*

$$h^0(\mathcal{O}(K - D)) = h^0(\Omega(-D)).$$

Proof of the corollary. Fix some $\omega_0 \in \Omega_{\mathrm{mer}}(S)$ and let K be its divisor. The map

$$\begin{aligned} \mathcal{M}(S) &\to \Omega(S), \\ f &\mapsto f\omega_0 \end{aligned}$$

is a $\mathbb{C}$-linear bijection. As $\mathrm{div}(f\omega_0) = \mathrm{div}(f) + K$, we get

$$\mathrm{div}(f\omega_0) - D \geq 0 \quad \Leftrightarrow \quad \mathrm{div}(f) + K - D \geq 0,$$

so this bijection maps $\mathcal{O}(K-D)(S)$ to $\Omega(-D)(S)$, hence the claim. $\square$

Proof of Serre-Duality. By Lemma 3.10.7 for every $g \in C_c^\infty(\mathbb{C})$ there exists $f \in C^\infty(\mathbb{C})$ such that

$$\frac{\partial}{\partial \overline{z}} f = g.$$

Let (U_i) be a holomorphic atlas und (u_i) be a partition of unity. Let $\eta \in \Omega^2(S)$ be a smooth 2-form. In holomorphic local coordinates we have $\eta = g dz \wedge d\overline{z}$. So for u_i being part of the partition of unity, there exists $f \in C^\infty(\mathbb{C})$ with

$$\overline{\partial}(f_i dz) = u_i \eta.$$

Choose another partition of unity v_i with $v_i|_{\operatorname{supp}(u_i)} = 1$ and set $\omega = \sum_i v_i f_i dz_i$. Then $\overline{\partial}\omega = \eta$. We have shown that the sequence of sheaves

$$0 \to \Omega_{\mathrm{hol}} \to \Omega^{1,0} \xrightarrow{\overline{\partial}} \Omega^2 \to 0$$

is exact. As in Proposition 4.3.11 one shows that $H^1(S, \Omega^{1,0}) = 0$. So the cohomology sequence shortens to

$$0 \to H^0(S, \Omega_{\mathrm{hol}}) \to H^0(S, \Omega^{1,0}) \xrightarrow{\overline{\partial}} H^0(S, \Omega^2) \xrightarrow{\delta} H^1(S, \Omega_{\mathrm{hol}}) \to 0.$$

We define a linear map

$$\begin{aligned} \operatorname{Res} : H^1(S, \Omega_{\mathrm{hol}}) &\to \mathbb{C}, \\ \delta(\omega) &\mapsto \frac{1}{2\pi i} \int_S \omega. \end{aligned}$$

In order to show well-definedness, we note that for $\eta \in H^0(S, \Omega^{1,0})$ Stokes's Theorem implies that $\int_S \overline{\partial}\eta = \int_S d\eta = \int_{\partial S} \eta = 0$. This means that the map Res, defined on $H^0(S, \Omega^2)$, vanishes on the image of $\overline{\partial}$. This image is the kernel of δ, so Res factors over δ, hence is well-defined as written above. The map

$$\begin{aligned} \Omega_{\mathrm{hol}}(-D)(U) \times \mathcal{O}(D)(U) &\to \Omega_{\mathrm{hol}}(U), \\ (\omega, f) &\mapsto f\omega \end{aligned}$$

induces a bilinear map

$$\tilde{b} : H^0(S, \Omega_{\mathrm{hol}}(-D)) \times H^1(S, \mathcal{O}(D)) \to H^1(S, \Omega_{\mathrm{hol}})$$

We define a bilinear form

$$b = \mathrm{Res} \circ \tilde{b}.$$

We claim that b is a **perfect pairing**, i.e., That it identifies either space with the dual of the other. This in particular implies that both spaces have the same dimension, i.e., Serre Duality.

Definition 5.3.4. Let $\underline{U} = (U_i)_{i \in I}$ be an open covering of S. A **Mittag-Leffler cochain** with respect to $\underline{U}$ is a cochain $\mu \in C^0(\underline{U}, \Omega_{\mathrm{mer}})$, i.e., a family $(\mu_i)_{i \in I}$ of meromorphic differential forms $\mu_i \in \Omega_{\mathrm{mer}}(U_i)$, with the property that $\mu_i - \mu_j$ is holomorphic on $U_i \cap U_j$.

Let μ be a Mittag-Leffler cochain. Then $\check{d}\mu \in Z^1(S, \Omega_{\mathrm{hol}})$, hence defines a cohomology class which we denote by $\check{d}\mu \in H^1(S, \Omega_{\mathrm{hol}})$.

If μ_i is locally of the form $f_i dz$ and if $p \in U_i$, we define

$$\mathrm{Res}_p(\mu_i) = \mathrm{Res}_p f.$$

This number is well-defined and we set

$$\mathrm{Res}(\mu) = \sum_{p \in S} \mathrm{Res}_p(\mu_{i(p)},$$

where $i(p) \in I$ is any index with $p \in U_{i(p)}$. We now have two maps which we denote by Res.

Lemma 5.3.5. *If μ is a Mittag-Leffler cochain, then*

$$\mathrm{Res}(\mu) = \mathrm{Res}(\check{d}\mu).$$

Proof. Consider the exact cohomology sequence

$$0 \to H^0(S, \Omega_{\mathrm{hol}}) \to H^0(S, \Omega^{1,0}) \xrightarrow{\bar{\partial}} H^0(S, \Omega^2) \xrightarrow{\delta} H^1(S, \Omega_{\mathrm{hol}}) \to 0.$$

Let $\eta \in \Omega^2(S)$ be a 2-form with $\delta(\eta) = \check{d}\mu$. Since $d\mu$ is a cocycle of Ω_{hol}, and so of $\Omega^{1,0}$ and since $H^1(S, \Omega^{1,0}) = 0$, there is a $g \in C^0(\underline{U}, \Omega^{1,0})$ with $\check{d}g = \check{d}\mu$. The exterior derivative of the holomorphic form $\mu_i - \mu_j$ vanishes. But it coincides with the exterior derivative of $g_i = g_j$. So these exterior derivatives glue to a global 2-form η, which by the definition of the connecting homomorphism satisfies $\delta(\eta) = \check{d}\mu$. Therefore we have to show:

$$\operatorname{Res}(\mu) = \frac{1}{2\pi i}\int_S \eta.$$

We first show that only neighborhoods of the poles of μ contirbute to the integral. Let $\{p_1, \dots, p_n\}$ the poles of μ and let $Y = S \setminus \{p_1, \dots, p_n\}$. On $Y \cap U_i \cap U_j$ we have $g_i - \mu_i = g_j - \mu_j$. So the $h_i = g_i - \mu_i$ glue to a global $h \in \Omega^{1,0}(Y)$ with $h|_{U_i \cap Y} = h_i$.

For every point p_k we schoose an index $i(k)$ with $p_k \in U_{i(k)}$. Around these points we choose charts (V_k, z_k) with $V_k \subset U_{i(k)}$ and $z_k(p_k) = 0$. We decrease these neighborhoods so that they become pairwise disjoint. We also construct $f_k \in C^\infty(S)$, which are constantly 1 on an even smaller neighborhood $V_k' \Subset V_k$ and are supported inside V_k. Let $f^c = 1 - \sum_{k=1}^n f_k$. As the support of f^c is disjoint with the set of poles, we can apply Stokes's Theorem and we get $\int_S d(f^c h) = 0$. We now consider the neighborhoods V_k'. Outside of p_k we have $d(f_k h) = d(h) = dg_{i(k)}$. As the g_i are defined on all of V_k', the same holds for the derivatives. So we can integrate η or $d(h_k)$ over all of S. Therefore

$$\int_S \eta = \sum_{k=1}^n \int_{V_k} d(f_k h) = \sum_{k=1}^n d(f_k g_{i(k)} - df_k \mu_{i(k)}) = \sum_{k=1}^n \int_{V_k} -f(f_k \mu_{i(k)},$$

where we have applied Stokes once again. The residue Theorem gives

$$\int_{V_k} -d(f_k \mu_{i(k)} = \int_{V_k'} -d(\mu_{i(k)} = \int_{\partial V_j'} -\mu_{i(k)} = 2\pi i \operatorname{Res}_{p_j} \mu_{i(k)}.$$

Adding up thiese contributions yields the lemma. □

Lemma 5.3.6. *The map*

$$b_* : H^0(S, \Omega_{\text{hol}}(-D)) \to H^1(S, \mathcal{O}(D))^*$$

defined by the bilinear form b is injective.

