9. *The real affine plane* has as its "points" all ordered pairs (x, y) of real numbers. A "line" is determined by an ordered triple (u, v, w) of real numbers such that either $u \neq 0$ or $v \neq 0$, and it is defined as the set of all "points" (x, y) satisfying the linear equation $ux + vy + w = 0$. "Incidence" is defined as set membership. Verify that all the axioms for an affine plane are satisfied by this interpretation.

10. A "point" $[x, y, z]$ in *the real projective plane* is determined by an ordered triple (x, y, z) of real numbers that are not all zero, and it consists of all the ordered triples of the form (kx, ky, kz) for all real numbers $k \neq 0$; thus, $[kx, ky, kz] = [x, y, z]$. A "line" in the real projective plane is determined by an ordered triple (u, v, w) of real numbers that are not all zero, and it is defined as the set of all "points" $[x, y, z]$ whose coordinates satisfy the linear equation $ux + vy + wz = 0$. "Incidence" is defined as set membership. Verify that all the axioms for a projective plane are satisfied by this interpretation. Prove that by taking $z = 0$ as the equation of the "line at infinity," by assigning the affine "point" (x, y) the homogeneous coordinates $[x, y, 1]$, and by assigning affine "lines" to projective "lines" in the obvious way, the real projective plane becomes isomorphic to the projective completion of the real affine plane (see Major Exercise 4). Prove that the models in Exercise 10d are also isomorphic to the real projective plane.

11. The following statement is by the French mathematician Desargues: "If the vertices of two triangles correspond in such a way that the lines joining corresponding vertices are concurrent, then the intersections of corresponding sides are collinear." (See Figure 2.10.) This statement is independent of the axioms for projective planes: it holds in the real projective plane, but there exists other projective planes in which it fails. Report on this independence result (see Artzy or Stevenson).

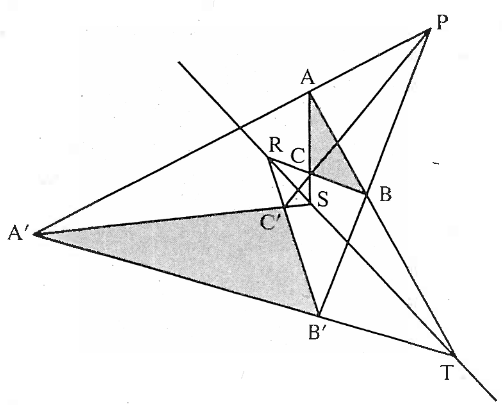


Figure 2.10
Desargues' theorem.

Hilbert's Axioms

The value of Euclid's work as a masterpiece of logic has been very grossly exaggerated.

Bertrand Russell

Flaws in Euclid

Having clarified our rules of reasoning (Chapter 2), let us return to the postulates of Euclid. In Exercises 9 and 10 of Chapter 1 we saw that Euclid neglected to state his assumptions that points and lines exist, that not all points are collinear, and that every line has at least two points lying on it. We made these assumptions explicit in Chapter 2 (p. 42) by adding two more axioms of incidence.

In Exercises 6 and 7, Chapter 1, we saw that some assumptions about "betweenness" are needed. In fact, Euclid never mentioned this notion explicitly, but tacitly assumed certain facts about it that are obvious in diagrams. In Chapter 2 we saw the danger of reasoning from diagrams, so these tacit assumptions will have to be made explicit.

Quite a few of Euclid's proofs are based on reasoning from diagrams. To make these proofs rigorous, a much larger system of explicit axioms is needed. Many such axiom systems have been proposed. We will present a modified version of David Hilbert's system of axioms. Hilbert's system was not the first, but his axioms are perhaps the most intuitive and are certainly the closest in spirit to Euclid's.*

* Let us not forget that no serious work toward constructing new axioms for Euclidean geometry had been done until the discovery of non-Euclidean geometry shocked mathematicians into reexamining the foundations of the former. We have the paradox of non-Euclidean geometry helping us to better understand Euclidean geometry!

David Hilbert

During the first quarter of the twentieth century Hilbert was considered the leading mathematician of the world.* He made outstanding, original contributions to a wide range of mathematical fields as well as to physics. He is perhaps best-known for his research in the foundations of geometry as well as the foundations of algebraic number theory, infinite dimensional spaces, and mathematical logic. A great champion of the axiomatic method, he axiomatized all of the above subjects except for physics (although he did succeed in providing physicists with very valuable mathematical techniques). He was also a mathematical prophet; in 1900 he predicted 23 of the most important mathematical problems of this century.

He has been quoted as saying: "One must be able to say at all times—instead of points, lines and planes—tables, chairs and beer mugs." In other words, since no properties of points, lines, and planes may be used in a proof other than the properties given by the axioms, you may as well call these undefined entities by other names.

Hilbert's axioms are divided into five groups: incidence, betweenness, congruence, continuity, and parallelism. We have already seen the three axioms of incidence in Chapter 2 (p. 42). In the next sections we will deal successively with the other groups of axioms.

Axioms of Betweenness

To further illustrate the need for axioms of betweenness, consider the following attempted proof of the theorem that base angles of an isosceles triangle are congruent. This is not Euclid's proof, which is flawed in other ways (see Golos, p. 57), but is an argument found in some high school geometry texts.

PROOF:

Given $\triangle ABC$ with $AC \cong BC$.

To prove $\measuredangle A \cong \measuredangle B$ (see Figure 3.1).

(1) Let the bisector of $\measuredangle C$ meet AB at D (every angle has a bisector).

(2) In triangles $\triangle ACD$ and $\triangle BCD$, $AC \cong BC$ (hypothesis).

(3) $\measuredangle ACD \cong \measuredangle BCD$ (definition of bisector of an angle).

(4) $CD \cong CD$ (things that are equal are congruent).

* I heartily recommend the warm and colorful biography of Hilbert by Constance Reid. It is nontechnical and conveys the excitement of the time when Göttingen was the capital of the mathematical world.

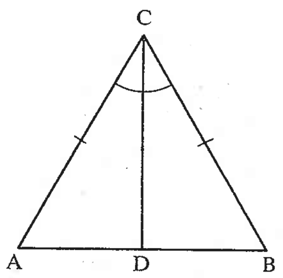


Figure 3.1

(5) $\triangle ACD \cong \triangle BCD$ (SAS).
(6) Therefore, $\measuredangle A \cong \measuredangle B$ (corresponding angles of congruent triangles).■

Consider the first step, whose justification is that every angle has a bisector. This is a correct statement and can be proved separately. But how do we know that the bisector of $\measuredangle C$ meets $\overleftrightarrow{AB}$, or if it does, how do we know that the point of intersection D lies *between* A and B? This may seem obvious, but if we are to be rigorous, it requires proof. For all we know, the picture might look like this:

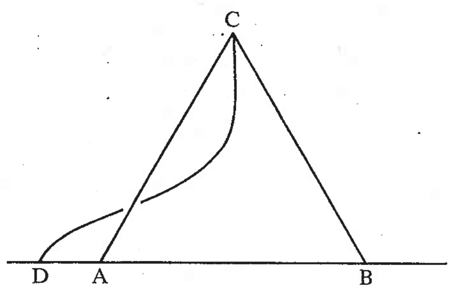


Figure 3.2

If this were the case, Steps 2–5 would still be correct, but we could conclude only that $\measuredangle B$ is congruent to $\measuredangle CAD$, not to $\measuredangle CAB$, since $\measuredangle CAD$ is the angle in $\triangle ACD$ that corresponds to $\measuredangle B$.

Once we state our axioms of betweenness, it will be possible to prove (after a considerable amount of work) that the bisector of $\measuredangle C$ does meet $\overleftrightarrow{AB}$ in a point D between A and B, so the above argument will be repaired

(see crossbar theorem, p. 69). There is, however, an easier proof of the theorem (given in the next section). We will use the shorthand notation

$$A * B * C$$

to abbreviate the statement "point B is between point A and point C."

BETWEENNESS AXIOM 1. If $A * B * C$, then A, B, and C are three distinct points all lying on the same line, and $C * B * A$.

The first part of this axiom fills the gap mentioned in Exercise 6, Chapter 1. The second part ($C * B * A$) makes the obvious remark that "between A and C" means the same as "between C and A"—it doesn't matter whether A or C is mentioned first.

BETWEENNESS AXIOM 2. Given any two distinct points B and D, there exist points A, C, and E lying on $\overleftrightarrow{BD}$ such that $A * B * D$, $B * C * D$, and $B * D * E$.

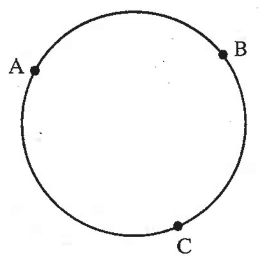
This axiom insures that there are points between B and D and that the line $\overleftrightarrow{BD}$ does not end at either B or D.

Figure 3.3

BETWEENNESS AXIOM 3. If A, B, and C are three distinct points lying on the same line, then one and only one of the points is between the other two.

In a sense, this axiom insures that a line is "straight"; if the points were on a circle, you would then have to say that each is between the other two (or none is between the other two—it would depend on which of the two arcs you look at—Figure 3.4).

Before stating the last betweenness axiom, let us examine some consequences of the first three. Recall that the *segment* AB is defined as the set of all points between A and B together with the endpoints A



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and B. The *ray* $\overrightarrow{AB}$ is defined as the set of all points on the segment AB together with all points C such that $A * B * C$. The second axiom insures that such points C exist, so the ray $\overrightarrow{AB}$ is larger than the segment AB. We can now prove the formulas you encountered in Exercise 7, Chapter 1.

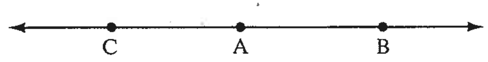
PROPOSITION 3.1. For any two points A and B: (i) $\overrightarrow{AB} \cap \overrightarrow{BA} = AB$, and (ii) $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$.

PROOF OF (i):

(1) By definition of segment and ray, $AB \subset \overrightarrow{AB}$ and $AB \subset \overrightarrow{BA}$, so by definition of intersection, $AB \subset \overrightarrow{AB} \cap \overrightarrow{BA}$.
(2) Conversely, let the point C belong to the intersection of $\overrightarrow{AB}$ and $\overrightarrow{BA}$; we wish to show that C belongs to AB.
(3) If $C = A$ or $C = B$, C is an endpoint of AB. Otherwise, A, B, and C are three collinear points (by definition of ray and Axiom 1), so exactly one of the relations $A * C * B$, $A * B * C$, or $C * A * B$ holds (Axiom 3).
(4) If $A * B * C$ holds, then C is not on $\overrightarrow{BA}$; if $C * A * B$ holds, then C is not on $\overrightarrow{AB}$. In either case, C does not belong to both rays.
(5) Hence, the relation $A * C * B$ must hold, so C belongs to AB. ■

The proof of (ii) is similar and is left as an exercise. (To be truly precise, we should say that $\overrightarrow{AB} \cup \overrightarrow{BA}$ equals the set of points lying on the line $\overleftrightarrow{AB}$, not that it equals the line itself—because "line" is undefined, we do not know that a line is a set of points.)

Recall next that if $C * A * B$, then $\overrightarrow{AC}$ is said to be *opposite* to $\overrightarrow{AB}$:



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By Axiom 1, points A, B, and C are collinear, and by Axiom 3, C does not belong to $\overrightarrow{AB}$, so rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are distinct. This definition is therefore in agreement with the definition given in Chapter 1. Axiom 2 guarantees that every ray $\overrightarrow{AB}$ has an opposite ray $\overrightarrow{AC}$.

It seems clear from the above figure that every point P lying on the line l through A, B, C must either belong to ray $\overrightarrow{AB}$ or to the opposite ray $\overrightarrow{AC}$. This statement seems similar to the second assertion of Proposition 3.1, but it is actually more complicated; we are now discussing *four* points A, B, C, and P, whereas previously we had to deal with only three points at a time. In fact, we encounter here another "pictorially obvious" assertion that cannot be proved without introducing another axiom (see Exercise 17).

Suppose we call the assertion "$C * A * B$ and P collinear with A, B, $C \Rightarrow P \in \overrightarrow{AC} \cup \overrightarrow{AB}$" *the line separation property*. Some mathematicians take this property as another axiom (e.g., Golos). However, it is considered inelegant in mathematics to assume more axioms than are necessary (although we pay for elegance by having to work harder to prove results that appear obvious). So we will not assume the line separation property as an axiom; instead, we will prove it as a consequence of our previous axioms and our last betweenness axiom, called *the plane separation axiom.*

DEFINITION. Let l be any line, A and B any points that do not lie on l. If $A = B$ or if segment AB contains no point lying on l, we say A and B are *on the same side of* l, whereas if $A \neq B$ and segment AB does intersect l, we say that A and B are *on opposite sides* of l [see Figure 3.6]. The law of the excluded middle (Rule 10) tells us that A and B are either on the same side or on opposite sides of l.

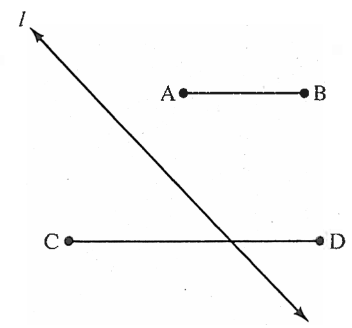


Figure 3.6
A and B are on the same side of l; C and D are on opposite sides of l.

BETWEENNESS AXIOM 4 (Separation). For every line l and for any three points A, B, and C not lying on l:

(i) if A and B are on the same side of l and B and C are on the same side of l, then A and C are on the same side of l [Figure 3.7].

(ii) if A and B are on opposite sides of l and B and C are on opposite sides of l, then A and C are on the same side of l [Figure 3.8].

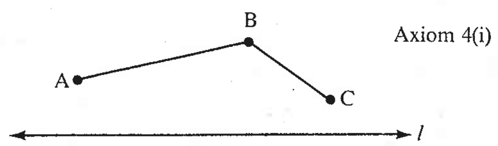


Figure 3.7

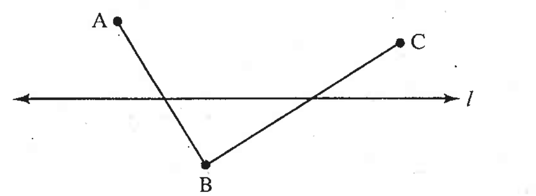


Figure 3.8

Axiom 4(i) indirectly guarantees that our geometry is two-dimensional, since it would not necessarily hold in three-space. (Line l could be outside the plane of this page and cut through segment AC; this interpretation shows that if we assumed the line separation property as an axiom, we could not prove the plane separation property.) Betweenness Axiom 4 is also needed to make sense out of Euclid's fifth postulate, which talks about two lines meeting on one "side" of a transversal. We can now define a *side* of a line l as the set of all points that are on the same side of l as some particular point not lying on l. (This definition may seem circular because we use the word "side" twice, but it is not; we have already defined the compound expression "on the same side.") Another expression commonly used for a "side of l" is a *half-plane bounded by l*.

PROPOSITION 3.2. Every line bounds exactly two half-planes and these half-planes have no point in common.

PROOF:

(1) For every line l there exists a point A not lying on l (by Incidence Axiom 3).

(2) There exists a point O lying on l (by Incidence Axiom 2).

(3) There exists a point B such that $B * O * A$ (Betweenness Axiom 2).

(4) Then A and B are on opposite sides of l (by definition), so l has at least two sides.

(5) Let C be any point distinct from A and B and not lying on l. If C and B are not on the same side of l, then C and A are on the same side of l (by the law of excluded middle and Betweenness Axiom 4(ii)).

(6) Hence, l has exactly two sides (by Steps 4 and 5).

(7) If these sides had a point C in common (RAA hypothesis), then by Betweenness Axiom 4(i) A and B would be on the same side of l, contradiction. ■

We next apply the plane separation property to study betweenness relations among four points.

PROPOSITION 3.3. Given $A * B * C$ and $A * C * D$. Then $B * C * D$ and $A * B * D$. [See Figure 3.9.]

