

ON THE SPEED OF A PLANAR RANDOM WALK AVOIDING ITS PAST CONVEX HULL

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ABSTRACT. We consider a random walk in \mathbb{R}^2 which takes steps uniformly distributed on the unit circle centered around the walker's current position but avoids the convex hull of its past positions. This model has been introduced and studied by Angel, Benjamini and Virág. We show a large deviation estimate for the distance of the walker from the origin, which implies that the walker has positive lim inf speed.

1. INTRODUCTION

Angel, Benjamini and Virág introduced and studied in [1] the following model of a random walk $(X_n)_{n \geq 0}$ in \mathbb{R}^2 , which they called the *rancher*. The walker starts at the origin $X_0 = 0$. Suppose it has already taken n steps ($n \geq 0$) and is currently at X_n . Then its next position X_{n+1} is uniformly distributed on the unit circle centered around X_n but conditioned so that the straight line segment $\overline{X_n, X_{n+1}}$ from X_n to X_{n+1} does not intersect the convex hull K_n of the past positions $\{X_0, X_1, \dots, X_n\}$, see Figure 1.

Note that $(X_n)_{n \geq 0}$ is not Markovian since in general one needs to know the whole history of the process in order to determine the transition probabilities for the next step. This makes this model more difficult to analyze than a Markovian random walk, a property it shares with many other self-interacting processes, see [1] and also [2] for references. We do not claim that this model is of particular importance or that its definition is very natural. However, in the class of all self-interacting processes it seems to be among the few models which are at least partially analyzable and still have some striking properties.

Several such properties have been conjectured by Angel, Benjamini and Virág in [1]. Based on simulations and heuristics, the authors of [1] believe

2000 Mathematics Subject Classification. 60K35. Secondary 60G50, 52A22.

Key words: convex hull, large deviations, law of large numbers, random walk, self avoiding, speed.

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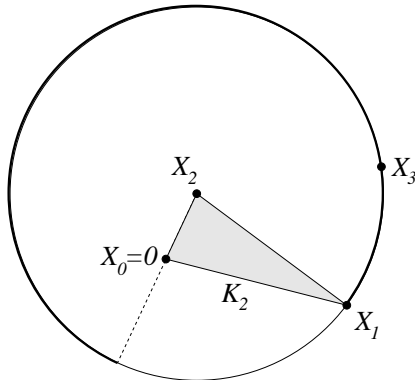


FIGURE 1. Three steps of the walk. X_3 is uniformly distributed on the bold arc of the circle with radius 1, centered in X_2 .

e.g. that the direction of the walk converges, i.e. that $X_n/\|X_n\|$ converges P -a.s. to a random direction, see [1, Conjecture 1]. Moreover, if we denote by w_n as a measure for transversal fluctuations the maximal distance of a point in K_n from the line $\overline{X_0, X_n}$, then Angel, Benjamini and Virág conjecture that w_n grows like $n^{3/4}$, see [1, Conjecture 2].

As far as results are concerned, the only major rigorous result which has been proved so far for this model to the best of our knowledge is that the walk has positive lim sup speed, i.e. there is a constant $c > 0$ such that P -a.s. $\limsup \|X_n\|/n > c$ as $n \rightarrow \infty$, see [1, Theorem 1]. Here (Ω, \mathcal{F}, P) is the underlying probability space.

The purpose of the present paper is to improve this result by showing the following.

Theorem 1. *There is a constant $c_1 > 0$ such that*

$$(1) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log P [\|X_n\| \leq c_1 n] < 0$$

and consequently,

$$(2) \quad \liminf_{n \rightarrow \infty} \frac{\|X_n\|}{n} \geq c_1 \quad P\text{-a.s.}$$

In particular, (2) proves [1, Conjecture 4]. We expect but were not able to prove that the speed $\lim \|X_n\|/n$ exists and is P -a.s. constant, as conjectured in [1, Conjecture 5].

Let us now describe how the present article is organized. The next section introduces general notation and gives a short overview of the proof. In Section 3 we introduce some sub- and supermartingales, which enable us in Section 4 to bound exponential moments of the time it takes the diameter of the convex hull K_n to increase. From this we deduce in Section 5 estimates