Proof. For every $\omega \in H^0(S, \Omega_{\text{hol}}(-D))$ we have to find a suitable covering $\underline{U}$ and a cocycle $f \in Z^1(\underline{U}, \mathcal{O}(D))$, such that $\operatorname{Res}(f\omega) \neq 0$. Let $p \in S$ be a point not in the support of the divisor D and let (U_0, z) be a chart around p with $z(p) = 0$. In the chart we write $\omega = f(z)dz$ with a holomorphic function f. We can shrink U_0 so that f has no zeros except possibly at $z = 0$. Let $f_0 = (zf(z))^{-1}$, so $f_0(z)\omega = dz/z$ on U_0.

Let $U_1 = S \setminus \{p\}$. Then $\mu = (dz/z, 0)$ is a Mittag-Leffler cochain with respect to $\underline{U} = (U_0, U_1)$, and we have $\operatorname{Res}(\mu) = 1$. So we are looking for $(f_{ij}) \in C^1(\underline{U}, \mathcal{O}(D))$, such that $f\omega = d\mu$. The cocycle given by $f_{01} = f_0|_{U_0 \cap U_1}$ does just this. □

To finish the proof of the Serre-Duality Theorem, we need to show the surjectivity of b_*.

Let B be an arbitrary divisor and $0 \neq g \in H^0(S, \mathcal{O}(B))$. The multiplication map $f \mapsto fg$ is a sheaf homomorphism $\mathcal{O}((D - B) \to \mathcal{O}(D)$ and so a homomorphism ψ of cohomology groups $H^1(S, \mathcal{O}(D - B)) \to H^1(S, \mathcal{O}(D))$.

Lemma 5.3.7. *The map*

$$\psi : H^1(S, \mathcal{O}(D - B)) \to H^1(S, \mathcal{O}(D))$$

is surjective.

Proof. If $A = \operatorname{div}(g)$, then multiplication by g gives an isomorphism $\mathcal{O}(D + A) \to \mathcal{O}(D)$ and ψ is the composition of the inclusion $\mathcal{O}(D - B) \to \mathcal{O}(D + A)$ with this isomorphism. Since sheaf isomorphisms induce isomorphisms on cohomology, it suffices to show that the map $H^1(S, \mathcal{O}(D - B)) \to H^1(S, \mathcal{O}(D + A))$ is surjective. This is done in Lemma 5.2.4, part (b). □

Definition 5.3.8. Let $B \geq 0$ and $D' = D - B$, as well as $g \equiv 1$. In this case we write

$$c_{D'}^D : H^1(S, \mathcal{O}(D))^* \to H^1(S, \mathcal{O}(D'))^*$$

for the map dual to ψ. Then $c_{D'}^D$ is injective. By means of the definitions one sees that the following diagram with exact rows and columns

$$\begin{tikzcd}
0 \arrow[r] & H^1(S, \mathcal{O}(D))^* \arrow[r, "c_{D'}^D"] & H^1(S, \mathcal{O}(D'))^* \\
0 \arrow[r] & H^0(S, \Omega_{\text{hol}}(-D)) \arrow[r] \arrow[u, "b_*^D"'] & H^0(S, \Omega_{\text{hol}}(-D')) \arrow[u, "b_*^{D'}"'] \\
& 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

commutes.

Lemma 5.3.9. *Let $\ell \in H^1(S, \mathcal{O}(D'))^*$ be in the image of $c_{D'}^D$ as well as in the image of $b_*(D')$. Then it lies in the image of $H^0(S, \mathcal{O}(-D))$. This means, if $\ell = c_{D'}^D(\ell_D) = b_*^{D'}(\omega)$, then $\omega \in H^0(S, \Omega_{\text{hol}}(-D))$.*

Proof. By definition we have $\ell(\xi) = b(\omega, \xi)$ for all $\xi \in H^1(S, \mathcal{O}(D'))$.

Assume, the claim is wrong. Then there is $p \in S$ with $\operatorname{ord}_p(\omega) < D(p)$. But one has $\operatorname{ord}_p(\omega) \geq D'(p)$. We want to construct a cohomology class $\xi \in H^1(S, \mathcal{O}(D'))$, which maps to zero in $H^1(S, \mathcal{O}(D))$, such that $\ell_D(\xi) = 0$, but on the other hand $\ell(\xi) = b(\omega, \xi) \neq 0$ holds. Then we get a contradiction to $\ell = b_*^{D'}(\omega) = c_{D'}^D(\ell_D)$.

For this we use the same covering construction as in the proof of injectivity. Let (U_0, z) be a chart around p with $z(p) = 0$ and write $\omega = f dz$ with a meromorphic function on U_0. We can decrease U_0 such that the support of D and D' in U_0 only consists of the point p and that in U_0 the function f has no pole or zero other than p. We consider the covering $\underline{U} = (U_0, U_1 = S \setminus \{p\})$ and $f_0 = (zf)^{-1}$. Then $\operatorname{ord}_p(f_0) + D(p) = -1 - \operatorname{ord}_p(\omega) + D(p) \geq 0$, so $(f_0, 0) \in C^0(S, \mathcal{O}(D))$. The coboundary of this lies in $Z^1(S, \mathcal{O}(D))$ and at the same time is $Z^1(S, \mathcal{O}(D'))$, since $U_0 \cap U_1$ is disjoint to the supports of both divisors. So let $\xi \in H^1(S, \mathcal{O}(D))$ be its cohomology class. By construction we have $\ell_D(\xi) = 0$ as claimed. Finally we have $b(\omega, \xi) = \operatorname{Res}(dz/z, 0) = 1$, **Contradiction!** □

Proof of surjectivity of the map $b_* : H^0(S, \Omega_{\mathrm{hol}}(-D)) \to H^1(S, \mathcal{O}(D))^*$: Let a non-zero linear form $\ell \in H^1(S, \mathcal{O}(D))^*$ be given and let $B = B_n = n[x]$, where we will determine n later. Further let $D_n = D - B_n$. We now fix the element $\ell \in H^1(S, \mathcal{O}(D))^*$ and consider the set

$$\Lambda_n = \left\{ \ell \circ g : g \in H^0(S, \mathcal{O}(B_n)\right\} \subset H^1(S, \mathcal{O}(D - B_n))^*.$$

Assume we already know $\Lambda_n \cap \operatorname{Im}(b_*^{D_n}) \neq 0$, i.e., there is g, such that $g\ell = b_*^{D_n}(\ell_n)$. Then let A be the divisor of g, so $1/g \in H^0(S, \mathcal{O}(A))$ and let $D' = D_n - A - D - (B_n + A)$. Then

$$c_{D'}^D(\ell) = \frac{1}{g}(g\ell) = \frac{1}{g} b_*^{D_n}(\omega) = b_*^{D'}\left(\frac{1}{g}\omega\right)$$

and we are in the situation of Lemma 5.3.9. This lemma says that $\frac{1}{g}\omega \in H^0(S, \Omega_{\mathrm{hol}}(-D))$ and $\ell = b_*^D\left(\frac{1}{g}\omega\right)$.

So we have to force in the non-empty intersection. For this we estimate the dimensions. By Corollary 5.2.6 one has

$$\dim \Lambda_n = h^0(S, \mathcal{O}(B_n)) \geq n + 1 - g_{\mathrm{coh}}.$$

By means of the same corollary we get

$$\dim \operatorname{Im}(b_*^{D_n}) = h^0(S, \Omega_{\mathrm{hol}}(-D_n)) \geq n - \deg(D) + 2g - 2 + 1 - g_{\mathrm{coh}}.$$

The ambient space, in which these subspaces intersect, has dimension

$$h^1(S, \mathcal{O}(D - B_n)) = g - 1 - \deg(D) + n.$$

These dimension grow linearly in n and the leading coefficient of this asymptotic is 1. For large n the spaces Λ_n and $\mathrm{Im}(b_*^{D_n})$ have non-trivial intersection. □

Theorem 5.3.10. *Let S be a compact Riemann surface. Then*

$$g(S) = g_{\mathrm{coh}}(S).$$

Together with Serre's Duality Theorem, this finishes the proof of the Riemann-Roch Theorem.