PROOF:

(1) A, B, C, and D are four distinct collinear points (see Exercise 1).

(2) There exists a point E not on the line through A, B, C, D (Incidence Axiom 3).

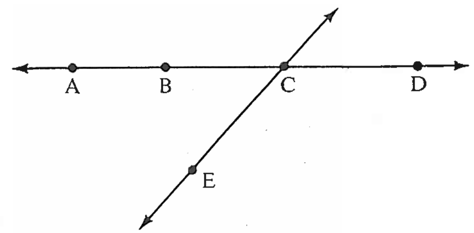


Figure 3.9

(3) Consider line $\overleftrightarrow{EC}$. Since (by hypothesis) AD meets this line in point C, A and D are on opposite sides of $\overleftrightarrow{EC}$.
(4) We claim A and B are on the same side of $\overleftrightarrow{EC}$. Assume on the contrary that A and B are on opposite sides of $\overleftrightarrow{EC}$ (RAA hypothesis).
(5) Then $\overleftrightarrow{EC}$ meets $\overleftrightarrow{AB}$ in a point between A and B (definition of "opposite sides").
(6) That point must be C (Proposition 2.1).
(7) Thus, $A*B*C$ and $A*C*B$, which contradicts Betweenness Axiom 3.
(8) Hence, A and B are on the same side of $\overleftrightarrow{EC}$ (RAA conclusion).
(9) It follows from Steps 3 and 8 and Proposition 3.2 that B and D must be on opposite sides of $\overleftrightarrow{EC}$.
(10) Hence, the point C of intersection of lines $\overleftrightarrow{EC}$ and $\overleftrightarrow{BD}$ lies between B and D (definition of "opposite sides"; Proposition 2.1, i.e., that the point of intersection is unique).

A similar argument involving $\overleftrightarrow{EB}$ proves that $A*B*D$ (Exercise 2b). ■

Finally we prove the *line separation property.*

PROPOSITION 3.4. If $C*A*B$ and l is the line through A, B, and C (Betweenness Axiom 1), then for every point P lying on l, P lies either on ray $\overrightarrow{AB}$ or on the opposite ray $\overrightarrow{AC}$.

PROOF:
(1) Either P lies on $\overrightarrow{AB}$ or it does not (law of excluded middle).
(2) If P does lie on $\overrightarrow{AB}$, we are done, so assume it doesn't; then $P*A*B$ (Betweenness Axiom 3).
(3) If $P = C$ then P lies on $\overrightarrow{AC}$ (by definition), so assume $P \neq C$; then exactly one of the relations $C*A*P$, $C*P*A$, or $P*C*A$ holds (Betweenness Axiom 3 again).
(4) Suppose the relation $C*A*P$ holds (RAA hypothesis).
(5) We know (by Betweenness Axiom 3) that exactly one of the relations $P*C*B$, $C*P*B$, or $C*B*P$ holds.
(6) If $P*B*C$, then combining this with $P*A*B$ (Step 2) gives $A*B*C$ (Proposition 3.3), contradicting the hypothesis.
(7) If $C*P*B$, then combining this with $C*A*P$ (Step 4) gives $A*P*B$ (Proposition 3.3), contradicting Step 2.
(8) If $B*C*P$, then combining this with $B*A*C$ (hypothesis and Betweenness Axiom 1) gives $A*C*P$ (Proposition 3.3), contradicting Step 4.
(9) Since we obtain a contradiction in all three cases, $C*A*P$ does not hold (RAA conclusion).
(10) Therefore, $C*P*A$ or $P*C*A$ (Step 3), which means that P lies on the opposite ray $\overrightarrow{AC}$. ■

The next theorem states a visually obvious property that Pasch discovered Euclid to be using without proof.

PASCH'S THEOREM. If $\triangle ABC$ is any triangle and l is any line intersecting side AB in a point between A and B, then l also intersects either side AC or side BC [see Figure 3.10]. If C does not lie on l, then l does not intersect both AC and BC.

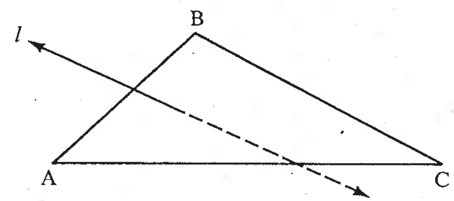


Figure 3.10

Intuitively, this theorem says that if a line "goes into" a triangle through one side, it must "come out" through another side.

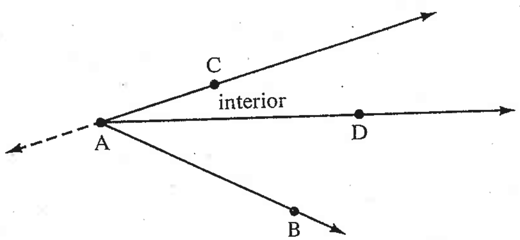
PROOF:
(1) Either C lies on l or it does not; if it does, the theorem holds (law of excluded middle).
(2) A and B do not lie on l, and the segment AB does intersect l (hypothesis).
(3) Hence, A and B lie on opposite sides of l (by definition).
(4) From Step 1 we may assume that C does not lie on l, in which case C is either on the same side of l as A or on the same side of l as B (separation axiom).
(5) If C is on the same side of l as A, then C is on the opposite side from B, which means that l intersects BC and does not intersect AC; similarly if C is on the same side of l as B, then l intersects AC and does not intersect BC (separation axiom). ■

Here are some more results on betweenness and separation that you will be asked to prove in the exercises.

PROPOSITION 3.5. Given A * B * C. Then AC = AB ∪ BC and B is the only point common to segments AB and BC.

PROPOSITION 3.6. Given A * B * C. Then B is the only point common to rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and $\overrightarrow{AB} = \overrightarrow{AC}$.

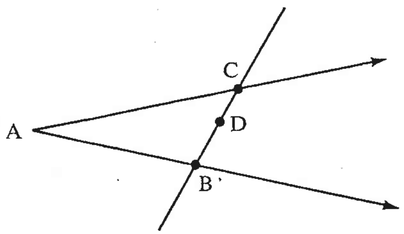
DEFINITION. Given an angle ∢CAB, define a point D to be in the *interior* of ∢CAB if D is on the same side of $\overleftrightarrow{AC}$ as B and if D is also on the same side of $\overleftrightarrow{AB}$ as C. (Thus, the interior of an angle is the intersection of two half-planes.)



ure 3.11

PROPOSITION 3.7. Given an angle ∢CAB and point D lying on line $\overleftrightarrow{BC}$. Then D is in the interior of ∢CAB if and only if B * D * C [see Figure 3.12].

Warning! Do not assume that every point in the interior of an angle lies on a segment joining a point on one side of the angle to a point on the other side. In fact, this assumption is false in hyperbolic geometry.



gure 3.12

PROPOSITION 3.8. If D is in the interior of ∢CAB, then (a) so is every other point on ray $\overrightarrow{AD}$ except A; (b) no point on the opposite ray to $\overrightarrow{AD}$ is in the interior of ∢CAB; (c) if C * A * E, then B is in the interior of ∢DAE [see Figure 3.13].

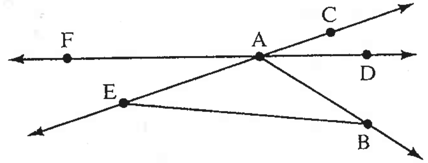


Figure 3.13

DEFINITION. Ray $\overrightarrow{AD}$ is *between* rays $\overrightarrow{AC}$ and $\overrightarrow{AB}$ if $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not opposite rays and D is interior to ∢CAB. (By Proposition 3.8a, this definition does not depend on the choice of point D on $\overrightarrow{AD}$.)

CROSSBAR THEOREM. If $\overrightarrow{AD}$ is between $\overrightarrow{AC}$ and $\overrightarrow{AB}$, then $\overrightarrow{AD}$ intersects segment BC [see Figure 3.14].

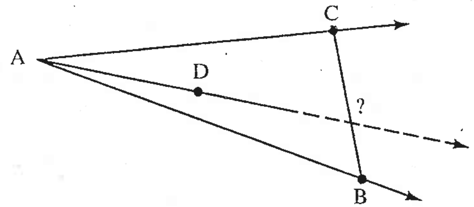


Figure 3.14

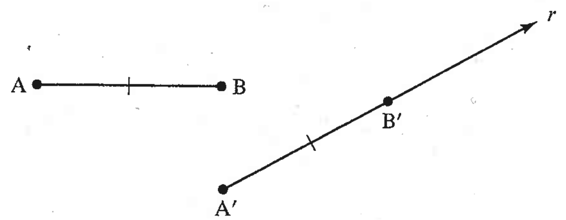
DEFINITION. The *interior* of a triangle is the intersection of the interiors of its three angles. Define a point to be *exterior* to the triangle if it is not in the interior and does not lie on any side of the triangle.

PROPOSITION 3.9. (a) If a ray r emanating from an exterior point of △ABC intersects side AB in a point between A and B, then r also intersects side AC or side BC. (b) If a ray emanates from an interior point of △ABC, then it intersects one of the sides, and if it does not pass through a vertex, it intersects only one side.

Axioms of Congruence

Recall that "congruent" is the last of our undefined terms; it is either a relation between segments or a relation between angles. We are accustomed to congruence as a relation between triangles, but we can now define this as follows: Two triangles are *congruent* if a one-to-one correspondence can be set up between their vertices so that corresponding sides are congruent and corresponding angles are congruent. When we write $\triangle ABC \cong \triangle DEF$ we understand that A corresponds to D, B to E, and C to F. Similar definitions can be given for congruence of quadrilaterals, pentagons, etc.

CONGRUENCE AXIOM 1. If A and B are distinct points and if A′ is any point, then for each ray r emanating from A′ there is a *unique* point B′ on r such that $B' \neq A'$ and $AB \cong A'B'$.



Intuitively speaking, this axiom says you can "move" the segment AB so that it lies on the ray r with A superimposed on A′, B superimposed on B′. (In Major Exercise 2, Chapter 1, you showed how to do this with a straightedge and collapsible compass.)

CONGRUENCE AXIOM 2. If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Moreover, every segment is congruent to itself.

This axiom replaces Euclid's first Common Notion, since it says that segments congruent to the same segment are congruent to each other. It also replaces the fourth Common Notion, since it says that segments that coincide are congruent.

CONGRUENCE AXIOM 3. If $A * B * C$, $A' * B' * C'$, $AB \cong A'B'$, and $BC \cong B'C'$, then $AC \cong A'C'$.

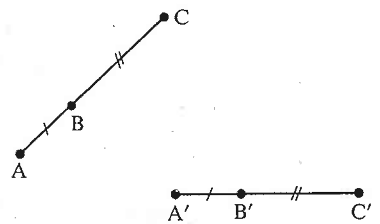


Figure 3.16

This axiom replaces the second Common Notion, since it says that if congruent segments are "added" to congruent segments, the sums are congruent. Here, "adding" means juxtaposing segments along the same line. For example, using Congruence Axioms 1 and 3, you can lay off a copy of a given segment AB two, three, ..., n times, to get a new segment $n \cdot AB$.

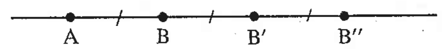


Figure 3.17
$AB'' = 3 \cdot AB$.

CONGRUENCE AXIOM 4. Given any angle $\sphericalangle BAC$ (where, by definition of "angle," $\overrightarrow{AB}$ is not opposite to $\overrightarrow{AC}$), and given any ray $\overrightarrow{A'B'}$ emanating from a point A′, then there is a *unique* ray $\overrightarrow{A'C'}$ on a given side of line $\overleftrightarrow{A'B'}$ such that $\sphericalangle B'A'C' \cong \sphericalangle BAC$.

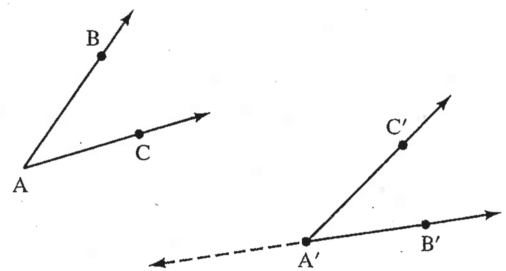


Figure 3.18

This axiom can be paraphrased to state that a given angle can be "laid off" on a given side of a given ray in a unique way (see Major Exercise 1g, Chapter 1).

CONGRUENCE AXIOM 5. If $\sphericalangle A \cong \sphericalangle B$ and $\sphericalangle A \cong \sphericalangle C$, then $\sphericalangle B \cong \sphericalangle C$. Moreover, every angle is congruent to itself.

This is the analog for angles of Congruence Axiom 2 for segments; the first part asserts the transitivity and the second part the reflexivity of the congruence relation. Combining them, we can prove the symmetry of this relation: $\sphericalangle A \cong \sphericalangle B \Rightarrow \sphericalangle B \cong \sphericalangle A$.

PROOF: $\sphericalangle A \cong \sphericalangle B$ (hypothesis) and $\sphericalangle A \cong \sphericalangle A$ (reflexivity) imply (substituting A for C in Congruence Axiom 5) $\sphericalangle B \cong \sphericalangle A$ (transitivity).∎

It would seem natural to assume next an "addition axiom" for congruence of *angles* analogous to Congruence Axiom 3 (the addition axiom for congruence of segments). We won't do this, however, because such a result can be proved using the next congruence axiom (see Proposition 3.19).

CONGRUENCE AXIOM 6 (SAS). If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of another triangle, then the two triangles are congruent.

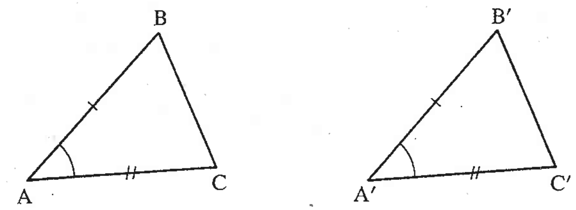


Figure 3.19

This is the side-angle-side criterion for congruence of triangles with which you are familiar. As mentioned, Euclid attempted to prove SAS as a theorem. His argument was essentially as follows: Move $\triangle A'B'C'$ so as to place point A′ on point A and $\overleftrightarrow{A'B'}$ on $\overleftrightarrow{AB}$. Since $AB \cong A'B'$, by hypothesis, point B′ must fall on point B. Since $\sphericalangle A \cong \sphericalangle A'$, $\overleftrightarrow{A'C'}$ must fall on $\overleftrightarrow{AC}$, and since $AC \cong A'C'$, point C′ must coincide with point C. Hence, B′C′ will coincide with BC and the remaining angles will coincide with the remaining angles, so the triangles will be congruent.

This argument is called *superposition*. It derives from the experience of drawing two triangles on paper, cutting out one, and placing it on top of the other. Although this is a good way to convince a novice in geometry to accept SAS, it is not a proof, and Euclid reluctantly used it in only one other theorem. It is not a proof because Euclid never stated an axiom that allows figures to be moved around without changing their size and shape.

Some modern writers introduce "motion" as an undefined term and lay down axioms for this term. (In fact, in Pieri's foundations of geometry, "point" and "motion" are the only undefined terms.) Or else, the geometry is first built up on a different basis, "distances" introduced, and a "motion" defined as a one-to-one transformation of the plane onto itself that preserves distance. Euclid can be vindicated by either approach. In fact, Felix Klein, in his 1872 Erlanger *Programm*, defined geometry as the study of those properties of figures that remain invariant under a particular group of transformations. (For an approach to geometry using motion as a foundation, see Ewald; see also the Major Exercises in Chapter 7.)

The "motions" mentioned in the previous paragraph are all mathematical abstractions. The physicist Baron von Helmholtz took a different approach. He maintained that geometry requires us to assume the actual existence of rigid bodies with free mobility in space. Geometry is then dependent on mechanics. Naturally, this viewpoint was rejected by most mathematicians.

As an application of SAS, the simple proof of Pappus (300 A.D.) for the theorem on base angles of an isosceles triangle follows.

PROPOSITION 3.10. If in $\triangle ABC$ we have $AB \cong AC$, then $\sphericalangle B \cong \sphericalangle C$.