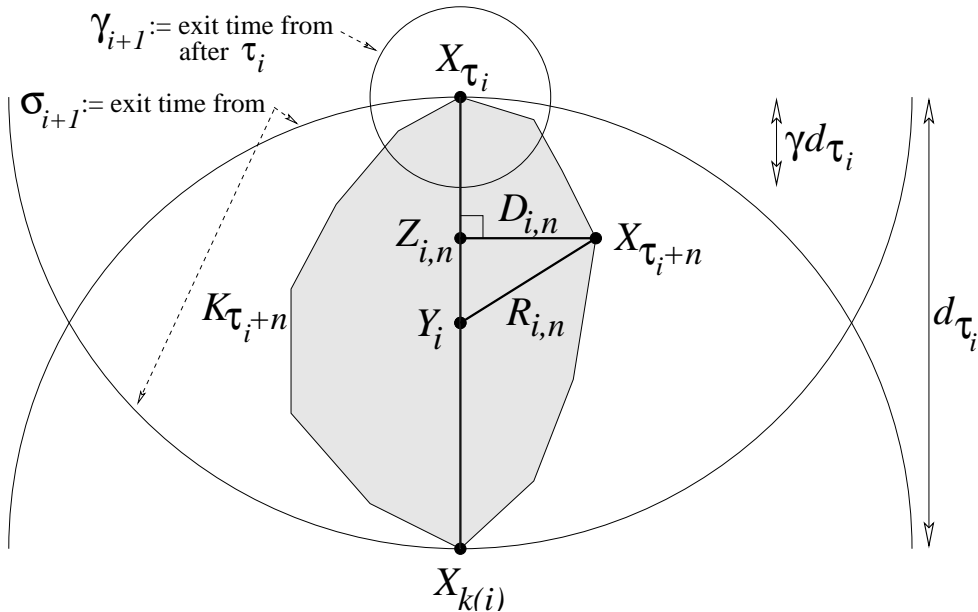


FIGURE 2. General notation

for the diameter of K_n similar to the ones claimed in Theorem 1 for $\|X_n\|$ and show how this implies Theorem 1.

2. NOTATION AND OUTLINE OF PROOF

We denote by d_n the diameter of K_n . Since $(K_n)_n$ is an increasing sequence of sets, $(d_n)_n$ is non-decreasing. The ladder times τ_i at which the process $(d_n)_{n \geq 0}$ strictly increases are defined recursively by

$$\tau_0 := 0 \quad \text{and} \quad \tau_{i+1} := \inf\{n > \tau_i : d_n > d_{\tau_i}\} \quad (i \geq 0).$$

It follows from [1] that P -a.s. $\tau_i < \infty$ for all $i \geq 0$. In Section 4 we will prove this fact by showing that the differences

$$\Delta_i := \tau_{i+1} - \tau_i \quad (i \geq 0),$$

between two successive finite ladder times have some finite exponential moments. Note that $\tau_1 = 1$ since $d_0 = 0$ and $d_1 = 1$ and observe that the τ_i 's are stopping times with respect to the canonical filtration $(\mathcal{F}_n)_{n \geq 0}$ generated by $(X_n)_{n \geq 0}$. Since the diameter of a bounded convex set is the distance between two of its extremal points there is for all $i \geq 1$ with $\tau_i < \infty$ a (P -a.s. unique) $0 \leq k(i) < \tau_i$ such that $d_{\tau_i} = \|X_{\tau_i} - X_{k(i)}\|$, see Figure 2. For $x \in \mathbb{R}^2$ and $r > 0$ we denote by $B(x, r)$ the closed disk with center x and radius r . If $\tau_i < \infty$ then

$$\sigma_{i+1} := \inf\{n \geq 0 \mid X_n \notin B(X_{\tau_i}, d_{\tau_i}) \cap B(X_{k(i)}, d_{\tau_i})\}$$

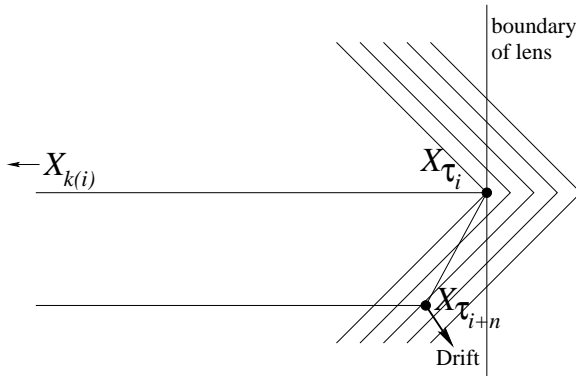


FIGURE 3. Idealized picture, assuming $d_{\tau_i} = \infty$. Whenever the walker crosses a wedge for the first time, which occurs at a positive rate, it has in the next step a positive drift towards the boundary of the lens.

is the exit time of the walk from the large lens shaped region shown in Figure 2, which we shall refer to as the *lens created at time τ_i* . Observe that K_{τ_i} is contained in the lens created at time τ_i . Moreover,

$$(3) \quad \tau_{i+1} \leq \sigma_{i+1}$$

since if $\sigma_{i+1} < \infty$, $X_{\sigma_{i+1}}$ has a distance from either X_{τ_i} or $X_{k(i)}$ greater than d_{τ_i} .