Proof. The Cohomological Theorem of Riemann-Roch, together with Serre-Duality, says for the divisor $D = 0$,

$$\ell(0) - \ell(K) = 1 - g_{\mathrm{coh}}.$$

For the divisor $D = K$ is says

$$\ell(K) - \ell(0) = \deg(K) + 1 - g_{\mathrm{coh}}.$$

Together this gives

$$\deg(K) = 2g_{\mathrm{coh}} - 2.$$

By Proposition 5.1.8 part (b) we have $\deg(K) = 2g - 2$, so $g = g_{\mathrm{coh}}$. □

* * *

6 Non-compact surfaces

6.1 The Dirichlet problem

Definition 6.1.1. Let U be an open subset of a Riemann surface S. Recall that a smooth function $u \in C^\infty(S)$ is harmonic, iff $d'd''u = 0$. In any holomorphic coordinate $z = x + iy$ this is equivalent to

$$\Delta u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0.$$

Lemma 6.1.2. *Let $D \subset S$ be a simply connected domain and let u be a harmonic function on D.*

(a) *The function u is the real part of a (uniquely determined) holomorphic function.*

(b) *(Maximum principle) If u attains a local maximum in D, then u is constant.*

Proof. (a) By Theorem 3.10.6, the harmonic differential form du equals $\mathrm{Re}(\omega)$ for some holomorphic form ω. As D is simply-connected, we have $du = \omega = df$ for some holomorphic function f. Therefore $u = f + \text{const}$.

(b) let $u = \mathrm{Re}(f)$ for a holomorphic f. By $\left|e^f\right| = e^u$, the holomorphic function e^f attains a local maximum in D, hence is constant. □

Definition 6.1.3. The **Dirichlet-problem** for an open set $U \subset S$ is the following: Suppose that $\partial U \neq \emptyset$ and let $f : \partial U \to \mathbb{R}$ be a continuous function. Find a continuous function $u : \overline{U} \to \mathbb{R}$ which is harmonic on U and satisfies $u \equiv f$ on ∂U.

If $\overline{U}$ is compact, then a solution, if it exists, is unique, since for two solutions u_1, u_2 the harmonic function $u_1 - u_2$ has boundary values zero. By the maximum principle it equals zero throughout.

Theorem 6.1.4. *Let $B_R(0)$ be the open disk around zero of radius $R > 0$ and let $f : \partial B_R(0) \to \mathbb{R}$ be continuous. Let*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} f\left(Re^{i\theta}\right) d\theta \qquad (*)$$

for $|z| < R$ and $u(z) = f(z)$ for $|z| = R$. Then u is continuous on $\overline{B_R(0)}$ and harmonic in $B_R(0)$.

Proof. To simplify the notation, we apply the biholomorphic map $z \mapsto Rz$ to reduce to the case $R = 1$. For $z \neq w$ let

$$P(z, w) = \frac{|w|^2 - |z|^2}{|w - z|^2}, \qquad F(z, w) = \frac{w + z}{w - z}.$$

Then $P(z, w) = \mathrm{Re}\Big(F(z, w)\Big)$. So $(*)$ reads

$$\begin{aligned} u(z) &= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\left(F(e^{i\theta}, z)\right) f\left(e^{i\theta}\right) d\theta \\ &= \mathrm{Re}\left(\frac{1}{2\pi i} \int_0^{2\pi} F(e^{i\theta}, z)\, f\left(e^{i\theta}\right) d\theta\right) \\ &= \mathrm{Re}\left(\frac{1}{2\pi i} \int_{|w|=1} F(w, z)\, f(w)\, \frac{dw}{w}\right). \end{aligned}$$

Therefore u is the real part of a holomorphic function and thus harmonic. It remains top show continuity on the boundary.

For $f = 1$ and $|z| < 1$ the residue theorem yields

$$\begin{aligned} \mathrm{Re}\left(\frac{1}{2\pi i} \int_{|w|=1} F(w, z) \frac{dw}{w}\right) &= \mathrm{Re}\left(\frac{1}{2\pi i} \int_{|w|=1} \frac{w + z}{w - z} \frac{dw}{w}\right) \\ &= 2 - 1 = 1. \end{aligned}$$

For $z_0 \in \partial\mathbb{D}$, letting $w = e^{-\theta}$ one gets

$$u(z) - f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} P(w, z)\Big(f(w) - f(z_0)\Big)\, d\theta.$$

Let $\varepsilon > 0$ and choose $\delta > 0$ such that $|w - z_0| < 2\delta$ implies $|f(w) - f(z_0)| < \varepsilon$. Let

$$M = \sup_{|z| \le R} |f(z)| + \sup_{|z|<R,\ |w|=R} P(w, z)$$

and let $A \subset [0, 2\pi)$ be the set of all θ such that $|e^{i\theta} - z_0| < 2\delta$ and let B be the rest. Let $|z| < 1$ such that $|z - z_0| < \alpha < \delta$ for some α with $0 < \alpha < \varepsilon\delta^2$. Then

$$|u(z) - u(z_0)| \le \underbrace{\frac{\varepsilon}{2\pi} \int_A P(w, z)\, d\theta}_{\le \varepsilon M} + \frac{M}{2\pi} \int_B P(w, z)\, d\theta$$

Now for $w = e^{i\theta}$ with $\theta \in B$ we have on the one hand $|w - z| \ge \delta$ and on the other

$1 - |z| < \alpha$, so

$$P(w, z) = \frac{1 - |z|^2}{|w - z|^2} = \frac{(1 - |z|)(1 + |z|)}{|w - z|^2} \leq \frac{2\alpha}{\delta^2}$$

and hence

$$\begin{aligned} |u(z) - u(z_0)| &\leq \varepsilon M + \frac{M}{2\pi} \frac{2\alpha}{\delta^2} 2\pi \\ &< \varepsilon 3M. \end{aligned}$$

The claim follows. □

Corollary 6.1.5. *Let $u : B_R(0) \to \mathbb{R}$ be a harmonic function. Then*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |z|^2}{re^{i\theta} - z|^2} u(re^{i\theta} \, d\theta$$

for all $|z| < r < R$. In particular, u satisfies the **Mean Value Principle***:*

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) \, d\theta.$$

Proof. This follows from the theorem because of the uniqueness of the solution of the Dirichlet problem. □

Corollary 6.1.6. *Let $u_n : B_R(0) \to \mathbb{R}$, $n \in \mathbb{N}$ be a sequence of harmonic functions converging locally uniformly to a function u. Then u is harmonic, too.*

Proof. For $|z| < r < R$ one has

$$u_n(z) = \frac{1}{2\pi} \int_0^{2\pi} P(re^{i\theta}, z) \, u_n(re^{i\theta}) \, d\theta.$$

Since u_n converges uniformly on the integration path, the same integral equation holds for u, which therefore is harmonic by the theorem. □

Theorem 6.1.7 (Harnack's theorem)**.** *Let $M \in \mathbb{R}$ and let (u_n) be a sequence of harmonic functions satisfying*

$$u_1 \leq u_2 \leq \cdots \leq M.$$

Then (u_n) converges locally uniformly to a harmonic function u.

Proof. It suffices to show uniform convergence on $B_\rho(0)$ for a given $0 < \rho < R$. Choose a $\rho < r < R$. Let $\varepsilon > 0$ and set $\varepsilon' = \varepsilon(r-\rho)/(r+\rho)$. Since the sequence $u_n(0)$ is increasing and bounded, there exists $m_0 \in \mathbb{N}$ such that

$$u_n(0) - u_m(0) < \varepsilon' \qquad \text{for every } m_0 \le m \le n.$$

We apply the integral formula to the positive harmonic function $u_n - u_m$. Let $|z| \le \rho$. Since

$$0 \le P(re^{i\theta}) \le \frac{r+|z|}{r-|z|} \le \frac{r+\rho}{r-\rho},$$

it follows

$$\begin{aligned} u_n(z) - u_m(z) &= \frac{1}{2\pi}\int_0^{2\pi} P(re^{i\theta}, z)\Big(u_n(re^{i\theta}) - u_m(re^{i\theta})\Big)\,d\theta \\ &\le \frac{r+\rho}{r-\rho}\frac{1}{2\pi}\int_0^{2\pi}\Big(u_n(re^{i\theta}) - u_m(re^{i\theta})\Big)\,d\theta \\ &= \frac{r+\rho}{r-\rho}\Big(u_n(0) - u_m(0)\Big) < \varepsilon. \end{aligned}$$

So the sequence converges locally uniformly and the limit is harmonic by Corollary 6.1.6. □

Definition 6.1.8. For an open set $Y \subset S$ let $\mathrm{Reg}(Y)$ denote the set of all subdomains $D \Subset Y$ such that the Dirichlet problem for D can be solved for every given continuous function on the boundary. If (U, ϕ) is a holomorphic chart with $U \subset Y$, then any set $D \Subset U$, such that $\phi(D)$ is a disk, lies in $\mathrm{Reg}(Y)$.

For a continuous function $u : Y \to \mathbb{R}$ and $D \in \mathrm{Reg}(Y)$, let $P_D u$ be the unique continuous function, which coincides with u on $Y \setminus D$ and is harmonic in D.