PROOF:

(1) Consider the correspondence of vertices $A \leftrightarrow A$, $B \leftrightarrow C$, $C \leftrightarrow B$. Under this correspondence, two sides and the included angle of $\triangle ABC$ are congruent respectively to the corresponding sides and included

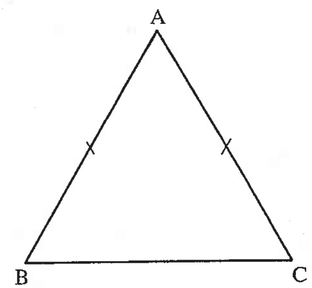


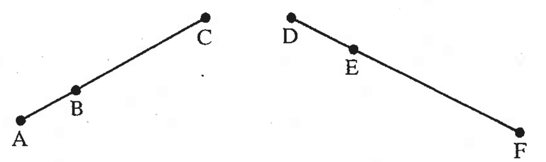
Figure 3

angle of $\triangle ACB$ (by hypothesis and Congruence Axiom 5 that an angle is congruent to itself).

(2) Hence, $\triangle ABC \cong \triangle ACB$ (SAS), so the corresponding angles $\sphericalangle B$ and $\sphericalangle C$ are congruent (by definition of congruence of triangles).■

Here are some more familiar results on congruence. We will prove some of them; if the proof is omitted, see the exercises.

PROPOSITION 3.11 (Segment Subtraction). If $A * B * C$, $D * E * F$, $AB \cong DE$, and $AC \cong DF$, then $BC \cong EF$.



.21

PROPOSITION 3.12. Given $AC \cong DF$, then for any point B between A and C, there is a unique point E between D and F such that $AB \cong DE$.

PROOF:

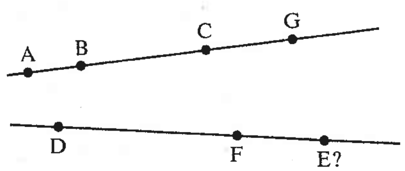
(1) There is a unique point E on $\overrightarrow{DF}$ such that $AB \cong DE$ (Congruence Axiom 1).

(2) Suppose E were not between D and F (RAA hypothesis).

(3) Then either $E = F$ or $D * F * E$ (definition of $\overrightarrow{DF}$).

(4) If $E = F$, then B and C are two distinct points on $\overrightarrow{AC}$ such that $AC \cong DF \cong AB$ (hypothesis, Step 1), contradicting the uniqueness part of Congruence Axiom 1.

(5) If $D * F * E$, then there is a point G on the ray opposite to $\overrightarrow{CA}$ such that $FE \cong CG$ (Congruence Axiom 1).



!2

(6) Then $AG \cong DE$ (Congruence Axiom 3).

(7) Thus, there are two distinct points B and G on $\overrightarrow{AC}$ such that $AG \cong DE \cong AB$ (Steps 1, 5, and 6), contradicting the uniqueness part of Congruence Axiom 1.

(8) $D * E * F$ (RAA conclusion).■

DEFINITION. $AB < CD$ (or $CD > AB$) means that there exists a point E between C and D such that $AB \cong CE$.

PROPOSITION 3.13 (Segment Ordering). (a) Exactly one of the following conditions holds (trichotomy): $AB < CD$, $AB \cong CD$, or $AB > CD$. (b) If $AB < CD$ and $CD \cong EF$, then $AB < EF$. (c) If $AB > CD$ and $CD \cong EF$, then $AB > EF$. (d) If $AB < CD$ and $CD < EF$, then $AB < EF$ (transitivity).

PROPOSITION 3.14 Supplements of congruent angles are congruent.

PROPOSITION 3.15. (a) Vertical angles are congruent to each other. (b) An angle congruent to a right angle is a right angle.

PROPOSITION 3.16. For every line l and every point P there exists a line through P perpendicular to l.

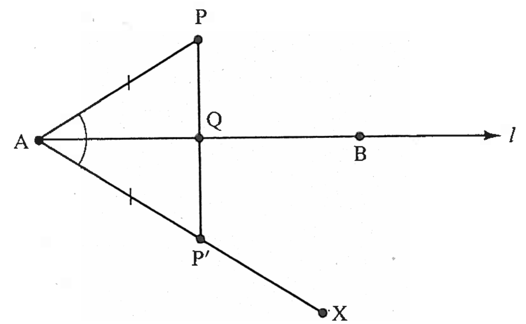


Figure 3.23

PROOF:

(1) Assume first that P does not lie on l and let A and B be any two points on l (Incidence Axiom 2).

(2) On the opposite side of l from P there exists a ray $\overrightarrow{AX}$ such that $\measuredangle XAB \cong \measuredangle PAB$ (Congruence Axiom 4).
(3) There is a point P′ on $\overrightarrow{AX}$ such that $AP' \cong AP$ (Congruence Axiom 1).
(4) PP′ intersects l in a point Q (definition of opposite sides of l).
(5) If Q = A, then $\overleftrightarrow{PP'} \perp l$ (definition of $\perp$).
(6) If Q ≠ A, then $\triangle PAQ \cong \triangle P'AQ$ (SAS).
(7) Hence, $\measuredangle PQA \cong \measuredangle P'QA$ (corresponding angles), so $\overleftrightarrow{PP'} \perp l$ (definition of $\perp$).
(8) Assume now that P lies on l. Since there are points not lying on l (Incidence Axiom 3), we can drop a perpendicular from one of them to l (Steps 5 and 7), thereby obtaining a right angle.
(9) We can lay off an angle congruent to this right angle with vertex at P and one side on l (Congruence Axiom 4); the other side of this angle is part of a line through P perpendicular to l (Proposition 3.15b). ■

It is natural to ask whether the perpendicular to l through P constructed in Proposition 3.16 is unique. If P lies on l, the uniqueness part of Congruence Axiom 4 guarantees that the perpendicular is unique. If P does not lie on l, we will not be able to prove uniqueness for the perpendicular until the next chapter. In elliptic geometry there is a point P called the "pole" of l such that every line through P is perpendicular to l! To visualize this, think of l as the equator on a sphere and P as the north pole; every great circle through the north pole is perpendicular to the equator (Figure 3.24).

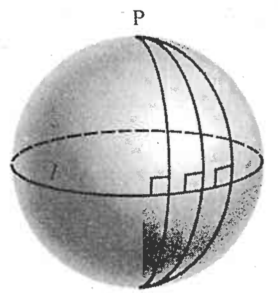


Figure 3.24

PROPOSITION 3.17 (ASA Criterion for Congruence). Given △ABC and △DEF with $\measuredangle A \cong \measuredangle D$, $\measuredangle C \cong \measuredangle F$, and $AC \cong DF$. Then $\triangle ABC \cong \triangle DEF$.

PROPOSITION 3.18 (Converse of Proposition 3.10). If in △ABC we have $\measuredangle B \cong \measuredangle C$, then $AB \cong AC$ and △ABC is isosceles.

PROPOSITION 3.19 (Angle Addition). Given $\overrightarrow{BG}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, $\overrightarrow{EH}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$, $\measuredangle CBG \cong \measuredangle FEH$, and $\measuredangle GBA \cong \measuredangle HED$. Then $\measuredangle ABC \cong \measuredangle DEF$.

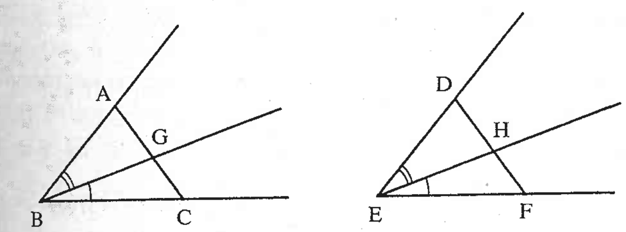


Figure 3.25

PROOF:
(1) By the crossbar theorem, we assume G is chosen so that $A * G * C$.
(2) By Congruence Axiom 1, we assume D, F, and H chosen so that $AB \cong ED$, $GB \cong EH$, and $CB \cong EF$.
(3) Then $\triangle ABG \cong \triangle DEH$ and $\triangle GBC \cong \triangle HEF$ (SAS).
(4) $\measuredangle DHE \cong \measuredangle AGB$, $\measuredangle FHE = \measuredangle CGB$ (Step 3), and $\measuredangle AGB$ is supplementary to $\measuredangle CGB$ (Step 1).
(5) $\measuredangle DHE$ is supplementary to $\measuredangle FHE$ (Step 4 and Proposition 3.14) so D, H, and F are collinear.
(6) $D * H * F$ (Proposition 3.7, using the hypothesis on $\overrightarrow{EH}$).
(7) $AC \cong DF$ (Steps 3 and 6, Congruence Axiom 3).
(8) $\triangle ABC \cong \triangle DEF$ (SAS, Steps 2, 3, and 7).
(9) $\measuredangle ABC \cong \measuredangle DEF$ (corresponding angles). ■

PROPOSITION 3.20 (Angle Subtraction). Given $\overrightarrow{BG}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, $\overrightarrow{EH}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$, $\measuredangle CBG \cong \measuredangle FEH$, and $\measuredangle ABC \cong \measuredangle DEF$. Then $\measuredangle GBA \cong \measuredangle HED$.

DEFINITION. $\measuredangle ABC < \measuredangle DEF$ means there is a ray $\overrightarrow{EG}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$ such that $\measuredangle ABC \cong \measuredangle GEF$ (see Figure 3.26).

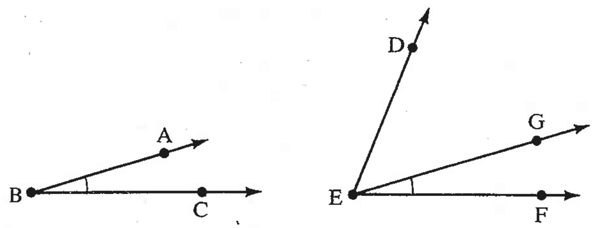


Figure 3.26

PROPOSITION 3.21 (Ordering of Angles). (a) Exactly one of the following conditions holds (trichotomy): $\sphericalangle P < \sphericalangle Q$, $\sphericalangle P \cong \sphericalangle Q$, or $\sphericalangle Q < \sphericalangle P$. (b) if $\sphericalangle P < \sphericalangle Q$ and $\sphericalangle Q \cong \sphericalangle R$, then $\sphericalangle P < \sphericalangle R$. (c) If $\sphericalangle P > \sphericalangle Q$ and $\sphericalangle Q \cong \sphericalangle R$, then $\sphericalangle P > \sphericalangle R$. (d) If $\sphericalangle P < \sphericalangle Q$ and $\sphericalangle Q < \sphericalangle R$, then $\sphericalangle P < \sphericalangle R$.

PROPOSITION 3.22 (SSS Criterion for Congruence). Given $\triangle ABC$ and $\triangle DEF$. If $AB \cong DE$, $BC \cong EF$, and $AC \cong DF$, then $\triangle ABC \cong \triangle DEF$.

The AAS criterion for congruence will be given in the next chapter because its proof is more difficult. The next proposition was assumed as an axiom by Euclid, but can be proved from Hilbert's axioms.

PROPOSITION 3.23 (Euclid's Fourth Postulate). All right angles are congruent to each other.

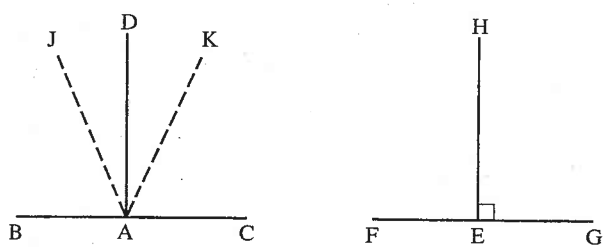


Figure 3.27

PROOF:

(1) Given $\sphericalangle BAD \cong \sphericalangle CAD$ and $\sphericalangle FEH \cong \sphericalangle GEH$ (two pairs of right angles, by definition). Assume the contrary, that $\sphericalangle BAD$ is not congruent to $\sphericalangle FEH$ (RAA hypothesis).

(2) Then one of these angles is smaller than the other, e.g., $\sphericalangle FEH < \sphericalangle BAD$ (Proposition 3.21a), so that by definition there is a ray $\overrightarrow{AJ}$ between $\overrightarrow{AB}$ and $\overrightarrow{AD}$ such that $\sphericalangle BAJ \cong \sphericalangle FEH$.

(3) $\sphericalangle CAJ \cong \sphericalangle GEH$ (Proposition 3.14).

(4) $\sphericalangle CAJ \cong \sphericalangle FEH$ (Steps 1 and 3, Congruence Axiom 5).

(5) There is a ray $\overrightarrow{AK}$ between $\overrightarrow{AD}$ and $\overrightarrow{AC}$ such that $\sphericalangle BAJ \cong \sphericalangle CAK$ (Step 1 and Proposition 3.21b).

(6) $\sphericalangle BAJ \cong \sphericalangle CAJ$ (Steps 2 and 4, and Congruence Axiom 5).

(7) $\sphericalangle CAJ \cong \sphericalangle CAK$ (Steps 5 and 6, and Congruence Axiom 5).

(8) Thus, we have $\sphericalangle CAD$ greater than $\sphericalangle CAK$ (by definition) and less than its congruent angle $\sphericalangle CAJ$ (Step 7 and Proposition 3.8c), which contradicts Proposition 3.21.

(9) $\sphericalangle BAD \cong \sphericalangle FEH$ (RAA conclusion). ■

Axioms of Continuity

These axioms are the most subtle and difficult to comprehend. They are needed to fill in a number of gaps in Euclid's *Elements*. A thorough discussion of them is beyond the scope of this book; it will suffice for our purposes to state them and make only brief remarks.

ARCHIMEDES' AXIOM. If AB and CD are any segments, then there is a number n such that if segment CD is laid off n times on the ray $\overrightarrow{AB}$ emanating from A, then a point E is reached where $n \cdot CD \cong AE$ and B is between A and E.

For example, if AB were π units long and CD of one unit length, you would have to lay off CD at least four times to get to a point E beyond the point B (see Figure 3.28).

The intuitive content of Archimedes' axiom is that if you arbitrarily choose one segment CD as a unit of length, then every other segment has

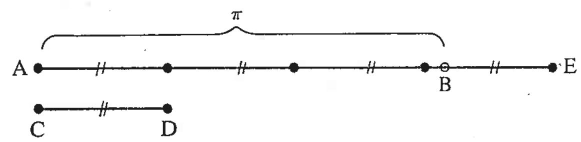


Figure 3.28

finite length with respect to this unit (in the notation of the axiom the length of AB with respect to CD as unit is at most n units). Another way to look at it is to choose AB as unit of length. The axiom says that no other segment can be infinitesimally small with respect to this unit (the length of CD with respect to AB as unit is at least $1/n$ units).

DEDEKIND'S AXIOM.* Suppose that the set of all points on a line l is the union $\Sigma_1 \cup \Sigma_2$ of two nonempty subsets such that no point of Σ_1 is between two points of Σ_2 and vice versa. Then there is a unique point O lying on l such that $P_1 * O * P_2$ if and only if $P_1 \varepsilon \Sigma_1$ and $P_2 \varepsilon \Sigma_2$.

Figure 3.29

Dedekind's axiom is a sort of converse to the line separation property (p. 63). That property says that any point O on l separates all the other points on l into those to the left of O and those to the right. Dedekind's axiom says that conversely, any separation of points on l into left and right is produced by a unique point O. A pair of subsets Σ_1 and Σ_2 with the properties in Dedekind's axiom is called a *Dedekind cut* of the line.

Of course, we must define what is meant by "left" and "right" here. The intuitive idea is obvious; it is left to you to write down a precise definition (Exercise 7).

Loosely speaking, the purpose of Dedekind's axiom is to insure that a line l has no "holes" in it, in the sense that for any point O on l and any positive real number x there exist unique points P_{-x} and P_x on l such that $P_{-x} * O * P_x$ and segments $P_{-x}O$ and OP_x both have length x (with respect to some unit segment of measurement).

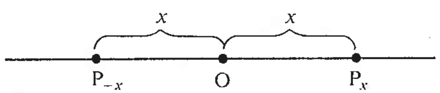


Figure 3.30

Without Dedekind's axiom there would be no guarantee, for example, of the existence of a segment of length π. With it, we can introduce a rectangular coordinate system into the plane and do geometry analytically, as Descartes and Fermat discovered in the seventeenth century. This

* This axiom was proposed by J. W. R. Dedekind in 1871; an analog of it is used in analysis

coordinate system enables us to prove that our axioms for Euclidean geometry are *categorical* in the sense that the system has a unique model (up to isomorphism—see Exercise 10, Chapter 2), namely, the usual Cartesian coordinate plane of all ordered pairs of real numbers.

If we omitted Dedekind's axiom, then another model would be the so-called *surd plane*, a plane that is used to prove the impossibility of trisecting every angle with a straightedge and compass (see Moise, p. 228 ff.). The categorical natural of all the axioms is proved in Borsuk-Szmielew (p. 276 ff.).

It can actually be proved that Archimedes' axiom is a consequence of Dedekind's and the other axioms (see Major Exercise 1 or Borsuk-Szmielew, p. 154). To see why Dedekind's axiom is needed, consider the argument Euclid gives to justify his very first proposition.

EUCLID'S PROPOSITION 1. Given any segment, there is an equilateral triangle having the given segment as one of its sides.

EUCLID'S PROOF:

(1) Let AB be the given segment. With center A and radius AB, let the circle BCD be described (Postulate III).
(2) Again with center B and radius BA, let the circle ACE be described (Postulate III).
(3) From a point C in which the circles cut one another, draw the segments CA and CB (Postulate I).
(4) Since A is the center of the circle CDB, AC is congruent to AB (definition of circle).
(5) Again, since B is the center of circle CAE, BC is congruent to BA (definition of circle).
(6) Since CA and CB are each congruent to AB (Steps 4 and 5), they are congruent to each other (first Common Notion).
(7) Hence, △ABC is an equilateral triangle (by definition) having AB as one of its sides. ■

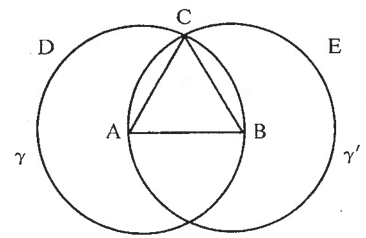


Figure 3

Since every step has apparently been justified, you may not see the gap in the proof. It occurs in the first three steps, especially in the third step, which explicitly states that C is a point in which the circles cut each other. (The second step states this implicitly by using the same letter "C" to denote part of the circle, as in the first step.) The point is: how do we know that such a point C exists?

If you believe it is obvious from the diagram that such a point C exists, you are right—but you are not allowed to use the diagram to justify this! We aren't saying that the circles constructed do not cut each other; we're saying only that another axiom is needed to *prove* that they do.

The gap can be filled by proving the following *circular continuity principle*:

CIRCULAR CONTINUITY PRINCIPLE. If a circle γ has one point inside and one point outside another circle γ', then the two circles intersect in two points.

Here a point P is defined as *inside* a circle with center O and radius OR if OP < OR (*outside* if OP > OR). In Figure 3.31, point B is inside circle γ', and the point B′ (not shown) such that A is the midpoint of BB′ is outside γ'. The circular continuity principle is a consequence of Dedekind's axiom (see Heath, p. 238).

Another gap in the *Elements* occurs in Euclid's method for dropping a perpendicular to a line (his twelfth proposition). In his construction Euclid tacitly assumes the following *elementary continuity principle:*

ELEMENTARY CONTINUITY PRINCIPLE. If one endpoint of a segment is inside a circle and the other outside, then the segment intersects the circle.

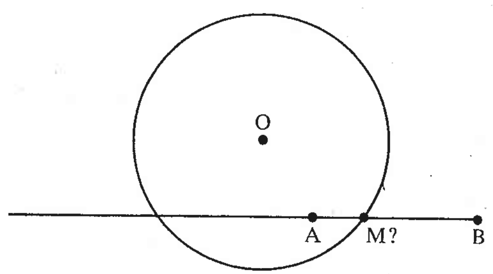


Figure 3.32

Let us sketch a proof of this proposition using Dedekind's axiom:

By definition of "inside" and "outside" of a circle, if O is the center of the circle and OR a radius, we have OA < OR < OB. We define a Dedekind cut for the line $\overleftrightarrow{AB}$ as follows: let Σ_1 be the set of all points P of segment AB such that OP < OR, together with all points on the ray opposite to $\overrightarrow{AB}$; let Σ_2 be the set of all points P of AB such that OP ≅ OR or OP > OR, together with all points on the ray opposite to $\overrightarrow{BA}$. If $\overleftrightarrow{AB}$ is directed so that A is to the left of B, then every point of Σ_1 is to the left of every point of Σ_2. Hence, by Dedekind's axiom, there is a unique point M on $\overleftrightarrow{AB}$ such that $P_1 * M * P_2$ for all $P_1 \varepsilon \Sigma_1$ and $P_2 \varepsilon \Sigma_2$. We can then show by an RAA argument that OM ≅ OR, i.e., M lies on the circle (see Major Exercise 2). ■

It is apparent that such reasoning with Dedekind's axiom is not elementary. If you have difficulty following it, don't be discouraged—it will not play an important role in this book. We will use the continuity axioms only on those few occasions when they prove absolutely essential. On other occasions they can be avoided. For example, Euclid used the existence of equilateral triangles on a given base to prove the existence of angle bisectors and midpoints (see Heath, Propositions 9 and 10), so that the existence of the bisectors and midpoints would seem to depend indirectly on continuity. But there is an ingenious way to prove the existence of angle bisectors and midpoints without using continuity (see Exercises 12–14, Chapter 4).

One reason Dedekind's axiom is not elementary is that it refers to variable sets of points that form Dedekind cuts. Alfred Tarski has proposed a different system of axioms for Euclidean geometry, in which the only variables are points and every statement is made in terms of the undefined term "point" and the undefined relations "betweenness" and "equidistance." With this restricted language, Tarski has been able to express all the results found in traditional elementary geometry textbooks. However, Dedekind's axiom cannot be expressed in this language. Tarski proposes to replace Dedekind's axiom with the elementary continuity principle. He calls this system "the geometry of elementary constructions" because the only existence statements that are valid in the system are those that can be proved by straightedge-and-compass constructions (see his paper "What is elementary geometry?" in Henkin, Suppes and Tarski).

Axiom of Parallelism

If we were to stop with the axioms we now have, we could do quite a bit of geometry, but we still couldn't do all of Euclidean geometry. We would be able to do *absolute geometry,* a misleading name first used by Janos Bolyai and still widely used. I prefer the name suggested by W. Prenowitz and M. Jordan, *neutral geometry,* so called because in doing this geometry we remain neutral about the one axiom from Hilbert's list left to be considered—historically the most controversial axiom of all.

HILBERT'S AXIOM OF PARALLELISM. For every line *l* and every point P not lying on *l* there is at most one line *m* through P such that *m* is parallel to *l*.

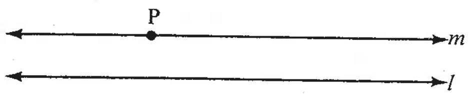


Figure 3.33

Note that this axiom is weaker than the Euclidean parallel postulate introduced in Chapter 1 (p. 17). This axiom asserts only that *at most* one line through P is parallel to *l*, whereas the Euclidean parallel postulate asserts in addition that *at least* one line through P is parallel to *l*. The reason "at least" is omitted from Hilbert's axiom is that it can be proved from the other axioms (see Corollary 2 to Theorem 4.1, p. 97); it is therefore unnecessary to assume this as part of an axiom. This observation is important because it implies that the elliptic parallel property (no parallel lines exist) is inconsistent with the axioms of neutral geometry. Thus, a different set of axioms is needed for the foundation of elliptic geometry (see Appendix B).

The axiom of parallelism completes our list of sixteen axioms for Euclidean geometry. In referring to these axioms we will use the following shorthand: the incidence axioms will be denoted by I-1, I-2, and I-3; the betweenness axioms by B-1, B-2, B-3, and B-4; the congruence axioms by C-1, C-2, C-3, C-4, C-5 and C-6. The continuity axioms and the parallelism axiom will be referred to by name.

Review Exercise

Which of the following statements are correct?

(1) Hilbert's axiom of parallelism is the same as the Euclidean parallel postulate given in Chapter 1.

(2) A * B * C is logically equivalent to C * B * A.

(3) In Axiom B-2 it is unnecessary to assume the existence of a point E such that B * D * E because this can be proved from the rest of the axiom and Axiom B-1, by interchanging the roles of B and D and taking E to be A.

(4) If A, B, and C are distinct collinear points, it is possible that *both* A * B * C *and* A * C * B.

(5) The "line separation property" asserts that a line has two sides.

(6) If points A and B are on opposite sides of a line *l*, then a point C not on *l* must either be on the same side of *l* as A or on the same side of *l* as B.

(7) If line *m* is parallel to line *l*, then all the points on *m* lie on the same side of *l*.

(8) If we were to take Pasch's theorem as an axiom instead of the Separation Axiom B-4, then B-4 could be proved as a theorem.

(9) The notion of "congruence" for two triangles is not defined in this chapter.

(10) It is an immediate consequence of Axiom C-2 that if AB ≅ CD, then CD ≅ AB.

(11) One of the congruence axioms asserts that if congruent segments are "subtracted" from congruent segments, the differences are congruent.

(12) In the statement of Axiom C-4 the variables A, B, C, A′, and B′ are quantified universally, and the variable C′ is quantified existentially.

(13) One of the congruence axioms is the side-side-side (SSS) criterion for congruence of triangles.

(14) Euclid attempted unsuccessfully to prove the SAS criterion for congruence by a method called "superposition."

(15) We can use Pappus' method to prove the converse of the theorem on base angles of an isosceles triangle if we first prove the angle-side-angle (ASA) criterion for congruence.

(16) Archimedes' axiom is independent of the other fifteen axioms for Euclidean geometry given in this book.

(17) $AB < CD$ means that there is a point E between C and D such that $AB \cong CE$.

(18) Neutral geometry used to be called "absolute geometry"; it is the geometry you have when the axiom of parallelism is excluded from the system of axioms given here.

Exercises on Betweenness

1. Given $A * B * C$ and $A * C * D$. Prove that:
   (a) A, B, C, D are four distinct points;
   (b) A, B, C, D are collinear (the proof of (a) requires an axiom).

2. (a) Finish the proof of Proposition 3.1 by showing that $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$.
   (b) Finish the proof of Proposition 3.3 by showing that $A * B * D$.
   (c) Prove the converse of Proposition 3.3 by applying the symmetry of betweenness (B-1).

3. Given $A * B * C$.
   (a) Use Proposition 3.3 to prove that $AB \subset AC$. Interchanging A and C, deduce $CB \subset CA$; which axiom justifies this interchange?
   (b) Use Axiom B-4 to prove that $AC \subset AB \cup BC$. (Hint: if P is a fourth point on AC, use another line through P to show $P \varepsilon AB$ or $P \varepsilon BC$.)
   (c) Finish the proof of Proposition 3.5. (Hint: if $P \neq B$ and $P \varepsilon AB \cap BC$, use another line through P to get a contradiction.)

4. Given $A * B * C$.
   (a) If P is a fourth point collinear with A, B, and C, use Proposition 3.3 and an axiom to prove that $\sim A * B * P \Rightarrow \sim A * C * P$.
   (b) Deduce that $\overrightarrow{BA} \subset \overrightarrow{CA}$ and, symmetrically, $\overrightarrow{BC} \subset \overrightarrow{AC}$.
   (c) Use this result, Proposition 3.1a, Proposition 3.3, and Proposition 3.5 to prove that B is the only point that $\overrightarrow{BA}$ and $\overrightarrow{BC}$ have in common.

5. Given $A * B * C$. Prove that $\overrightarrow{AB} = \overrightarrow{AC}$, completing the proof of Proposition 3.6. Deduce that every ray has a *unique* opposite ray.

6. In Axiom B-2 we were given distinct points B and D and we asserted the existence of points A, C, and E such that $A * B * D$, $B * C * D$, and $B * D * E$. We can now show that it was not necessary to assume the existence of a point C between B and D because we can prove from our other axioms (including the rest of Axiom B-2) and from Pasch's theorem (which was proved without using Axiom B-2) that C exists.* Your job is to justify each step in the proof (some of the steps require a separate RAA argument).

* Regarding superfluous hypotheses, there is a story that Napoleon, after examining a copy of Laplace's *Celestial Mechanics*, asked Laplace why there was no mention of God in the work. The author replied, "I have no need of this hypothesis."

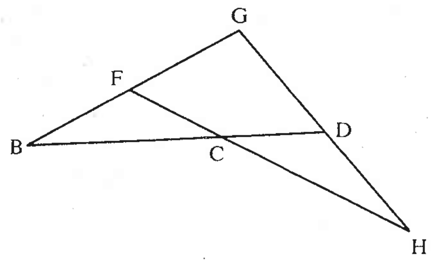


Figure 3.34

PROOF:
(1) There exists a line $\overleftrightarrow{BD}$ through B and D.
(2) There exists a point F not lying on $\overleftrightarrow{BD}$.
(3) There exists a line $\overleftrightarrow{BF}$ through B and F.
(4) There exists a point G such that $B * F * G$.
(5) Points B, F, and G are collinear.
(6) G and D are distinct points and D, B, and G are not collinear.
(7) There exists a point H such that $G * D * H$.
(8) There exists a line $\overleftrightarrow{GH}$.
(9) H and F are distinct points.
(10) There exists a line $\overleftrightarrow{FH}$.
(11) D does not lie on $\overleftrightarrow{FH}$.
(12) B does not lie on $\overleftrightarrow{FH}$.
(13) G does not lie on $\overleftrightarrow{FH}$.
(14) Points D, B, and G determine a triangle $\triangle DBG$ and $\overleftrightarrow{FH}$ intersects side BG in a point between B and G.
(15) H is the only point lying on both $\overleftrightarrow{FH}$ and $\overleftrightarrow{GH}$.
(16) No point between G and D lies on $\overleftrightarrow{FH}$.
(17) Hence, $\overleftrightarrow{FH}$ intersects side BD in a point C between D and B.
(18) Thus, there exists a point C between D and B. ■

7. Given a line l. Fix two points A and B on it. Stipulate arbitrarily that A is *to the left* of B. Using this convention and the relation of betweenness, give a precise definition of when a point C on l is "to the left" of another point D on l. (There will be many cases to consider, differing as to how C and D are located relative to A and B.)

8. From the three-point model (p. 44) we saw that if we used only the axioms of incidence we could not prove that a line has more than two points lying on it. Using the betweenness axioms as well, prove that every line has at least five points lying on it. Give an informal argument to show that every segment (*a fortiori*, every line) has an infinite number of points lying on it (a formal proof requires the technique of mathematical induction).

9. Given a line l, a point A on l, and a point B not on l. Then every point of the ray $\overrightarrow{AB}$ (except A) is on the same side of l as B. (Hint: use an RAA argument.)

10. Prove Proposition 3.7.

11. Prove Proposition 3.8 (Hint: for Proposition 3.8c prove in two steps that E and B lie on the same side of $\overleftrightarrow{AD}$, first showing that EB does not meet $\overrightarrow{AD}$, then showing that EB does not meet the opposite ray $\overrightarrow{AF}$. Use Exercise 9.)

12. Prove the crossbar theorem. (Hint: assume the contrary, which means that B and C lie on the same side of $\overleftrightarrow{AD}$. Use Proposition 3.8c to derive a contradiction.)

13. Prove Proposition 3.9. (Hint: for Proposition 3.9c use Pasch's theorem and Proposition 3.7; see Figure 3.35. For Proposition 3.9b let the ray emanate from point D in the interior of △ABC. Use the crossbar theorem and Proposition 3.7 to show that $\overrightarrow{AD}$ meets BC in a point E such that A * D * E. Apply Pasch's theorem to △ABE and △AEC; see Figure 3.36.)