Lemma 6.1.9. *Let $Y \subset S$ be open and $D \in \mathrm{Reg}(Y)$. The map $P_D : C(Y, \mathbb{R}) \to C(Y, \mathbb{R})$ is $\mathbb{R}$-linear and monotonic, i.e., for $u, v \in C(Y, \mathbb{R})$ one has*

$$u \le v \qquad \Rightarrow \qquad P_D(u) \le P_D(v).$$

A given function $U \in C(Y, \mathbb{R})$ is harmonic iff $P_D(u) = u$ for every D.

Proof. Linearity is clear and monotonicity follows from Theorem 6.1.4. □

Definition 6.1.10. A function $U \in C(Y, \mathbb{R})$ is called **subharmonic**, if

$$u \le P_D(u)$$

holds for every $D \in \mathrm{Reg}(Y)$.

A function $u \in C(Y, \mathbb{R})$ is calle **locally subharmonic**, if every point $y \in Y$ possesses an open neighbourhood $U \subset Y$, such that u is subharmonic on U.

Lemma 6.1.11. *If u, v are subharmonic and $\lambda \geq 0$, then $u + v$, λu and $\max(u, v)$ are subharmonic.*

Proof. Only the maximum is non-trivial. First, $u, v \leq \max(u, v)$ implies that $P_D U, P_D v \leq P_D(\max(u, v)$ and hence

$$\max(P_D u, P_D v) \leq P_D\Big(\max(u, v)\Big).$$

Next $u \leq P_D u$ and $v \leq P_D v$ imply $\max(u, v) \leq \max(P_D u, P_D v)$. Together this yields

$$\max(u, v) \leq P_D\Big(\max(u, v)\Big),$$

as claimed. □

Theorem 6.1.12 (Maximum Principle for subharmonic functions)**.** *Let $Y \subset S$ be a domain and $u : Y \to \mathbb{R}$ a locally subharmonic function. If u attains its maximum at some point $y_0 \in Y$, the u is constant.*

Proof. Let $c = u(y_0)$ and

$$A = \{x \in Y : u(x) = c\}.$$

If $A \neq Y$, there exists a point $a \in \partial A \cap Y$. Since u is continuous, $u(a) = c$. In every neighbourhood of a there is x such that $u(x) < c$. Using a local chart which maps a to zero, we view u as a map $B_R(0) \to \mathbb{R}$ for some $R > 0$ and we find $0 < r < R$ such that with $D = B_r(0)$ we have $u(0) \leq P_D u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta})\, d\theta < c = u(0)$, a contradiction! We conclude that $A = Y$, so u is constant. □

Corollary 6.1.13. *If $u : Y \to \mathbb{R}$ is locally subharmonic, then it is subharmonic.*

Proof. Let $D \in \mathrm{Reg}(Y)$. Since $P_D(u)$ is harmonic in D, the function $v = u - P_D(u)$ is locally subharmonic in D. Since v vanishes on the boundary, we get $v \leq 0$, i.e., $u \leq P_D(u)$. □

Lemma 6.1.14. *If u is subharmonic on Y and $B \in \mathrm{Reg}(Y)$, then $P_B(u)$ is also subharmonic.*

Proof. Set $v = P_B u$ and suppose $D \in \mathrm{Reg}(Y)$ is arbitrary. We have to show $v \le P_D v$. On $Y \smallsetminus D$ one has $P_D v = v$ and on $Y \smallsetminus B$ one has $u = v$ and since $v \ge u$ one has

$$v = u \le P_D(u) \le P_D(v).$$

So one gets $v - P_D v \le 0$ on $Y \smallsetminus (B \cap D)$. Since $v - P_D v$ is harmonic on $B \cap D$, it follows that

$$v - P_D v \le 0 \quad \text{on} \quad B \cap D.$$

Hence $v \le P_D v$ on all of Y. □

Lemma 6.1.15 (Perron)**.** *Suppose that $M \subset C(Y, \mathbb{R})$ is a non-empty set of subharmonic functions with the following properties*

(a) $u, v \in M \;\Rightarrow\; \sup(u, v) \in M$,

(b) $P_D(M) \subset M$ *for every* $D \in \mathrm{Reg}(Y)$,

(c) *there exists* $K > 0$ *such that* $u \le K$ *for every* $u \in M$.

Then the function $u^* : Y \to \mathbb{R}$ *defined by*

$$u^*(x) = \sup_{u \in M} u(x)$$

is harmonic on Y.

Proof. Let $a \in Y$ and let $D \in \mathrm{Reg}(Y)$ be a neighbourhood of a. Choose a sequence $u_n \in M$ with

$$\lim_n u_n(a) = u^*(a).$$

By (a) we may assume $u_1 \le u_2 \le \dots$. Let $v_n = P_d u_n$, then $v_1 \le v_2 \le \dots$. By Harnack's theorem the sequence (v_n) converges on D to a harmonic function $v : D \to \mathbb{R}$ and one has

$$v(a) = u^*(a) \quad \text{and} \quad v \le u^* \text{ on } D.$$

We claim that $v(x) = u^*(x)$ for every $x \in D$. To see this, fix some $x \in D$ and let $w_n \in M$ be a sequence with $\lim_n w_n(x) = u^*(x)$. Because of (a) and (b) we may assume that

$$v_n \le w_n = P_D w_n \quad \text{and } w_n \le w_{n+1}$$

for every $n \in \mathbb{N}$. Hence the sequence (w_n) converges on D to a harmonic function $w : D \to \mathbb{R}$ with

$$v \le w \le u^*.$$

Since $v(a) = w(a) = u^*(a)$, the maximum principle applied to the harmonic function $w - v \geq 0$ implies $v(y) = w(y)$ for every $y \in D$. In particular,

$$u(x) = w(x) = u^*(x).$$

The claim is proven. $\square$

Definition 6.1.16. Let $Y \subset S$ be a domain and $f : \partial Y \to \mathbb{R}$ be a bounded function. Set

$$K = \sup_{x \in \partial Y} f(x).$$

Let $\mathcal{P}_f$ be the set of functions $u \in C(\overline{Y}, \mathbb{R})$ such that

(i) u is subharmonic in Y,

(ii) $u \leq f$ on ∂Y.

$\mathcal{P}_f$ is called the **Perron class** of f. By the lemma

$$u^* = \sup_{u \in \mathcal{P}_f} u$$

is harmonic on Y.

Definition 6.1.17. A boundary point $x \in \partial Y$ is called **regular**, if there is an open neighbourhood U of x and a function $\beta \in C(\overline{Y} \cap U, \mathbb{R})$ such that

(i) β is subharmonic on $U \cap Y$,

(ii) $\beta(x) = 0$ and $\beta < 0$ on $\overline{Y} \cap U \setminus \{x\}$.

Such a function β is called a **barrier** at x.

Lemma 6.1.18. *Let $x \in \partial Y$ be a regular point, V an open neighbourhood of x and $m \leq c$ real numbers. Then there exists a function $v \in C(\overline{Y}, \mathbb{R})$ with*

(a) *v is subharmonic on Y,*

(b) *$v(x) = c, \quad v \leq c$ on the set $\overline{Y} \cap V$,*

(c) *$v \equiv m$ on the set $\overline{Y} \setminus V$.*

Proof. WLOG we may assume $c = 0$. Let U be an open neighbourhood of x and $\beta \in C(\overline{Y} \cap U, \mathbb{R})$ a barrier at x. We may shring V if necessary and so we can assume

$V \Subset U$. Then

$$\sup\left\{\beta(y) : y \in \partial V \cap \overline{Y}\right\} < 0.$$

Hence there exists a constant $k > 0$ such that

$$k\beta < m \quad \text{on the set} \quad \partial V \cap \overline{Y}.$$

let

$$v = \begin{cases} \max(m, k\beta) & \text{on } \overline{Y} \cap V \\ m & \text{on } \overline{Y} \setminus V. \end{cases}$$

Then v is continuous on $\overline{Y}$, locally subharmonic on Y, thus subharmonic and also satisfies (b) and (c). □

Lemma 6.1.19. *Suppose that Y is an open subset of S and $f \in C(\partial Y, \mathbb{R})$ a bounded function and*

$$u^* = \sup\left\{u : u \in \mathcal{P}_f\right\},$$

where $\mathcal{P}_f$ is the Perron class of f. Then for every regular boundary point $x \in \partial Y$ we have

$$\lim_{\substack{y \to x \\ y \in Y}} u^*(y) = f(x).$$

Proof. For given $\varepsilon > 0$ there exists a relatively compact open neighbourhood V of x with

$$f(x) - \varepsilon \le f(y) \le f(x) + \varepsilon$$

for every $y \in \partial Y \cap V$. Let $k, K \in \mathbb{R}$ with $k \le f \le K$.