14. Prove that a line cannot be contained in the interior of a triangle.

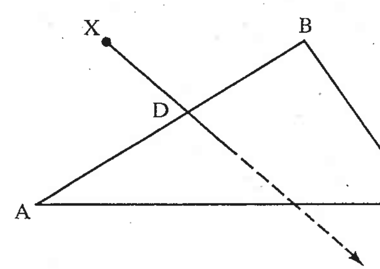


Figure 3.35

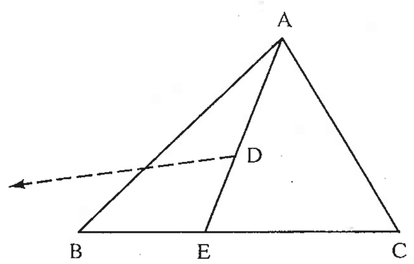


Figure 3.36

15. If a, b, and c are rays, let us say that they are *coterminal* if they emanate from the same point, and let us use the notation $a * b * c$ to mean that b is between a and c (as defined on p. 69). The analog of Axiom B-1 states that if $a * b * c$, then a, b, c are distinct, coterminal, and $c * b * a$; this analog is obviously correct. State the analogs of Axioms B-2 and B-3 and Proposition 3.3 and tell which parts of these analogs are correct. (Beware of opposite rays!)

16. Find an interpretation of the incidence axioms and the first two betweenness axioms for which Axiom B-3 fails in the following way: there exist three collinear points, no one of which is between the other two. (Hint: in the usual Euclidean model, introduce a new betweenness relation A * B * C to mean that B is the midpoint of AC.)

17. Find an interpretation of the incidence axioms and the first three betweenness axioms for which the line separation property (Proposition 3.4) fails. (Hint: in the usual Euclidean model, pick a point P that is between A and B in the usual Euclidean sense and specify that A will now be considered to be between P and B. Leave all other betweenness relations among points alone. Show that P lies neither on ray $\overrightarrow{AB}$ nor its opposite ray $\overrightarrow{AC}$.)

18. Define the *dyadic rational plane* as that which consists of all ordered pairs (x, y) of rational numbers with at most a power of 2 in the denominator. With incidence and betweenness defined as in Major Exercise 6 below, show that the incidence axioms, the first three betweenness axioms, and the line separation property all hold in the dyadic rational plane; show that Pasch's theorem fails (e.g., the lines $3x + y = 1$ and $y = 0$ do not meet in this plane).

19. A set of points S is called *convex* if whenever two points A and B are in S, the entire segment AB is contained in S. Prove that a half-plane, the interior of an angle, and the interior of a triangle are all convex sets, whereas the exterior of a triangle is not convex. Is a triangle a convex set?

Exercises on Congruence

20. Justify each step in the following proof of Proposition 3.11:

PROOF:

(1) Assume on the contrary that BC is not congruent to EF.
(2) Then there is a point G on $\overrightarrow{EF}$ such that BC ≅ EG.
(3) G ≠ F.
(4) Since AB ≅ DE, adding gives AC ≅ DG.
(5) However, AC ≅ DF.
(6) Hence, DF ≅ DG.

(7) Therefore, F = G.
(8) Our assumption has led to a contradiction, hence, $BC \cong EF$. ■

21. Prove Proposition 3.13a. (Hint: in case AB and CD are not congruent, there is a unique point $F \neq D$ on $\overrightarrow{CD}$ such that $AB \cong CF$ (reason?). In case $C * F * D$, show that $AB < CD$. In case $C * D * F$, use Proposition 3.12 and some axioms to show that $CD < AB$.)

22. Use Proposition 3.12 to prove Proposition 3.13b and c.

23. Use the previous exercise and Proposition 3.3 to prove Proposition 3.13d.

24. Justify each step in the following proof of Proposition 3.14:

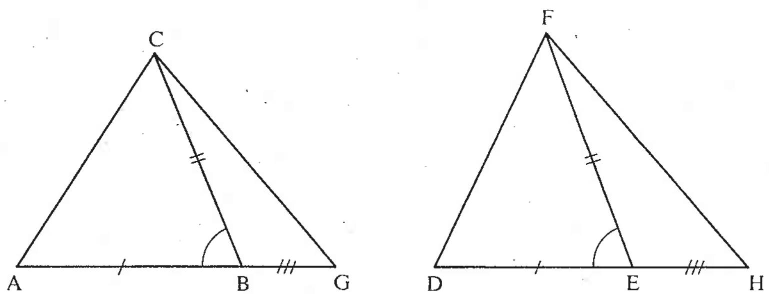


Figure 3.37

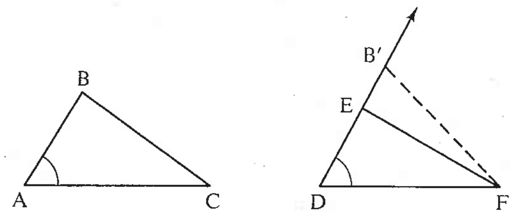
PROOF:
Given $\sphericalangle ABC \cong \sphericalangle DEF$. To prove $\sphericalangle CBG \cong \sphericalangle FEH$.
(1) The points A, C, and G being given arbitrarily on the sides of $\sphericalangle ABC$ and the supplement $\sphericalangle CBG$ of $\sphericalangle ABC$, we can choose the points D, F, and H on the sides of the other angle and its supplement so that $AB \cong DE$, $CB \cong FE$, and $BG \cong EH$.
(2) Then, $\triangle ABC \cong \triangle DEF$.
(3) Hence, $AC \cong DF$ and $\sphericalangle A \cong \sphericalangle D$.
(4) Also, $AG \cong DH$.
(5) Hence, $\triangle ACG \cong \triangle DFH$.
(6) Therefore, $CG \cong FH$ and $\sphericalangle G \cong \sphericalangle H$.
(7) Hence, $\triangle CBG \cong \triangle FEH$.
(8) It follows that $\sphericalangle CBG \cong \sphericalangle FEH$, as desired. ■

25. Deduce Proposition 3.15 from Proposition 3.14.

26. Justify each step in the following proof of Proposition 3.17 (see Figure 3.38):

PROOF:
Given $\triangle ABC$ and $\triangle DEF$ with $\sphericalangle A \cong \sphericalangle D$, $\sphericalangle C \cong \sphericalangle F$, and $AC \cong DF$. To prove $\triangle ABC \cong \triangle DEF$.

(1) There is a unique point B′ on ray $\overrightarrow{DE}$ such that $DB' \cong AB$.
(2) $\triangle ABC \cong \triangle DB'F$.
(3) Hence, $\sphericalangle DFB' \cong \sphericalangle C$.
(4) This implies $\overrightarrow{FE} = \overrightarrow{FB'}$.
(5) In that case, $B' = E$.
(6) Hence, $\triangle ABC \cong \triangle DEF$. ■



Figure

27. Prove Proposition 3.18.

28. Prove that an equiangular triangle (all angles congruent to one another) is equilateral.

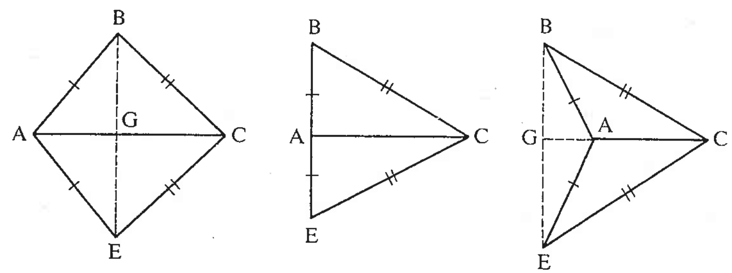
29. Prove Proposition 3.20. (Hint: use Axiom C-4 and Proposition 3.19.)

30. Given $\sphericalangle ABC \cong \sphericalangle DEF$ and $\overrightarrow{BG}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Prove that there is a unique ray $\overrightarrow{EH}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$ such that $\sphericalangle ABG \cong \sphericalangle DEH$. (Hint: show that D and F can be chosen so that $AB \cong DE$ and $BC \cong EF$, and that G can be chosen so that $A * G * C$. Use Propositions 3.7 and 3.12 and SAS to get H. See Figure 3.25, p. 77.)

31. Prove Proposition 3.21 (imitate Exercises 21–23).

32. Prove Proposition 3.22. (Hint: use three congruence axioms to reduce to the case where A = D, C = F, and the points B and E are on opposite sides of $\overleftrightarrow{AC}$. Then consider the three cases in Figure 3.39 separately.)

33. If $AB < CD$, prove that $2AB < 2CD$.



Figure

34. Let Q^2 be the *rational plane* of all ordered pairs (x, y) of rational numbers with the usual interpretations of the undefined geometric terms used in analytic geometry. Show that Axiom C-1 fails in Q^2. (Hint: the segment from $(0, 0)$ to $(1, 1)$ cannot be laid off on the x axis from the origin.)

35. Let R^2 be the *real plane* of all ordered pairs (x, y) of real numbers, with the usual interpretations of the undefined geometric terms used in analytic geometry. In this plane there is a notion of the length of a segment once a standard of measurement is chosen. Let us decide to measure all lengths in inches except for segments on the x axis, which we will measure in feet, and let us *redefine congruence of segments* to mean the two segments have the same "length" in this perverse way of measuring. Incidence, betweenness, and congruence of angles will have the same meaning they usually have. Show that the first five congruence axioms still hold in this interpretation but that SAS fails (see Figure 3.40). Show that angle addition (Proposition 3.19) still holds in this interpretation.

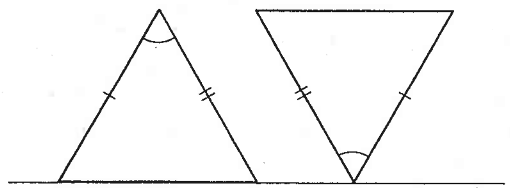


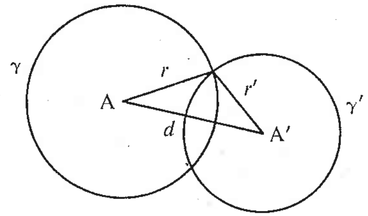
Figure 3.40

Major Exercises

1. Prove that Dedekind's axiom implies Archimedes' axiom. (Hint: divide the set of points on $\overleftrightarrow{AB}$ into two subsets, Σ_1 the set of all points that can be "reached" by laying off copies of segment CD starting at A, and Σ_2 the set of points that are inaccessible. Show that (Σ_1, Σ_2) is a Dedekind cut if Σ_2 is nonempty, and let O be the point of $\overleftrightarrow{AB}$ furnished by Dedekind's axiom. By an RAA argument, show that O can be reached; also by an RAA argument, show that a point to the right of O can be reached, so that Σ_2 must be empty.)

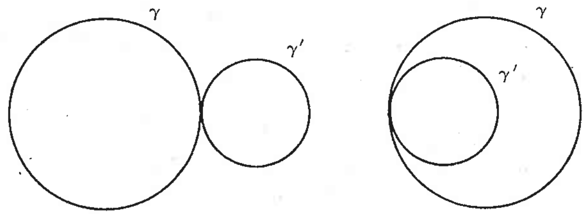
2. In order to finish the argument proving the elementary continuity principle (p. 83), i.e., to show that the point M obtained from Dedekind's axiom actually lies on the circle, we must assume that a length can be assigned to every segment and that for these lengths the *triangle inequality* holds, i.e., the sum of the lengths of two sides of a triangle exceeds the length of the third side (see Chapter 4). Using this, show that if OM < OR, there is some $M' \varepsilon \Sigma_2$ such that OM' < OR; similarly, show that if OM > OR, there is some $M' \varepsilon \Sigma_1$ such that OM' > OR, so that we must have OM ≅ OR.

3. Let γ be a circle with center A and radius of length r. Let γ' be another circle with center A', radius of length r', and let d be the distance from A to A'. There is a hypothesis about the numbers r, r', and d that insures that the circles γ and γ' intersect in two distinct points. Figure out (or guess) what this hypothesis is. (Hint: it's a statement that certain numbers obtained from r, r', and d are less than certain others.)



Figure

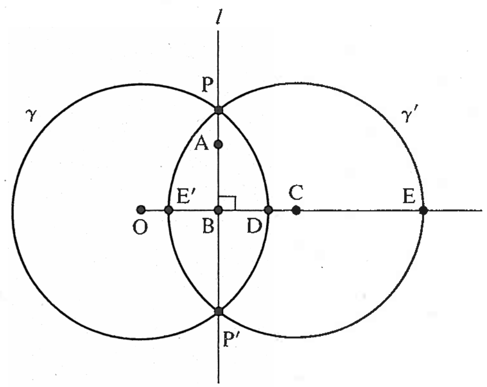
What hypothesis on r, r', and d insures that γ and γ' intersect in only one point, i.e., that the circles are tangent to each other? (See Figure 3.42.)



Figure

4. Report on T. L. Heath's proof for the circular continuity principle.

5. Assuming the circular continuity principle (instead of the stronger Dedekind's axiom), you can show that a line passing through the interior of a circle intersects the circle in two points; namely, in Figure 3.43, line l is assumed to pass through point A interior to circle γ with center O. Point B is taken to be the foot of the perpendicular from O to l, point C is the reflection of O across l, point D lies on γ and $\overrightarrow{OB}$. Then points E and E' are chosen collinear with O and B so that CE ≅ OD ≅ CE'. Prove that all these points satisfy the betweenness relations shown in the figure, that the circle γ' with center C and radius CE intersects γ in two points P, P', and that these points lie on the original line l. (In Exercise 17, Chapter 5, you will show that the circular continuity principle implies the elementary continuity principle).



igure 3.43

6. Incidence, points, and lines in the real plane R^2 were given in Major Exercise 9, Chapter 2. Distance is given by the usual Pythagorean formula

$$d(AB) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

where $A = (a_1, a_2)$, $B = (b_1, b_2)$. Define $A * B * C$ to mean $d(AC) = d(AB) + d(BC)$, and define $AB \cong CD$ to mean $d(AB) = d(CD)$. Define $\sphericalangle ABC \cong \sphericalangle DEF$ if A, C, D, and F can be chosen on the sides of these angles so that $AB \cong ED$, $CB \cong FE$, and $AC \cong DF$. With these interpretations, verify all the axioms for Euclidean geometry (see Moise, Chapter 26, or Borsuk and Szmielew, Chapter 4).

7. Suppose in Major Exercise 6 the field R of real numbers is replaced by an arbitrary *Euclidean field* F (an ordered field in which every positive number has a square root). Show that all the axioms for Euclidean geometry except Dedekind's and Archimedes' axioms are satisfied; show also that the circular continuity principle is satisfied.

Neutral Geometry | 4

If only it could be proved . . . that "there is a Triangle whose angles are together not *less* than two right angles"! But alas, *that* is an *ignis fatuus* that has never yet been caught!

C. L. Dodgson (Lewis Carroll)

Geometry Without the Parallel Axiom

In the exercises of the previous chapter you gained experience in proving some elementary results from Hilbert's axioms. Many of these results were taken for granted by Euclid. You can see that filling in the gaps and rigorously proving every detail is a long task. In any case, we must show that Euclid's postulates are consequences of Hilbert's. We have seen that Euclid's first postulate is the same as Hilbert's Axiom I-1. In our new language, Euclid's second postulate says the following: given segments AB and CD, there exists a point E such that $A * B * E$ and $CD \cong BE$. This follows immediately from Hilbert's Axiom C-1 applied to the ray emanating from B opposite to $\overrightarrow{BA}$.

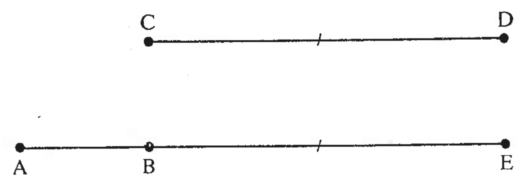


Figure 4.1

The third postulate of Euclid becomes a definition in Hilbert's system. The *circle* with *center* O and *radius* OA is defined as the set of all points P such that OP is congruent to OA. Axiom C-1 then guarantees that on every ray emanating from O there exists such a point P.