1.Step: Using Lemma 6.1.18, choose a function $v \in C(\overline{Y}, \mathbb{R})$ which is subharmonic on Y and satisfies

$$\begin{aligned} v(x) &= f(x) - \varepsilon, \\ v|_{\overline{Y} \cap V} &\le f(x) - \varepsilon, \\ v|_{\overline{Y} \setminus V} &= k - \varepsilon. \end{aligned}$$

Then $v|_{\partial Y} \le f$. That means that $v \in \mathcal{P}_f$ and hence $v \le u^*$. Therefore

$$\liminf_{Y \ni y \to x} u^*(y) \ge v(x) = f(x) - \varepsilon.$$

2. Step: By Lemma 6.1.18, there exists a function $w \in C(\overline{Y}, \mathbb{R})$ which is subharmonic

and satisfies

$$w(x) = -f(x),$$
$$w|_{\overline{Y}\cap V} \leq -f(x),$$
$$w|_{\overline{Y}\setminus V} = -K.$$

For every $u \in \mathcal{P}_f$ and $y \in \partial Y \cap V$ one has $u(y) \leq f(x) + \varepsilon$. Thus

$$u(y) + w(y) \leq \varepsilon$$

for $y \in \partial Y \cap V$. Also, we get

$$u(z) + w(z) \leq K - K = 0$$

for every $z \in \overline{Y} \cap \partial V$.

Applying the maximum principle to the function $u + w$, which is subharmonic, one gets

$$u + w \leq \varepsilon$$

on $\overline{Y} \cap V$. Thus

$$u|_{\overline{Y}\cap V} \leq \varepsilon - w|_{\overline{Y}\cap V}$$

holds for every $u \in \mathcal{P}_f$. Hence

$$\limsup_{\substack{y\to x\\ y\in Y}} u^*(y) \leq \varepsilon - w(x) = f(x) + \varepsilon.$$

The two steps imply the claim. □

Theorem 6.1.20. *Suppose that for an open set $Y \subset S$ all boundary points are regular. Then for every continuous bounded function $f : \partial Y \to \mathbb{R}$ the Dirichlet problem can be solved.*

Proof. This follows from the last Lemma. □

Proposition 6.1.21. *Let $Y \subset \mathbb{C}$ be open and $a \in \partial Y$. Suppose there exists a disk $B_r(m)$ in $\mathbb{C}$ with $r > 0$ such that $a \in \partial B_r(m)$ and $\overline{B_r(m)} \cap Y = \emptyset$. Then a is a regular boundary point of Y.*

Proof. Let $c = (a + m)/2$, then

$$\beta(z) = \log(r/2) - \log\left(|z - c|\right)$$

defines a barrier at a. $\square$

* * *

6.2 Regularity

Lemma 6.2.1. *Let $S \subset \mathbb{C}$ be open and star-shaped. Then for any given $f \in C^\infty(S)$ there exists $g \in C^\infty(S)$ such that $\Delta g = f$, where*

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \overline{z}}$$

is the **Laplace-Operator**.

Proof. Choose $g_1 \in C^\infty(S)$ such that $\frac{\partial}{\partial \overline{z}} g_1 = f$ and $g_1 \in C^\infty(S)$ such that $\frac{\partial}{\partial \overline{z}} g_2 = \overline{g}_1$, which is possible by Lemma 3.10.7. Then $g = \frac{1}{4}\overline{g_2}$ does the job. □

Lemma 6.2.2. *Let $U \subset \mathbb{C}$ be open and $K \subset U$ compact. For an open interval I in $\mathbb{R}$ let $g \in C^\infty(U \times I)$ have support in $K \times I$ and suppose that T is a distribution on U. Then the function $t \mapsto T[g(z,t)]$ is infinitely differentiable and satisfies*

$$\frac{\partial}{\partial t} T_z[g(z,t)] = T_z\left[\frac{\partial g(z,t)}{\partial t}\right].$$

Proof. It suffices to prove the formula, since the smoothness follows by iteration. As T is linear, we have

$$\begin{aligned}\frac{\partial}{\partial t} T_z[g(z,t)] &= \lim_{h\to 0} \frac{T_z[g(z,t+h)] - T_z[g(z,t)]}{h} \\ &= \lim_{h\to 0} T_z\left[\frac{g(z,t+h) - g(z,t)}{h}\right]\end{aligned}$$

For fixed $t \in I$ and sufficiently small $h \neq 0$ let

$$f_h(z) = \frac{1}{h}\Big(g(z,t+h) - g(z,t)\Big).$$

Then $f_h \in C^\infty(U)$ and tends to $\frac{\partial g(\cdot,t)}{\partial t}$ uniformly with all derivatives in a fixed compact L with $K \subset L \subset U$ where outside L f_h vanishes. Thus the continuity of the distribution T yields the claim. □

Definition 6.2.3. Let $\rho \in \mathbb{C}^\infty(\mathbb{C})$ with

(a) $\rho(z)$ only depends on $|z|$,

(b) $\mathrm{supp}(\rho) \subset \mathbb{D}$,

(c) $\rho \geq 0$,

(d) $\int_{\mathbb{C}} \rho(z)\, dx\, dy = 1$.

For $\varepsilon > 0$ let $\rho_\varepsilon(z) = \frac{1}{\varepsilon^2}\rho\left(\frac{z}{\varepsilon}\right)$. Then ρ_ε is supported in $\varepsilon\mathbb{D}$ and still satisfies (b) and (c).

Lemma 6.2.4. *Let $U \subset \mathbb{C}$ be open, $f \in C^\infty(U)$ and $\varepsilon > 0$. Let U_ε be the set of all $z \in U$ which have distance $> \varepsilon$ to the boundary ∂U of U.*

(a) *For every $\alpha \in \mathbb{N}_0^2$, on the set U_ε we have*

$$D^\alpha(f * \rho_\varepsilon) = D^\alpha f * \rho_\varepsilon.$$

(b) *If $z \in U_\varepsilon$ and f is harmonic on $B_\varepsilon(z)$, then*

$$f * \rho_\varepsilon(z) = f(z).$$

Proof. (a) follows from the interchange of integration and differentiation in Analysis 3 and for (b) one uses polar coordinates, the claim follows from the Mean Value Property □

Theorem 6.2.5 (Regularity Theorem)**.** *Let $U \subset \mathbb{C}$ be open and lat T be a distribution on U with $\Delta T = 0$, where*

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Then T is a smooth function.

Proof. For $w \in U_\varepsilon$ let. Let

$$h(w) = T_z(\rho_\varepsilon(z - w))$$

denote the distribution T applied to the function $z \mapsto \rho_\varepsilon(z - w)$. Let $f \in C_c^\infty(U)$ have support in U_ε. Then the convolution product $f * \rho_\varepsilon$ has compact support in U and we get

$$\begin{aligned} T[f * \rho_\varepsilon] &= T_w\left[\int_U f(z)\rho_\varepsilon(w - z)\, dx\, dy\right] \\ &= \int_Y f(z)h(z)\, dx\, dy. \end{aligned}$$

By Lemma 6.2.1 there exists $\psi \in C^\infty(\mathbb{C})$ with $\Delta\psi = f$. Then ψ is harmonic on

$V = \mathbb{C} \setminus \operatorname{supp} f$. By Lemma 6.2.4 (b) we get

$$\psi = \psi * \rho_\varepsilon \qquad \text{on} \quad V_\varepsilon.$$

Hence $\phi = \psi - \psi * \rho_\varepsilon$ has compact support in U and Lemma 6.2.4 (b) one has

$$T[f * \rho_\varepsilon] = T_w\left[\int_U f(z)\rho_\varepsilon(w-z)\,dx\,dy\right].$$

Since T is continuous and linear, you can interchange it with integration (Just write the integral as a limit of Riemann-sums.) So you get

$$\begin{aligned} T[f * \rho_\varepsilon] &= \int_U f(z)T_w[\rho_\varepsilon(w-z)]\,dx\,dy \\ &= \int_U f(z)h(z)\,dx\,dy. \end{aligned}$$

By Lemma 6.2.1, there exists a function $\psi \in C^\infty(\mathbb{C})$ with $\Delta\psi = f$. The function ψ thus is harmonic on $V = \mathbb{C} \setminus \operatorname{supp}(f)$. By Lemma 6.2.4 (b) it follows

$$\psi = \psi * \rho_\varepsilon$$

on V_ε. This means that $\phi = \psi - \psi * \rho_\varepsilon$ has compact support in U and by Lemma 6.2.4 (a) satisfies

$$\Delta\phi = \Delta(\psi - \psi * \rho_\varepsilon) = \Delta\psi - \Delta(\psi) * \rho_\varepsilon = f - f * \rho_\varepsilon.$$

Since $\Delta T = 0$, one has $T[\Delta\phi] = 0$, hence

$$T[f] = t[f * \rho_\varepsilon] = \int_U f(z)h(z)\,dx\,dy.$$

The theorem is proven. $\square$

Corollary 6.2.6. *Let T be a distribution on U with $\frac{\partial}{\partial \bar z}T = 0$. Then T is a holomorphic function.*

Proof. Since $\Delta T = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z}T = 0$, T is a smooth function by Theorem 6.2.5. But since $\frac{\partial}{\partial \bar z}T = 0$, this function is holomorphic. $\square$

* * *

6.3 Runge sets

Definition 6.3.1. Let S be a Riemann surface. An open subset $R \subset Y$ is called a **Runge set**, if none of the connected components of $Y \setminus R$ is compact. For instance, the unit disc $\mathbb{D} \subset \mathbb{C}$ is Runge in $\mathbb{C}$, but the puncture disk $\mathbb{D} \setminus \{0\}$ is not.

Theorem 6.3.2. *Suppose that S is a non-compact surface. Then there exists a sequence $Y_1 \Subset Y_1 \Subset \dots$ of relatively compact Runge domains with $\bigcup_j Y_j = S$, so that every Y_j has regular boundary.*

Proof. Since the topology of S is countably generated, there exists a sequence of relatively open domains with regular boundary $Y_1' \Subset Y_2' \Subset \dots$ with $\bigcup_j Y_j' = S$. Let Y_j be the union of Y_j' with all compact components of $S \setminus Y_j'$. $\square$

Lemma 6.3.3. *Let Z be an open subset of a Riemann surface S and $\Lambda : \Omega^{0,1}(S) \to \mathbb{C}$ be a continuous linear mapping with $\Lambda(d''g) = 0$ for every $g \in C^\infty(S)$ with $\operatorname{supp}(g) \Subset Z$. Then there exists a holomorphic 1-form $\sigma \in \Omega_{\mathrm{hol}}(S)$ such that*

$$\Lambda(\omega) = \int_Z \sigma \wedge \omega$$

for every $\omega \in \Omega^{0,1}(S)$ with $\operatorname{supp}(\omega) \Subset Z$.

Proof. Let $z : U \to V \subset \mathbb{C}$ be a chart on S which lies in Z. Identify U with V. For $\phi \in C_c^\infty(U)$, let $\tilde{\phi} \in \Omega^{0,1}(S)$ with $\tilde{\phi} = \phi\, d\bar{z}$ on U and zero on $S \setminus U$. Then the mapping

$$\Lambda_U : C_c^\infty(U) \to \mathbb{C}, \qquad \phi \mapsto \Lambda(\tilde{\phi})$$

is a distribution on U which vanishes on all functions of the form $\phi = \frac{\partial g}{\partial \bar{z}}$ with $g \in C_c^\infty(U)$. This means one has $\frac{\partial}{\partial \bar{z}} S_U = 0$. Hence by Corollary 6.2.6 there exists a unique holomorphic function $h \in \mathcal{O}(U)$ with

$$\Lambda[\tilde{\phi}] = \int_U h(z)\phi(z)\, dz \wedge d\bar{z}$$

for every $\phi \in C_c^\infty(U)$. Setting $\sigma_U = h\, dz$ we get

$$\Lambda[\omega] = \int_U \sigma_U \wedge \omega$$

for every $\omega \in \Omega_c^{0,1}(U)$. If we do the same for another chart V, the uniqueness implies that σ_U and σ_V agree on $U \cap V$ and so they patch together to define a form σ which satisfies the claim. □

Theorem 6.3.4. *Let R be a relatively compact open Runge subset of a non-compact Riemann surface S. Then for every open set V with $R \subset V \Subset S$ the image of the restriction map $\mathcal{O}(V) \to \mathcal{O}(R)$ is dense in the topology of locally uniform convergence (of all derivatives).*

Proof. Let $\beta : C^\infty(V) \to C^\infty(R)$ be the restriction map. We show that if $T : C^\infty(R) \to \mathbb{C}$ is a continuous linear functional with $T|_{\beta(\mathcal{O}(V))} = 0$, then $T = 0$. The claim will then follow from the Hahn-Banach Theorem.

To prove this, define a linear mapping

$$\Lambda : \Omega^{0,1}(S) \to \mathbb{C}$$

as follows: By Corollary 3.10.8 for given $\omega \in \Omega^{0,1}(S)$ there exists a function $f \in C^\infty(V)$ with $d''f = \omega|_V$. Then set

$$\Lambda(\omega) = T\left(f|_Y\right).$$

By assumption, this map is independent of the choice of the function f, since is g is another solution, then $f - g$ is holomorphic on V and so $T(f - g) = 0$. We show that Λ is continuous. Consider the vector space

$$E = \left\{(\omega, f) \in \Omega^{0,1}(S) \times C^\infty(V) : d''f = \omega|_V\right\}.$$

Since $d'' : C^\infty(V) \to \Omega^{0,1}(V)$ is continuous, E is a closed subspace of $\Omega^{0,1}(S) \times C^\infty(V)$ and thus a locally convex space. Now the projection $\pi : E \to \Omega^{0,1}(S)$ is surjective and thus is an open map. Also the mapping $\beta \circ \pi : E \to C^\infty(R)$ is continuous. Since the diagram

$$\begin{tikzcd} E \arrow[r, "\beta\circ\pi"] \arrow[d, "\pi"'] & C^\infty(R) \arrow[d, "T"] \\ \Omega^{0,1}(S) \arrow[r, "\Lambda"] & \mathbb{C} \end{tikzcd}$$

commutes, Λ is continuous, as T is. Since any continuous linear functional on $C^\infty(R)$ has compact support, there exits a compact subset $K \subset R$ with

(a) $T(f) = 0$ for every $f \in C^\infty(R)$ with $\operatorname{supp}(f) \subset R \setminus K$

and a compact subset $L \subset S$ with

(b) $S(\omega) = 0$ for every $\omega \in \Omega^{0,1}(S)$ with $\operatorname{supp}(\omega) \subset X \setminus L$.

If g is a function with $\operatorname{supp}(g) \Subset X \setminus K$, then $\Lambda(d''g) = T(g|_R) = 0$. By Lemma 6.4.3 there exists a holomorphic 1-form $\sigma \in \Omega_{\text{hol}}(S \setminus K)$ such that

$$\Lambda[\omega] = \int_{S \setminus K} \sigma \wedge \omega$$

for every $\omega \in \Omega^{0,1}(S)$ with $\operatorname{supp}(\omega) \Subset S \setminus K$. Because of (b) above we get $\sigma|_{S \setminus (K \cup L)} = 0$. Let $h(K)$ be the union of K and every relatively compact connected component of $S \setminus K$. Then every connected component of $S \setminus h(K)$ meets $S \setminus (K \cup L)$. By the Identity Theorem this implies $\sigma|_{X \setminus h(K)} = 0$, i.e.,

(c) $\Lambda[\omega] = 0$ for every $\omega \in \Omega^{0,1}(S)$ with $\operatorname{supp}(\omega) \Subset S \setminus h(K)$.

Now suppose $f \in \mathcal{O}(R)$. We have to show $T[f] = 0$. Since R is Runge, $H(K) \subset R$. Hence there is a function $g \in C^\infty(S)$ with $f = g$ in a neighbourhood ogf $h(K)$ and $\operatorname{supp}(g) \Subset R$. Then $T[f] = T[g|_R]$ by (a) and $T[g|_R] = \Lambda[d''g]$ by the definition of Λ. Since g is holomorphic on a neighbourhood of $h(K)$, one has $\operatorname{supp}(d''g) \Subset S \setminus h(K)$ and thus $\Lambda[d''g] = 0$ by (c). Together we get $T[f] = 0$ for every $f \in \mathcal{O}(R)$. □

Theorem 6.3.5 (Runge Approximation Theorem)**.** *Let S be a non-compact Riemann surface and Y be an open subset whose complement contains no compact connected component. Then every holomorphic function on Y is a locally uniform limit of holomorphic functions on S.*

Proof. It suffices to consider the case when Y is relatively compact in X. Fix $f \in \mathcal{O}(Y)$, a compact subset $K \subset Y$ and some $\varepsilon > 0$. There exists an exhaustion $Y_1 \Subset Y_2 \Subset \dots$ by Runge domains of X with $Y_0 = Y \Subset Y_1$. By Theorem 6.3.4 there is a holomorphic function $f_1 \in \mathcal{O}(Y_1)$ with

$$\left\| f_1 - f \right\|_K < 2^{-1}\varepsilon,$$

where $\|.\|_K$ is the supremum norm on K. Again by Theorem 6.3.4 one gets a sequence of functions $f_n \in \mathcal{O}(Y_n)$ with

$$\left\| f_n - f_{n-1} \right\|_{\overline{Y_{n-2}}} < 2^{-n}\varepsilon$$

for every $n \geq 2$. For every $n \in \mathbb{N}$ the sequence $(f_\nu)_{\nu > n}$ converges uniformly on Y_n. Hence there exists a function $F \in \mathcal{O}(X)$, which on each Y_n is the limit of $(f_\nu)_{\nu > n}$. By construction we have $\left\|F - f\right\|_K < \varepsilon$. □

* * *

6.4 Weak solutions

Definition 6.4.1. Let D be a divisor on a surface S. A **solution** of D is a meromorphic function f on S with $\operatorname{div}(f) = D$.