The fourth postulate of Euclid—all right angles are congruent—becomes a theorem in Hilbert's system, as was shown in Proposition 3.23.

Euclid's parallel postulate is discussed later in this chapter (p. 107 ff). In this chapter we shall be interested in neutral geometry—by definition, all those geometric theorems that can be proved using only the axioms of incidence, betweenness, congruence, and continuity and without using the axiom of parallelism. Every result proved previously is a theorem in neutral geometry. You should review all the statements in the theorems, propositions, and exercises of Chapter 3 because they will be used throughout the book. Our proofs will be less formal henceforth.

What is the purpose of studying neutral geometry? We are not interested in studying it for its own sake. Rather, we are trying to clarify the role of the parallel postulate by seeing which theorems in the geometry do not depend on it, i.e., which theorems follow from the other axioms alone without ever using the parallel postulate in proofs. This will enable us to avoid many pitfalls and to see much more clearly the logical structure of our system. Certain questions that can be answered in Euclidean geometry, e.g., whether there is a unique parallel through a given point, may not be answerable in neutral geometry because its axioms do not give us enough information.

A large number of theorems can be proved in neutral geometry (287 in Borsuk and Szmielew). We have given many familiar ones in the text and exercises of the previous chapters. Be prepared for a few strange-looking theorems in this chapter.

Alternate Interior Angle Theorem

The next theorem requires a definition: let t be transversal to lines l and l', with t meeting l at B and l' at B′. Choose points A and C on l such that $A * B * C$; choose points A′ and C′ on l' such that A and A′ are on the same side of t and such that $A' * B' * C'$. Then the following four angles are called *interior*: $\sphericalangle A'B'B$, $\sphericalangle ABB'$, $\sphericalangle C'B'B$, $\sphericalangle CBB'$. The two pairs ($\sphericalangle ABB'$, $\sphericalangle C'B'B$) and ($\sphericalangle A'B'B$, $\sphericalangle CBB'$) are called pairs of *alternate interior angles* (see Figure 4.2).

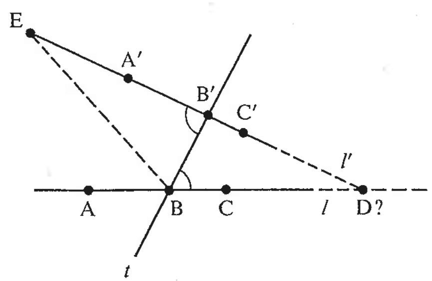


Figure 4.2

THEOREM 4.1 (Alternate Interior Angle Theorem). If two lines cut by a transversal have a pair of congruent alternate interior angles, then the two lines are parallel.

PROOF: Given $\sphericalangle A'B'B \cong \sphericalangle CBB'$. Assume on the contrary l and l' meet at a point D. Say D is on the same side of t as C and C′. There is a point E on $\overrightarrow{B'A'}$ such that $B'E \cong BD$ (Axiom C-1). Segment BB′ is congruent to itself, so that $\triangle B'BD \cong \triangle BB'E$ (SAS). In particular, $\sphericalangle DB'B \cong \sphericalangle EBB'$. Since $\sphericalangle DB'B$ is the supplement of $\sphericalangle EB'B$, $\sphericalangle EBB'$ must be the supplement of $\sphericalangle DBB'$ (Proposition 3.14 and Axiom C-4). This means that E lies on l, hence l and l' have the two points E and D in common, which contradicts Proposition 2.1 of incidence geometry. Therefore, $l \parallel l'$. ■

This theorem has two very important corollaries.

COROLLARY 1. Two lines perpendicular to the same line are parallel. Hence, the perpendicular dropped from a point P not on line l to l is *unique* (and the point at which the perpendicular intersects l is called its *foot*).

PROOF: If l and l' are both perpendicular to t, the alternate interior angles are right angles and hence are congruent (Proposition 3.23). ■

COROLLARY 2. If l is any line and P is any point not on l, there exists at least one line m through P parallel to l.

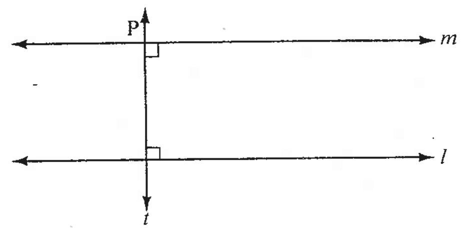


Figure 4.3

PROOF: There is a line t through P perpendicular to l, and again there is a unique line m through P perpendicular to t (Proposition 3.16). Since l and m are both perpendicular to t, Corollary 1 tells us that $l \parallel m$. (This construction will be used repeatedly.) ■

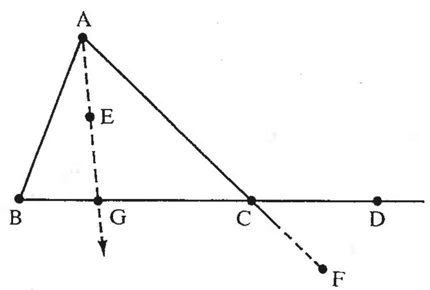
To repeat, there always exists a line m through P parallel to l—this has been proved in neutral geometry. But we don't know that m is *unique.* Although Hilbert's parallel postulate says that m is indeed unique, we are not assuming that postulate. We must keep our minds open to the strange possibility that there may be other lines through P parallel to l.

Warning! You are accustomed in Euclidean geometry to use the *converse* of Theorem 4.1, which states, "if two lines are parallel, then alternate interior angles cut by a transversal are congruent." We haven't proved this converse, so don't use it! (It turns out to be logically equivalent to the parallel postulate—see pp. 106–108.)

Exterior Angle Theorem

Before we continue our list of theorems, we must first make another definition: an angle supplementary to an angle of a triangle is called an *exterior angle* of the triangle; the two angles of the triangle not adjacent to this exterior angle are called the *remote interior angles.* The following theorem is a consequence of Theorem 4.1:

THEOREM 4.2 (Exterior Angle Theorem). An exterior angle of a triangle is greater than either remote interior angle [see Figure 4.4].



gure 4.4

To prove ∢ACD is greater than ∢B and ∢A:

PROOF: Consider the remote interior angle ∢BAC. If ∢BAC ≅ ∢ACD, then $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$ (Theorem 4.1), which contradicts the hypothesis that these lines meet at B. Suppose ∢BAC were greater than ∢ACD (RAA hypothesis). Then there is a ray $\overrightarrow{AE}$ between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ such that ∢ACD ≅ ∢CAE (by definition). This ray $\overrightarrow{AE}$ intersects BC in a point G (crossbar theorem, p. 69). But according to Theorem 4.1, lines $\overleftrightarrow{AE}$ and $\overleftrightarrow{CD}$ are parallel. Thus, ∢BAC cannot be greater than ∢ACD (RAA conclusion). Since ∢BAC is also not congruent to ∢ACD, ∢BAC must be less than ∢ACD (Proposition 3.21a).

For remote angle ∢ABC, use the same argument applied to exterior angle ∢BCF, which is congruent to ∢ACD by the vertical angle theorem (Proposition 3.15a).■

The exterior angle theorem will play a very important role in what follows. It was the sixteenth proposition in Euclid's *Elements.* Euclid's proof had a gap due to reasoning from a diagram. He considered the line $\overleftrightarrow{BM}$ joining B to the midpoint of AC and he constructed point B such that B ∗ M ∗ B′ and BM ≅ MB′ (Axiom C-1). He then assumed from the diagram that B′ lay in the interior of ∢ACD (see Figure 4.5).

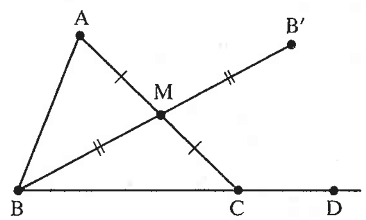


Figure 4.5

We won't give the rest of Euclid's argument (see Heath, p. 279), for there is the added difficulty that the existence of midpoints must first be justified (see Exercise 12). However, this gap in Euclid's argument can easily be filled with the tools we have developed. Namely, since segment BB′ interesects AC at M, B and B′ are on opposite sides of $\overleftrightarrow{AC}$ (by definition). Since BD meets $\overleftrightarrow{AC}$ at C, B and D are also on opposite sides of $\overleftrightarrow{AC}$. Hence, B′ and D are on the same side of $\overleftrightarrow{AC}$ (Axiom B-4). Next, B′ and M are on the same side of $\overleftrightarrow{CD}$, since segment MB′ does not contain the point B at which $\overleftrightarrow{MB'}$ meets $\overleftrightarrow{CD}$ (by construction of B′ and Axioms B-1 and B-3). Also, A and M are on the same side of $\overleftrightarrow{CD}$ because segment AM does not contain the point C at which $\overleftrightarrow{AM}$ meets $\overleftrightarrow{CD}$ (by definition of midpoint and Axiom B-3). So again, Separation Axiom B-4 insures that A and B′ are on the same side of $\overleftrightarrow{CD}$. By definition of "interior" (p. 68), we have shown that B′ lies in the interior of ∢ACD.

Note. In elliptic geometry the exterior angle theorem is false (see

Figure 3.24 where a triangle is shown with both an exterior angle and a remote interior angle that are right angles).

As a consequence of the exterior angle theorem (and our previous results), you can now prove as exercises the following familiar propositions (it is not necessary to use any continuity properties in your proofs).

PROPOSITION 4.1 (SAA Congruence Criterion). Given $AC \cong DF$, $\sphericalangle A \cong \sphericalangle D$; and $\sphericalangle B \cong \sphericalangle E$. Then $\triangle ABC \cong \triangle DEF$.

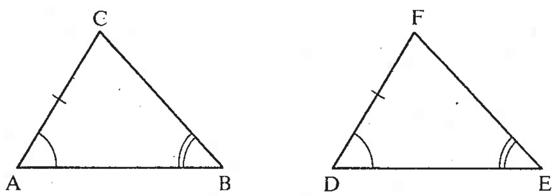


Figure 4.6

PROPOSITION 4.2. Two right triangles are congruent if the hypotenuse and a leg of one are congruent respectively to the hypotenuse and a leg of the other.

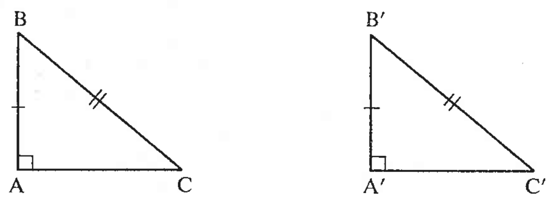


Figure 4.7

PROPOSITION 4.3 (Midpoints). Every segment has a unique midpoint.

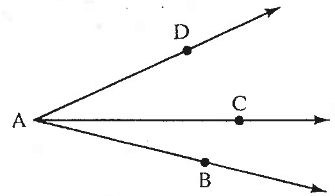
PROPOSITION 4.4 (Bisectors). (a) Every angle has a unique bisector. (b) Every segment has a unique perpendicular bisector.

PROPOSITION 4.5. In a triangle $\triangle ABC$, the greater angle lies opposite the greater side and the greater side lies opposite the greater angle, i.e., $AB > BC$ if and only if $\sphericalangle C > \sphericalangle A$.

PROPOSITION 4.6. Given $\triangle ABC$ and $\triangle A'B'C'$, if $AB \cong A'B'$ and $BC \cong B'C'$, then $\sphericalangle B < \sphericalangle B'$ if and only if $AC < A'C'$.

Measure of Angles and Segments

Thus far in our treatment of geometry we have refrained from using numbers that measure the sizes of angles and segments—this was in keeping with the spirit of Euclid. From now on, however, we will not be so austere. The next theorem (Theorem 4.3) asserts the possibility of measurement and lists its properties. The proof requires the axioms of continuity for the first time (in keeping with the elementary level of this book, the interested reader is referred to Borsuk and Szmielew, Chapter 3, §9 and 10).* In some popular treatments of geometry this theorem is taken as an axiom (ruler-and-protractor postulates—see Moise). The familiar notation $(\sphericalangle A)^\circ$ will be used for the number of degrees in $\sphericalangle A$, and the length of segment AB (with respect to some unit of measurement) will be denoted by $\overline{AB}$.



Figure

THEOREM 4.3.
A. There is a unique way of assigning a degree measure to each angle such that the following properties hold:

(0) $(\sphericalangle A)^\circ$ is a real number such that $0 \leqq (\sphericalangle A)^\circ < 180^\circ$.
(1) $(\sphericalangle A)^\circ = 90^\circ$ if and only if $\sphericalangle A$ is a right angle.
(2) $(\sphericalangle A)^\circ = (\sphericalangle B)^\circ$ if and only if $\sphericalangle A \cong \sphericalangle B$.
(3) If $\overrightarrow{AC}$ is interior to $\sphericalangle DAB$, then $(\sphericalangle DAB)^\circ = (\sphericalangle DAC)^\circ + (\sphericalangle CAB)^\circ$.

* The axioms of continuity are not needed if one merely wants to define the addition for congruence classes of segments and then prove the triangle inequality (Corollary 2 to Theorem 4.3; see Borsuk and Szmielew, pp. 103–108, for a definition of this operation). It is in order to prove Theorems 4.4 and 4.7 and Lemma 6.1 that we need the measurement of angles and segments by real numbers, and for such measurement Archimedes' axiom is required. However, parts 4 and 11 of Theorem 4.3, the proofs for which require Dedekind's axiom, are never used in proofs in this book. See Hessenberg and Diller for coordinatization without continuity axioms.

(4) For every real number x between 0 and 180, there exists an angle $\sphericalangle A$ such that $(\sphericalangle A)^\circ = x^\circ$.
(5) If $\sphericalangle B$ is supplementary to $\sphericalangle A$, then $(\sphericalangle A)^\circ + (\sphericalangle B)^\circ = 180^\circ$.
(6) $(\sphericalangle A)^\circ > (\sphericalangle B)^\circ$ if and only if $\sphericalangle A > \sphericalangle B$.

B. Given a segment OI, called *unit segment*. Then there is a unique way of assigning a length $\overline{AB}$ to each segment AB such that the following properties hold:

(7) $\overline{AB}$ is a positive real number and $\overline{OI} = 1$.
(8) $\overline{AB} = \overline{CD}$ if and only if $AB \cong CD$.
(9) $A * B * C$ if and only if $\overline{AC} = \overline{AB} + \overline{BC}$.
(10) $\overline{AB} < \overline{CD}$ if and only if $AB < CD$.
(11) For every positive real number x, there exists a segment AB such that $\overline{AB} = x$.

Using degree notation, $\sphericalangle A$ is defined as *acute* if $(\sphericalangle A)^\circ < 90^\circ$, and *obtuse* if $(\sphericalangle A)^\circ > 90^\circ$. Combining Theorems 4.2 and 4.3 gives the following corollary (see Exercise 2), which is essential for proving the Saccheri-Legendre theorem.

COROLLARY 1. The sum of the degree measures of any *two* angles of a triangle is less than 180°.

The only immediate application of segment measurement that we will make is in the proof of the next corollary, the famous "triangle inequality."

COROLLARY 2 (Triangle Inequality). If A, B, and C are three noncollinear points, then $\overline{AC} < \overline{AB} + \overline{BC}$.

PROOF:

(1) There is a unique point D such that $A * B * D$ and $BD \cong BC$ (Axiom C-1 applied to the ray opposite to $\overrightarrow{BA}$).

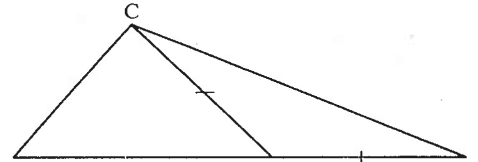


Figure 4.9

(2) Then $\sphericalangle BCD \cong \sphericalangle BDC$ (Proposition 3.10: base angles of an isosceles triangle).
(3) $\overline{AD} = \overline{AB} + \overline{BD}$ (Theorem 4.3(9)) and $\overline{BD} = \overline{BC}$ (Step 1 and Theorem 4.3(8)); substituting gives $\overline{AD} = \overline{AB} + \overline{BC}$.
(4) $\overrightarrow{CB}$ is between $\overrightarrow{CA}$ and $\overrightarrow{CD}$ (Proposition 3.7), hence, $\sphericalangle ACD > \sphericalangle BCD$ (by definition).
(5) $\sphericalangle ACD > \sphericalangle ADC$ (Steps 2 and 4; Proposition 3.21c).
(6) $AD > AC$ (Proposition 4.5).
(7) Hence, $\overline{AB} + \overline{BC} > \overline{AC}$ (Theorem 4.3(10); Steps 3 and 6). ■

Saccheri-Legendre Theorem

The following very important theorem also requires an axiom of continuity (Archimedes' axiom) for its proof.

THEOREM 4.4 (Saccheri-Legendre). The sum of the degree measures of the three angles in any triangle is *less than or equal to* 180°.

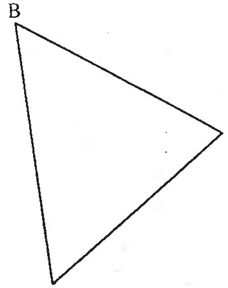


Figure 4.
$(\sphericalangle A)^\circ + (\sphericalangle B)^\circ + (\sphericalangle C)^\circ \leqq 18$

This result may strike you as peculiar, since you are accustomed to the notion of an *exact* sum of 180°. Nevertheless, this exactness cannot be proved in neutral geometry! Saccheri tried, but the best he could conclude was "less than or equal." (We will discuss this further in this chapter, pp. 108–112.) Max Dehn showed in 1900 that there is no way to prove this theorem without Archimedes' axiom.* The idea of the proof is as follows:

Assume, on the contrary, that the angle sum of $\triangle ABC$ is greater than $180°$, say $180° + p°$, where p is a positive number. It is possible (by a trick you will find in Exercise 15) to replace $\triangle ABC$ with another triangle that has the same angle sum as $\triangle ABC$ but in which one angle has at most half the number of degrees as $(\measuredangle A)°$. We can repeat this trick to get another triangle that has the same angle sum $180° + p°$ but in which one angle has at most one-fourth the number of degrees as $(\measuredangle A)°$. The Archimedean property of real numbers guarantees that if we repeat this construction enough times, we will eventually obtain a triangle that has angle sum $180° + p°$ but in which one angle has degree measure at most $p°$. The sum of the degree measures of the other *two* angles will be greater than or equal to $180°$, contradicting Corollary 1 to Theorem 4.3. This proves the theorem.

You should prove the following consequence of the Saccheri-Legendre theorem as an exercise.

COROLLARY 1. The sum of the degree measures of two angles in a triangle is less than or equal to the degree measure of their remote exterior angle [see Figure 4.11].

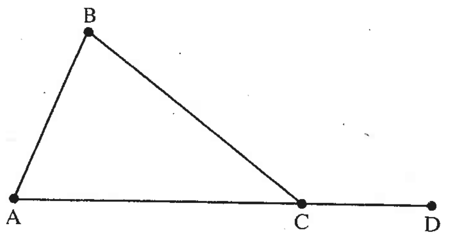


Figure 4.11
$(\measuredangle A)° + (\measuredangle B)° \leqq (\measuredangle BCD)°$.

It is natural to generalize the Saccheri-Legendre theorem to polygons other than triangles. For example, let us prove that the angle sum of a quadrilateral □ABCD is at most $360°$. Break □ABCD into two triangles $\triangle ABC$ and $\triangle ADC$ by the diagonal AC (see Figure 4.12). By the Saccheri-Legendre theorem,

$$(\measuredangle B)° + (\measuredangle BAC)° + (\measuredangle ACB)° \leqq 180° \quad \text{and}$$
$$(\measuredangle D)° + (\measuredangle DAC)° + (\measuredangle ACD)° \leqq 180°.$$

Theorem 4.3(3) gives us the equations

$$(\measuredangle BAC)° + (\measuredangle DAC)° = (\measuredangle BAD)° \quad \text{and}$$
$$(\measuredangle ACB)° + (\measuredangle ACD)° = (\measuredangle BCD)°$$

Using these equations, we add the two inequalities to obtain the desired inequality

$$(\measuredangle B)° + (\measuredangle D)° + (\measuredangle BAD)° + (\measuredangle BCD)° \leqq 360°.$$

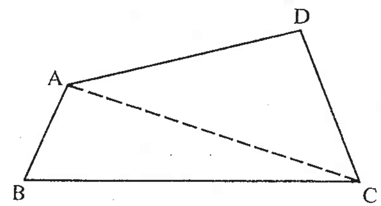


Figure 4.12

Unfortunately, there is a gap in this simple argument! To get the equations used above, we assumed by looking at the diagram (Figure 4.12) that C was interior to $\measuredangle BAD$ and that A was interior to $\measuredangle BCD$. But what if the quadrilateral looked like Figure 4.13?

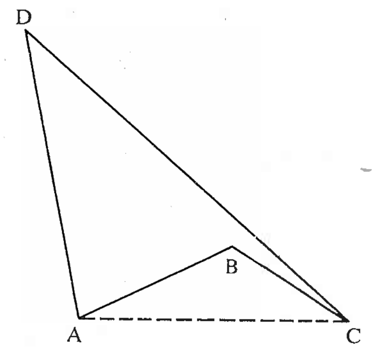


Figure 4.13

* See the reference in Major Exercise 1. The full significance of Archimedes' axiom was first grasped in the 1880s by M. Pasch and O. Stolz. G. Veronese and T. Levi-Civita developed the first non-Archimedean geometry.

In this case the equations would not hold. To prevent such a case, we must add an hypothesis; we must assume that the quadrilateral is "convex."

DEFINITION. Quadrilateral □ABCD is called *convex* if it has a pair of opposite sides, e.g., AB and CD, such that CD is contained in one of the half-planes bounded by $\overleftrightarrow{AB}$ and AB is contained in one of the half-planes bounded by $\overleftrightarrow{CD}$.*

The assumption made above is now justified by starting with a convex quadrilateral. Thus, we have proved the following corollary:

COROLLARY 2. The sum of the degree measures of the angles in any *convex* quadrilateral is at most 360°.

Note. The Saccheri-Legendre theorem is false in elliptic geometry (see Figure 3.24, p. 76). In fact, it can be proved in elliptic geometry that the angle sum of a triangle is always greater than 180°.

Equivalence of Parallel Postulates

We shall now prove the equivalence of Euclid's fifth postulate and Hilbert's parallel postulate. Note, however, that we are not proving either or both of the postulates; we are only proving that we *can* prove one *if* we first assume the other. We shall first state Euclid V (all the terms in the statement have now been defined carefully).

* It can be proved that this condition also holds for the other pair of opposite sides, AD and BC—see Exercise 23 in this chapter. The use of the word "convex" in this definition does not agree with its use in Exercise 19, Chapter 3; a convex quadrilateral is obviously not a "convex set" as defined in that exercise. However, we can define the *interior* of a convex quadrilateral □ABCD as follows: each side of □ABCD determines a half-plane containing the opposite side; the interior of □ABCD is then the intersection of the four half-planes so determined. You can then prove that the interior of a convex quadrilateral is a convex set (which is one of the problems in Exercise 25).

EUCLID'S POSTULATE V. If two lines are intersected by a transversal in such a way that the sum of the degree measures of the two interior angles on one side of the transversal is less than 180°, then the two lines meet on that side of the transversal.

THEOREM 4.5. Euclid's fifth postulate ⇔ Hilbert's parallel postulate.

PROOF: First, assume Hilbert's postulate. The situation of Euclid V is shown in Figure 4.14. $(\sphericalangle 1)° + (\sphericalangle 2)° < 180°$ (hypothesis) and $(\sphericalangle 1)° + (\sphericalangle 3)° = 180°$ (supplementary angles, Theorem 4.3(5)). Hence, $(\sphericalangle 2)° < 180° - (\sphericalangle 1)° = (\sphericalangle 3)°$. There is a unique ray $\overrightarrow{B'C'}$ such that $\sphericalangle 3$ and $\sphericalangle C'B'B$ are congruent alternate interior angles (Axiom C-4). By Theorem 4.1, $\overleftrightarrow{B'C'}$ is parallel to *l*. Since $m \neq \overleftrightarrow{B'C'}$, *m* meets *l* (Hilbert's postulate). To conclude, we must prove that *m* meets *l* on the same side of *t* as C'. Assume, on the contrary, that they meet at a point A on the opposite side. Then $\sphericalangle 2$ is an exterior angle of △ABB'. Yet it is smaller than the remote interior $\sphericalangle 3$. This contradiction of Theorem 4.2 proves Euclid V (RAA).

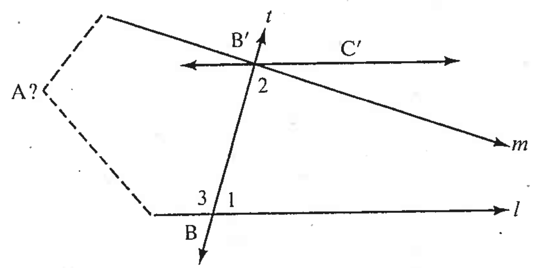


Figure 4.14

Conversely, assume Euclid V and refer to Figure 4.15, the situation of Hilbert's postulate. Let *t* be the perpendicular to *l* through P, and *m* the perpendicular to *t* through P. We know that $m \parallel l$ (Corollary 1 to Theorem 4.1). Let *n* be any other line through P. We must show that *n* meets *l*. Let $\sphericalangle 1$ be the acute angle *n* makes with *t* (which angle exists because $n \neq m$). Then $(\sphericalangle 1)° + (\sphericalangle PQR)° < 90° + 90° = 180°$. Thus, the hypothesis of Euclid V is satisfied. Hence, *n* meets *l*, proving Hilbert's postulate.■

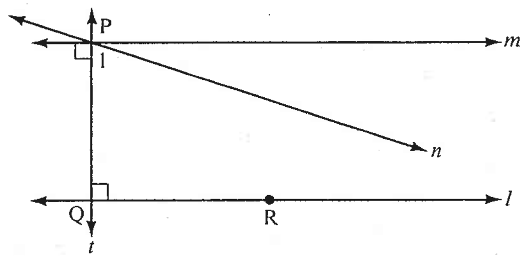


Figure 4.15

Since Hilbert's parallel postulate and Euclid V are logically equivalent in the context of neutral geometry, Theorem 4.5 allows us to use them interchangeably. You will prove as exercises that the following statements are also logically equivalent to the parallel postulate.

PROPOSITION 4.7. Hilbert's parallel postulate ⇔ if a line intersects one of two parallel lines, then it also intersects the other.

PROPOSITION 4.8. Hilbert's parallel postulate ⇔ converse to Theorem 4.1 (alternate interior angles).

PROPOSITION 4.9. Hilbert's parallel postulate ⇔ if t is transversal to l and m, $l \parallel m$, and $t \perp l$, then $t \perp m$.

PROPOSITION 4.10. Hilbert's parallel postulate ⇔ if $k \parallel l$, $m \perp k$, and $n \perp l$, then either $m = n$ or $m \parallel n$.

The next proposition is another statement logically equivalent to Hilbert's parallel postulate, but at this point we can only prove the implication in one direction (the other implication is proved in Chapter 6).

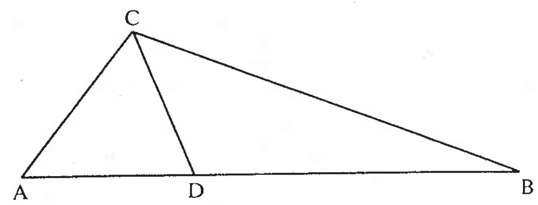
PROPOSITION 4.11. Hilbert's parallel postulate ⇒ the angle sum of every triangle is 180°.

Angle Sum of a Triangle

We define the *angle sum* of triangle △ABC as (∢A)° + (∢B)° + (∢C)°, which is a certain number of degrees ≦180° (by the Saccheri-Legendre theorem). We define the *defect* of any triangle to be 180° minus the angle sum. In Euclidean geometry we are accustomed to having no "defective" triangles, i.e. we are accustomed to having the defect equal zero (Proposition 4.11).

The main purpose of this section is to show that if *one* defective triangle exists, then *all* triangles are defective. Or, put in the contrapositive form, if one triangle has angle sum 180°, then so do all others. We are not asserting that one such triangle does exist, nor are we asserting the contrary; we are only examining the hypothesis that one might exist.

THEOREM 4.6. Let △ABC be any triangle and D a point between A and B. Then defect (△ABC) = defect (△ACD) + defect (△BCD) (*additivity of the defect*).



Figure

PROOF: Since $\overrightarrow{CD}$ is interior to ∢ACB (Proposition 3.7), (∢ACB)° = (∢ACD)° + (∢BCD)° (by Theorem 4.3(3)). Since ∢ADC and ∢BDC are supplementary angles, 180° = (∢ADC)° + (∢BDC)° (by Theorem 4.3(5)). To obtain the additivity of the defect, all we have to do is write down the definition of the defect (180° minus the angle sum) for each of the three triangles under consideration and substitute the two equations above (Exercise 1). ■

COROLLARY. Under the same hypothesis, the angle sum of △ABC is equal to 180° if and only if the angle sums each of △ACD and △BCD are equal to 180°.

PROOF: If △ACD and △BCD both have defect zero, then defect of △ABC = 0 + 0 = 0 (Theorem 4.6). Conversely, if △ABC has defect zero, then by Theorem 4.6 defect (△ACD) + defect (△BCD) = 0. But the defect of a triangle can never be negative (Saccheri-Legendre theorem). Hence, △ACD and △BCD each have defect zero (the sum of two non-negative numbers equals zero only when each equals zero). ■

Next, recall that by definition a *rectangle* is a quadrilateral whose four angles are right angles. Hence, the angle sum of a rectangle is 360°. Of course, we don't yet know whether rectangles exist in neutral geometry. (Try to construct one without using the parallel postulate or any statement logically equivalent to it—see Exercise 19.)

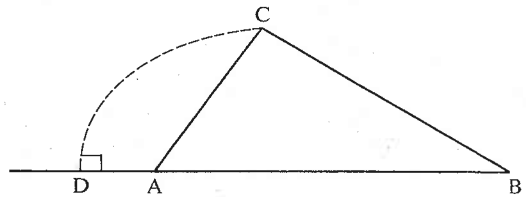
The next theorem is the result we seek. Its proof will be given in five steps.

THEOREM 4.7. If a triangle exists whose angle sum is 180°, then a rectangle exists. If a rectangle exists, then every triangle has angle sum equal to 180°.

PROOF:

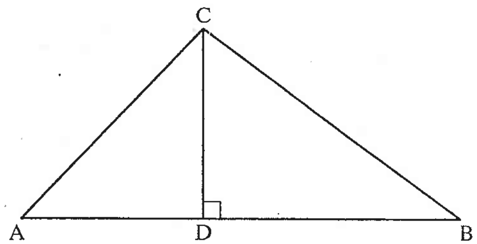
(1) Construct a *right* triangle having angle sum 180°.

Let $\triangle ABC$ be the given triangle with defect zero (hypothesis). Assume it is not a right triangle, otherwise we are done. At least two of the angles in this triangle are acute, since the angle sum of two angles in a triangle must be less than 180° (Corollary to Theorem 4.3), e.g., $\angle A$ and $\angle B$ are acute. Let CD be the altitude from vertex C (which exists by Proposition 3.16). We claim that D lies between A and B. Assume the contrary, e.g., $D * A * B$.



re 4.17

Then remote interior angle $\angle CDA$ is greater than exterior angle $\angle CAB$, contradicting Theorem 4.2. Similarly, if $A * B * D$, we get a contradiction. Thus, $A * D * B$ (Axiom B-3).



re 4.18

It now follows from the Corollary to Theorem 4.6 that each of the right triangles $\triangle ADC$ and $\triangle BDC$ have defect zero.