A **weak solution** is a function $f : S \setminus P \to \mathbb{C}$, where P is the set of poles of D, such that for every $a \in S$ there exists a local coordinate z mapping a to zero, and a smooth function ψ such that

$$f(z) = \psi(z) z^k$$

holds in a neighbourhood of a, where $k = D(a)$. Clearly, a weak solution is a proper solution iff f is holomorphic on $S \setminus P$.

Definition 6.4.2. For a curve $c : [0, 1] \to S$, let ∂c denote the divisor $c(1) = c(0)$.

Lemma 6.4.3. *Let c be a curve in S and let U be a relatively compact open neighbourhood of $c([0, 1])$. Then there exists a weak solution f of ∂c such that $f \equiv 1$ on $S \setminus U$.*

Proof. Consider first the case when U is the unit disk in $\mathbb{C}$. Let $a = c(0)$ and $b = c(1)$. There exists $0 < r < 1$ such that $c([0, 1]) \subset \{|z| < r\}$. The function $\log\big((z - b)/(z - a)\big) = \log(z - b) - \log(z - a)$ has a well-defined holomorphic branch in the annulus $A = \{r < |z| < 1\}$ as the indeterminacies of the two summands cancel. Choose a smooth function ψ on U such that $\psi|_{\{|z|\le r\}} = 1$ and $\psi|_{\{|z|\ge r'\}} = 0$ for some $r < r' < 1$. Set

$$f(z) = \begin{cases} \exp\left(\psi(z) \log \frac{z-b}{z-b}\right) & r < |z| < 1, \\ \frac{z-b}{z-a} & |z| \le r \end{cases}$$

and extend it by the constant 1 to all of S.

In the general case there exists a partition

$$0 = t_0 < t_1 < \cdots < t_n = 1,$$

And coordinates (U_j, z_j) for $j = 1, \ldots, n$ such that

(a) $c([t_{j-1}, t_j]) \subset U_j$,

(b) $z_j(U_j) = \mathbb{D}$.

Letting c_j denote the curve restricted to $[t_{j-1}, t_j]$ we get a weak solution f_j of the divisor of ∂c_j such that $f_j|_{S \setminus U_j} = 1$. The product $f = f_1 \cdots f_n$ satisfies the lemma. ◻

Lemma 6.4.4. *Every divisor D on a non-compact surface S has a weak solution.*

Proof. Choose a sequence $K_1, K_2, \dots$ of compact Runge subsets of S such that $K_j \subset \mathring{K}_{j+1}$ and $S = \bigcup_j K_j$. We claim that, given $a_0 \in S \smallsetminus K_j$ and the divisor A_0 with $A_0(a_0) = 1$ and $A_0(x) = 0$ for $x \neq a_0$, there exists a weak solution ϕ of A_0 with $\phi|_{K_j} = 1$.

Since K_j is Runge, the point a_0 lies in a connected component U of $S \smallsetminus K_j$ which is not relatively compact. Hence there exists a point $a_1 \in U \smallsetminus K_{j+1}$ and a curve c_0 in U, which connects a_0 to a_1. By Lemma 6.4.3 there is a weak solution ϕ_0 of the divisor ∂c_0 with $\phi_0|_{K_j} = 1$. Repeating the construction gives a sequence of points $a_\nu \in S \smallsetminus K_{j+\nu}$ and weak solutions ϕ_ν of the divisors $a_\nu - a_{\nu-1}$ with $\phi_\nu|_{K_{j+\nu-1}} = 1$. Thus the product

$$\phi = \prod_{\nu=0}^{\infty} \phi_\nu$$

converges, indeed, it is locally finite. This function ϕ is a weak solution of the divisor A_0.

Next suppose D is an arbitrary divisor on S. For $\nu \in \mathbb{N}$ set

$$D_\nu(x) = \begin{cases} D(x) & x \in K_{\nu+1} \smallsetminus K_\nu, \\ 0 & \text{otherwise}, \end{cases}$$

where $K_0 = \emptyset$. Since D_ν is a finite divisor, there exists a weak solution ψ_ν of D_ν and then

$$\psi = \prod_\nu \psi_n u$$

satisfies the claim. □

Theorem 6.4.5. *Let S be a non-compact Riemann surface. Then for any given $\omega \in \Omega^{0,1}(S)$ there exists a function $f \in C^\infty(S)$ with $d'' f = \omega$.*

Proof. For every relatively compact open set $U \Subset S$ there exists by Corollary 4.6.14 a function $f \in C^\infty(U)$ with $d''g = \omega|_U$. Suppose $Y_0 \Subset Y_1 \Subset \dots$ is an exhaustion of S by Runge domains. By induction on n we construct functions $f_n \in C^\infty(Y_n)$ such that

(a) $d'' f = \omega|_{Y_n}$,

(b) $\left\| f_{n+1} - f_n \right\|_{Y_{n-1}} \leq 2^{-n}$.

First choose $f_0 \in C^\infty(Y_0)$ as an arbitrary solution of the equation $d'' f_0 = \omega|_{U_0}$. Next suppose $f_0, \ldots, f_n$ have been constructed. There exists $g_{n+1} \in C^\infty(Y_{n+1})$ with $d''g_{n+1} = \omega|_{U_{n+1}}$. On Y_n the function $g_{n=1} - f_n$ is holomorphic. By Theorem 6.3.4 there exists $h \in \mathcal{O}(Y_{n+1})$ such that

$$\left\|(g_{n+1} - f_n) - h\right\|_{Y_{n-1}} \leq 2^{-n}.$$

Set $f_{n+1} = g_{n+1} - h$. Then '$d''f_{n+1} = d''g_{n+1} = \omega|_{Y_{n+1}}$ and

$$\left\|f_{n+1} - f_n\right\| \leq 2^{-n}.$$

The f_n now converge to a solution $f \in C^\infty(S)$ of the equation $d''f = \omega$. □

Theorem 6.4.6. *Let S be a non-compact Riemann surface. Then*

$$H^1(S, \mathcal{O}) = 0.$$

Proof. By the Dolbeault Theorem 4.5.3 one has $H^1(S, \mathcal{O}) \cong \Omega^{0,1}(S)/d''C^\infty(S)$. By Theorem 6.4.5 we have $\Omega^{0,1}(S) = d''C^\infty(S)$ und daher $H^1(S, \mathcal{O}) = 0$. □

Theorem 6.4.7. *On a non-compact Riemann surface, every divisor has a solution, i.e., is the divisor of a meromorphic function.*

Proof. Since the problem has a solution locally, there exists an open covering $\underline{U} = (U_i)_{i \in I}$ of S and meromorphic functions $f_i \in \mathcal{M}(U_i)$ such that the divisor of f_i coincides with D on U_i. We may assume that the U_i are simply connected. Then

$$\frac{f_i}{f_j} \in \mathcal{O}^\times(U_i \cap U_j)$$

for all $i, j \in I$. Now suppose ψ is a weak solution of D, which exists by Lemma 6.4.4. Then $\psi = \psi_i f_i$ on U_i, where the function ψ_i has no zeros. Since U_i is simply-connected, there exists a function $\phi_i \in C^\infty(U_i)$ with $\psi_i = e^{\phi_i}$, i.e., $\psi = e^{\phi_i} f_i$ on U_i. Then on $U_i \cap U_j$ one has

$$e^{\phi_j - \phi_i} = \frac{f_i}{f_j} \in \mathcal{O}^\times(U_i \cap U_j). \qquad (*)$$