(2) From a right triangle of defect zero construct a rectangle.

Let $\triangle CDB$ be a right triangle of defect zero with $\angle D$ a right angle. By Axiom C-4, there is a unique ray $\overrightarrow{CX}$ on the opposite side of $\overleftrightarrow{CB}$ from D such that $\angle DBC \cong \angle BCX$. By Axiom C-1, there is a unique point E on $\overrightarrow{CX}$ such that $CE \cong BD$.

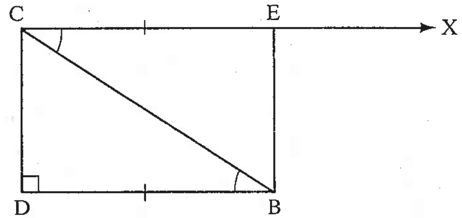


Figure 4.19

Then $\triangle CDB \cong \triangle BEC$ (SAS). Hence, $\triangle BEC$ is also a right triangle of defect zero with right angle at E. Also, since $(\angle DBC)° + (\angle BCD)° = 90°$ by our hypothesis, we obtain by substitution $(\angle ECB)° + (\angle BCD)° = 90°$ and $(\angle DBC)° + (\angle EBC)° = 90°$. Moreover, B is an interior point of $\angle ECD$, since the alternate interior angle theorem implies $\overleftrightarrow{CE} \parallel \overleftrightarrow{DB}$ and $\overleftrightarrow{CD} \parallel \overleftrightarrow{BE}$ and C is interior to $\angle EBD$ (for the same reason). Thus, we can apply Theorem 4.3(3) to conclude that $(\angle ECD)° = 90° = (\angle EBD)°$. This proves that $\square CDBE$ is a rectangle.

(3) From one rectangle, construct "arbitrarily large" rectangles.

More precisely, given any right triangle $\triangle D'E'C'$, construct a rectangle $\square AFBC$ such that $AC > D'C'$ and $BC > E'C'$. This can be done using Archimedes' axiom. We simply "lay off" enough copies of the rectangle we have to achieve the result (see Figures 4.20 and 4.21; you can make this "laying off" precise as an exercise).

(4) Prove that all *right* triangles have defect zero.

This is achieved by "embedding" an arbitrary right triangle $\triangle D'C'E'$ in a rectangle, as in Step 3, then showing successively (by twice applying the Corollary to Theorem 4.6) that $\triangle ACB$, $\triangle DCB$, and $\triangle DCE$ each have defect zero (see Figure 4.22).

(5) If every *right* triangle has defect zero, then *every* triangle has defect zero.

As in Step 1, drop an altitude to decompose an arbitrary triangle into two right triangles (Figure 4.18) and apply the corollary to Theorem 4.6. ■

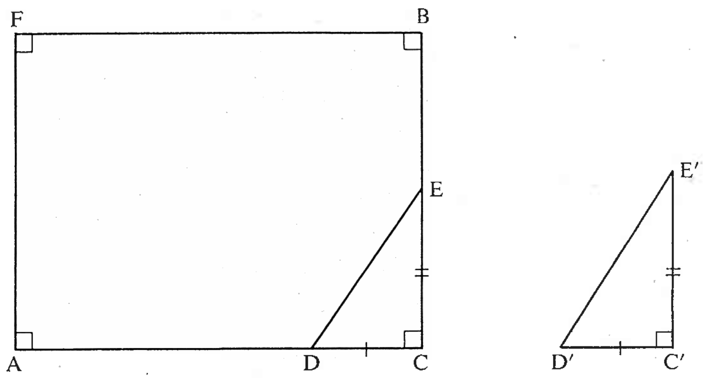


Figure 4.20

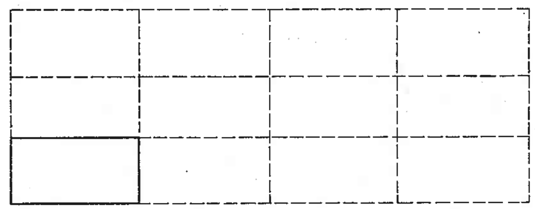


Figure 4.21

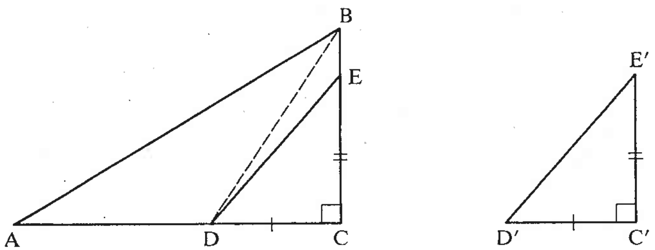


Figure 4.22

Historians credit Theorem 4.7 to Saccheri and Legendre, but we will not name it after them so as to avoid confusion with Theorem 4.4.

COROLLARY. If there exists a triangle with positive defect, then all triangles have positive defect.

Review Exercise

Which of the following statements are correct?

(1) If two triangles have the same defect, they are congruent.

(2) Euclid's fourth postulate is a theorem in neutral geometry.

(3) Theorem 4.5 shows that Euclid's fifth postulate is a theorem in neutral geometry.

(4) The Saccheri-Legendre theorem tells us that some triangles exist that have angle sum less than 180° and some triangles exist that have angle sum equal to 180°.

(5) The alternate interior angle theorem states that if parallel lines are cut by a transversal, then alternate interior angles are congruent to each other.

(6) It is impossible to prove in neutral geometry that quadrilaterals exist.

(7) The Saccheri-Legendre theorem is false in Euclidean geometry because in Euclidean geometry the angle sum of any triangle is never less than 180°.

(8) According to our definition of "angle," the degree measure of an angle cannot equal 180°.

(9) The notion of one ray being "between" two others is undefined.

(10) It is impossible to prove in neutral geometry that parallel lines exist.

(11) The definition of "remote interior angle" given on p. 98 is incomplete because it used the word "adjacent," which has never been defined.

(12) An exterior angle of a triangle is any angle that is not in the interior of the triangle.

(13) The SSS criterion for congruence of triangles is a theorem in neutral geometry.

(14) The alternate interior angle theorem implies, as a special case, that if a transversal is perpendicular to one of two parallel lines, then it is also perpendicular to the other.

(15) Another way of stating the Saccheri-Legendre theorem is to say that the defect of a triangle cannot be negative.

(16) The ASA criterion for congruence of triangles is one of the axioms for neutral geometry.

(17) The proof of Theorem 4.7 depends on Archimedes' axiom.

(18) If △ABC is any triangle and C is any of its vertices, and if a perpendicular is dropped from C to $\overleftrightarrow{AB}$, then that perpendicular will intersect $\overleftrightarrow{AB}$ in a point between A and B.

(19) It is a theorem in neutral geometry that given any point P and any line l, there is at most one line through P perpendicular to l.

(20) It is a theorem in neutral geometry that vertical angles are congruent to each other.

(21) The proof of Theorem 4.2 (on exterior angles) uses Theorem 4.1 (on alternate interior angles).

(22) The gap in Euclid's attempt to prove Theorem 4.2 can be filled using our axioms of betweenness.

Exercises

The following are exercises in neutral geometry, unless otherwise stated. This means that in your proofs you are allowed to use only those results that have been given previously (including results from previous exercises). You are not allowed to use the parallel postulate or other results from Euclidean geometry that depend on it.

1. (a) Finish the last step in the proof of Theorem 4.6. (b) Prove that congruent triangles have the same defect. (c) Prove the corollary to Theorem 4.7 (p. 110).
2. Prove Corollary 1 to Theorem 4.3 (p. 102) by applying Theorem 4.2 and parts 5 and 6 of Theorem 4.3.
3. State the converse to Euclid's fifth postulate. Prove this converse as a theorem in neutral geometry.
4. Prove Propostion 4.7.
5. Prove Proposition 4.8. (Hint: assume the converse to Theorem 4.1. Let m be the parallel to l through P constructed in the proof of Corollary 2 to Theorem 4.1 and let n be any parallel to l through P. Use the congruence of alternate interior angles and the uniqueness of perpendiculars to prove $m = n$. Assuming next the parallel postulate, use Axiom C-4 and an RAA argument to establish the converse to Theorem 4.1.)
6. Prove Proposition 4.9.
7. Prove Proposition 4.10.
8. Prove Proposition 4.11. (Hint: see Figure 4.23.)

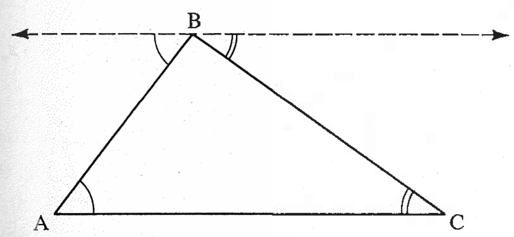


Figure 4.23

9. The following purports to be a proof in neutral geometry of the SAA criterion for congruence. Find the flaw.

   Given AC ≅ DE, ∢A ≅ ∢D, ∢B ≅ ∢E. Then ∢C ≅ ∢F, since $(\sphericalangle C)^\circ = 180^\circ - (\sphericalangle A)^\circ - (\sphericalangle B)^\circ = 180^\circ - (\sphericalangle D)^\circ - (\sphericalangle E)^\circ = (\sphericalangle F)^\circ$ (Theorem 4.3(2)). Hence, △ABC ≅ △DEF by ASA (Proposition 3.17).

10. Here is a correct proof of the SAA criterion. Justify each step. (1) Assume side AB is not congruent to side DE. (2) Then AB < DE or DE < AB. (3) If DE < AB, then there is a point G between A and B such that AG ≅ DE (see Figure 4.24). (4) Then △CAG ≅ △FDE. (5) Hence, ∢AGC ≅ ∢E. (6) It follows that ∢AGC ≅ ∢B. (7) This contradicts a certain theorem (which?). (8) Therefore, DE is not less than AB. (9) By a similar argument involving a point H between D and E, AB is not less than DE. (10) Hence, AB ≅ DE. (11) Therefore, △ABC ≅ △DEF.

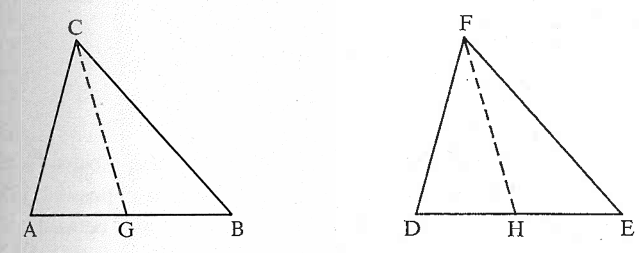


Figure 4.24

11. Prove Proposition 4.2. (Hint: see Figure 4.7, p. 100. On the ray opposite to $\overrightarrow{AC}$, lay off segment AD congruent to A′C′. First prove △DAB ≅ △C′A′B′, then use isosceles triangles and the SAA criterion to conclude.)

12. Here is a proof that segment AB has a midpoint. Justify each step (see Figure 4.25).

    (1) Let C be any point not on $\overleftrightarrow{AB}$. (2) There is a unique ray $\overrightarrow{BX}$ on the opposite side of $\overleftrightarrow{AB}$ from C such that ∢CAB ≅ ∢ABX. (3) There is a unique point D on $\overrightarrow{BX}$ such that AC ≅ BD. (4) D is on the opposite side of $\overleftrightarrow{AB}$ from C. (5) Let E be the point at which segment CD intersects $\overleftrightarrow{AB}$. (6) Assume E is

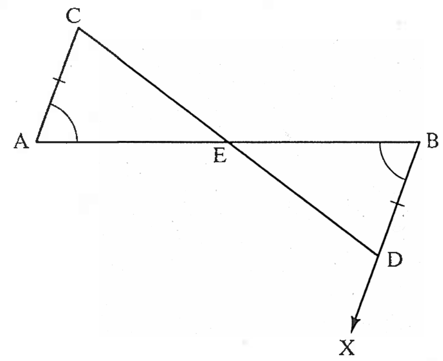


Figure 4.25

not between A and B. (7) Then either E = A, or E = B, or E * A * B, or A * B * E. (8) $\overleftrightarrow{AC}$ is parallel to $\overleftrightarrow{BD}$. (9) Hence, E ≠ A and E ≠ B. (10) Assume E * A * B (Figure 4.26).

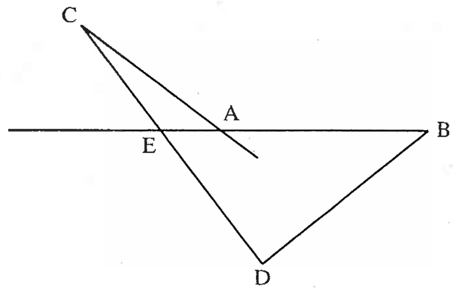


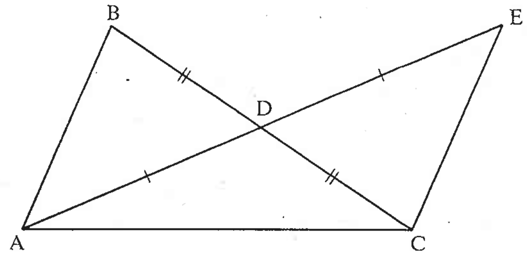
Figure 4.26

(11) Since $\overleftrightarrow{CA}$ intersects side EB of △EBD at a point between E and B, it must also intersect either ED or BD. (12) Yet this is impossible. (13) Hence, A is not between E and B. (14) Similarly, B is not between A and E. (15) Thus, A * E * B (see Figure 4.25). (16) Then ∢AEC ≅ ∢BED. (17) △EAC ≅ △EBD. (18) Therefore, E is a midpoint of AB.

13. Prove that segment AB has only one midpoint. (Hint: assume the contrary, and use Propositions 3.3, 3.5, and 3.13 to derive a contradiction.)

14. Prove Proposition 4.4 (on bisectors). (Hint: use midpoints.)

15. Prove the following result, needed to demonstrate the Saccheri-Legendre theorem (see Figure 4.27). Let D be the midpoint of BC and E the unique point on $\overrightarrow{AD}$ such that A * D * E and AD ≅ DE. Then △AEC has the same angle sum as △ABC, and either (∢EAC)° or (∢AEC)° is ≤ ½ (∢BAC)°. (Hint: first show that △BDA ≅ △CDE, then that (∢EAC)° + (∢AEC)° = (∢BAC)°.)

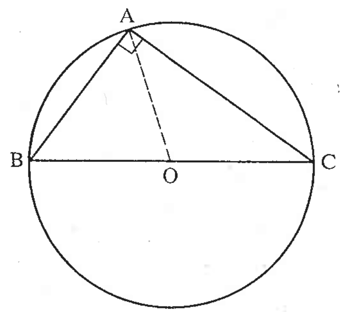


Figu

16. Prove Corollary 1 to the Saccheri-Legendre theorem.

17. Prove the following theorems:
   (a) Let γ be a circle with center O, and let A and B be two points on γ. The segment AB is called a *chord* of γ; let M be its midpoint. Then $\overleftrightarrow{OM}$ is perpendicular to $\overleftrightarrow{AB}$. (Hint: corresponding angles of congruent triangles are congruent.)
   (b) Let AB be a chord of the circle γ having center O. Prove that the perpendicular bisector of AB passes through the center O of γ.

18. A familiar Euclidean statement is "an angle inscribed in a semicircle is a right angle." Prove that this statement implies the existence of a right triangle △ABC whose angle sum is 180°.



Figu

19. Find the flaw in the following argument purporting to construct a rectangle. Let A and B be any two points. There is a unique line l through A perpendicular to $\overleftrightarrow{AB}$ (Proposition 3.16) and, similarly, there is a unique line m through B perpendicular to $\overleftrightarrow{AB}$. Take any point C on m other than B. There is a unique line through C perpendicular to l—let it intersect l at D. Then □ABCD is a rectangle.

20. On p. 99 we saw how the gap in Euclid's proof of Theorem 4.2 (on exterior angles) could be filled, and in Exercise 12 we justified Euclid's use of a midpoint. Now finish Euclid's argument to obtain another proof of Theorem 4.2.