This implies $\phi_{i,j} = \phi_i - \phi_j \in \mathcal{O}(U_i \cap U_j)$. The family $\phi_{i,j}$ is a cocycle in $Z^1(\underline{U}, \mathcal{O})$. As $H^1(S, \mathcal{O}) = 0$, this cocycle splits. Thus there exist holomorphic functions $g_i \in \mathcal{O}(U_i)$ with

$$\phi_{i,j} = \phi_i - \phi_j = g_i - g_j$$

on $U_i \cap U_j$. By $(*)$ we get $e^{g_j - g_i} = f_i / f_j$, so

$$e^{g_i} f_i = e^{g_j} f_j$$

holds on $U_i \cap U_j$. Hence there exists a global meromorphic function $f \in \mathcal{M}(S)$ with $f = e^{g_i} f_i$ on U_i, whence the claim. □

Corollary 6.4.8. *Let S be a non-compact Riemann surface. Then there exists a holomorphic form $\omega \in \Omega_{\mathrm{hol}}(S)$ which nowhere vanishes.*

Proof. Let g be a non-constant meromorphic function on S and $f \in \mathcal{M}(S)$ a function with divisor $-(dg)$. Then $\omega = f\,dg$ is a holomorphic 1-form without zeros. □

* * *

6.5 Riemann Mapping Theorem

Remark 6.5.1. Recall that if S is simply connected, every holomorphic differential form has a primitive, that is, we have

$$\Omega_{\text{hol}}(S) = d\mathcal{O}(S).$$

Lemma 6.5.2. *Let S be a Riemann surface with $\Omega_{\text{hol}}(S) = d\mathcal{O}(S)$.*

(a) *Every holomorphic function $f : S \to \mathbb{C}^\times$ has a holomorphic logarithm and hence a holomorphic square root.*

(b) *Every harmonic function $u : S \to \mathbb{R}$ is real part of a holomorphic function on S.*

Proof. (a) $\frac{1}{f}df$ is a holomorphic 1-form on S. Since $H^1_\partial(S) = 0$, there exists $g \in \mathcal{O}(S)$ with $dg = \frac{1}{f}df$. Then

$$d(fe^{-g}) = dfe^{-g} - fe^{-g}f^{-1}df = 0.$$

Hence fe^{-g} is constant. Adding a constant to g we can assume $e^g = f$. With $h = e^{g/2}$ one has $h^2 = f$.

(b) By Theorem 3.10.6 there exists a holomorphic form $\omega \in \Omega_{\text{hol}}(S)$ such that $du = \operatorname{Re}(\omega)$. Since $\Omega_{\text{hol}}(S) = d\mathcal{O}(S)$, one has $du = \operatorname{Re}(dg) = d\operatorname{Re}(g)$ for some $g \in \mathcal{O}(S)$. Thus $u = \operatorname{Re}(g) + \text{const}$. □

Theorem 6.5.3. *Let S be a non-compact Riemann surface and let $Y \Subset S$ be a domain with regular boundary, such that $\Omega_{\text{hol}}(Y) = d\mathcal{O}(Y)$. Then there exists a biholomorphic mapping $Y \xrightarrow{\cong} \mathbb{D}$.*

Proof. Choose a point $p \in Y$. By Theorem 6.4.7 there exists a holomorphic function g on X with a first order zero at p which does not vanish on $S \smallsetminus \{a\}$. By Theorem 6.1.20 there exists a continuous function $u : \overline{Y} \to \mathbb{R}$, harmonic in Y with

$$u(y) = \log|g(y)| \qquad \text{for every } y \in \partial Y \qquad (*)$$

By Lemma 6.5.2 u is the real part of a holomorphic function $h \in \mathcal{O}(Y)$. We claim that the function

$$f = e^{-h}g \in \mathcal{O}(Y)$$

maps Y biholomorphically onto the unit disl $\mathbb{D}$. First we show $f(Y) \subset \mathbb{D}$. For $y \in Y \setminus \{a\}$ one has

$$|f(y)| = |e^{h(y)}|\,|g(y)| = e^{\log|g(y)|}e^{-u(y)}.$$

Hence the the continuous function $\phi = |f|$ can be extended continuously to $\overline{Y}$ which is identically 1 on ∂Y by (∗). The Maximum Principle implies $|f(y)| < 1$ for every $y \in Y$, i.e., $f(Y) \subset \mathbb{D}$.

Next we show that the map $f : Y \to \mathbb{D}$ is proper. For this it suffices to show that for every $r < 1$ the preimage $Y(r)$ of the disk $\{|z| \le r\}$ is compact in Y. But

$$Y(r) = \left\{y \in Y : |f(y)| \le r\right\} = \left\{y \in \overline{Y} : \phi(y) \le r\right\},$$

so $Y(r)$ is a closed subset of the compact $\overline{Y}$, hence compact.

Since $f : Y \to \mathbb{D}$ is proper, each value is taken equally often by Theorem 3.4.3. But the value zero is taken exactly once. Hence $f : Y \to \mathbb{D}$ is bijective and hence biholomorphic. □

Lemma 6.5.4. *Let S be a non-compact Riemann surface with $\Omega_{\mathrm{hol}}(S) = d\mathcal{O}(S)$ and let $Y \subset S$ be a Runge domain. Then $\Omega_{\mathrm{hol}}(Y) = d\mathcal{O}(Y)$ as well.*

Proof. Let $\omega \in \Omega_{\mathrm{hol}}(Y)$. By Corollary 6.4.8 there exists a holomorphic 1-form $\omega_0 \in \Omega_{\mathrm{hol}}(S)$ without zeros. Then $\omega = f\omega_0$ for some $f \in \Omega(Y)$. By the Runge Approximation Theorem 6.3.5 there exists a sequence $f_n \in \mathcal{O}(S)$ converging locally uniformly on Y to f. Since every form $f_n\omega_0$ has a primitive, we have $\int_\gamma f_n\omega = 0$ for every closed curve γ in Y. In the limit, this yields $\int_\gamma \omega = 0$, hence ω has a primitive. □

Theorem 6.5.5 (Riemann Mapping Theorem)**.** *For a connected Riemann surface S the following are equivalent:*

(a) $\Omega_{\mathrm{hol}}(S) = d\mathcal{O}(S)$.

(b) *S is isomorphic to $\mathbb{C}, \widehat{\mathbb{C}}$ or $\mathbb{E}$.*

(c) *S is simply connected.*

Proof. The implications (b)⇒(c)⇒(a) are clear. So it remains to show (a)⇒(b).

If S is compact, then every holomorphic function on S is constant and so $d\mathcal{O} = 0$, hence (a) implies that $\Omega_{\mathrm{hol}}(S) = 0$, so S has genus 0, hence is biholomorphic to $\widehat{\mathbb{C}}$.

So now assume that S is non-compact. Let $Y_1 \Subset Y_1 \Subset \dots$ be an exhaustion of X by Runge domains with regular boundaries. By Lemma 6.5.4 we have $\Omega_{\text{hol}}(Y_n) = d\mathcal{O}(Y_n)$ for every n. So by Theorem 6.5.3, every Y_n is biholomorphic to the disk $\mathbb{D}$. Choose a point $p \in Y_0$ and a coordinate neighbourhood (U, z) of p. Then there exists a real number $rn > 0$ and a biholomorphic map

$$f_n : Y_n \to r_n\mathbb{D}$$

with

$$f_n(p) = 0 \quad \text{and} \quad \frac{df_n}{dz}(p) = 1.$$

We claim that $r_n \leq r_{n+1}$. To see this let

$$h_1 = f_{n+1} \circ f_n^{-1} : r_n\mathbb{D} \to r_{n+1}\mathbb{D}$$

and define $h : \mathbb{D} \to \mathbb{D}$ by $h(z) = \frac{1}{r_{n+1}} h_1(r_n z)$. Then $h(0) = 0$ and by Schwarz's Lemma we get $|h'(0)| \leq 1$. But $h'(0) = \frac{r_n}{r_{n+1}}$ and thus $r_n \leq r_{n+1}$. Let

$$R = \lim_{n\to\infty} r_n \in (0, \infty].$$

We claim that there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that for every $m \in \mathbb{N}$ the sequence $(f_{n_k})_{k\geq m}$ converges locally uniformly on Y_m. For fixed m one gets such a subsequence from the Theorem of Arzela-Ascoli in the holomorphic version (Analysis Theorem 20.4.7). Then, of this subsequece one takes a subsequence for Y_{m+1} and so on. The usual diagonal argument then yields the desired subsequence $(f_{n_k})_{k\in\mathbb{N}}$.

Let $f \in \mathcal{O}(S)$ be the limit of this sequence, i.e., on each Y_m the function f is the uniform limit of $f_{n_k}|_{Y_m}$. It follows that $f : S \to \mathbb{C}$ is injective and thus maps S biholomorphically to an open subset of $\mathbb{C}$, which by the Riemann mapping Theorem for $\mathbb{C}$ (Analysis) either equals $\mathbb{C}$ or is biholomorphic to $\mathbb{D}$. $\square$