We have now introduced enough notation to be able to outline the idea of the proof of Theorem 1.

Outline of Proof. To show that $(\|X_n\|)_n$ is growing at a positive rate we shall first show that $(d_n)_n$ is doing so. To this end we shall bound exponential moments of the Δ_i 's. This is the heart of the proof and done as follows.

Suppose that d_n is already quite large and that we have just reached a new ladder time τ_i , at which the diameter has increased. Due to (3) the diameter can stay constant only as long as the walk is hiding in the lens created at time τ_i . However, there are two mechanisms, corresponding to Lemmas 4 and 5, which ensure that the walk cannot hide in the lens for too long.

The first one, described in Lemma 4, works while the walk is still inside a small ball around X_{τ_i} , see Figure 2. The radius of this ball is chosen to be proportional to d_{τ_i} . This situation is depicted in an idealized form in Figure 3. In this regime the following happens: Whenever the walk crosses for the first time any of the wedge-shaped lines shown in Figure 3, it has a drift away from $X_{k(i)}$ towards the boundary of the lens. To see this consider the two boundary lines of the convex hull which are emanating from $X_{\tau_{i+n}}$ in Figure 3. Suppose we are to draw in Figure 3 the line going from $X_{\tau_{i+n}}$ to the left. In the picture this line has slope 0, i.e. it is parallel to $\overline{X_{\tau_i}, X_{k(i)}}$, but

of course it could have a different slope. However, this slope cannot be much smaller than 0 because X_{τ_i+n} is close to X_{τ_i} but far away from $X_{k(i)}$. On the other hand, the line emanating from X_{τ_i+n} to the right must have in Figure 3 a slope of at least 1. Together, these two lines create a drift downwards and more importantly to the right towards the boundary of the lens. (If the left line has a positive slope then the drift to the right is even stronger.)

So every time the walker crosses for the first time such a helpful wedge-shaped line it will be pushed towards the next wedge-shaped line. If the little ball around X_{τ_i} was infinitely large this would result in crossings of the wedge-shaped lines at a positive rate, which would propel the walk out of the lens very fast, i.e. this mechanism alone would give a finite exponential moment of Δ_i , see Lemma 4. However, since the ball's size is finite and proportional to d_{τ_i} , the walk can escape from this little ball before leaving the lens with a probability exponentially small in d_{τ_i} . (This will happen a finite number of times.)

Whenever this occurs, we rely on the second mechanism, see Lemma 5. The line $\overline{X_{\tau_i}, X_{k(i)}}$, which is part of the convex hull, creates a positive drift out of the lens. Since the diameter of the lens is of order d_{τ_i} it will take the walker a time of order d_{τ_i} to reach the boundary of the lens. In fact we shall show, see Lemma 5, that it is exponentially costly for the walk to stay in the lens for a time longer than the order of d_{τ_i} . This is enough to make sure that the walk is not slowed down too much by occasionally resisting the first mechanism and getting lost in the lens. \square

To make this idea precise we need to introduce more notation. The point

$$Y_i := \frac{X_{\tau_i} + X_{k(i)}}{2} \quad (i \geq 1)$$

will serve as the “center” of K_{τ_i} and the “radius”

$$R_{i,n} := \|X_{\tau_i+n} - Y_i\| \quad (i \geq 1, n \geq 0)$$

is the distance of X_{τ_i+n} from this center. The orthogonal projection of X_{τ_i+n} onto the straight line passing through X_{τ_i} and $X_{k(i)}$ will be called $Z_{i,n}$ ($i \geq 1, n \geq 0$). The distance of X_{τ_i+n} from this line is denoted by

$$D_{i,n} := \|X_{\tau_i+n} - Z_{i,n}\| \quad (i \geq 1, n \geq 0).$$

For the following definitions we assume $i, n \geq 1$ and $\tau_i + n < \tau_{i+1}$. In particular, due to (3), we assume that at time $\tau_i + n$ the walk has not yet left the lens created at time τ_i . This implies that $Z_{i,n} \in \overline{X_{\tau_i}, X_{k(i)}}$ and that X_{τ_i} and $X_{k(i)}$ are still boundary points of K_{τ_i+n} , as shown in Figures 2 and 4. Hence if we start in X_{τ_i+n} and follow the two boundary line segments emanating from X_{τ_i+n} we will eventually reach X_{τ_i} and $X_{k(i)}$. The boundary line segment whose continuation leads first to K_{τ_i} and then to $X_{k(i)}$ is called

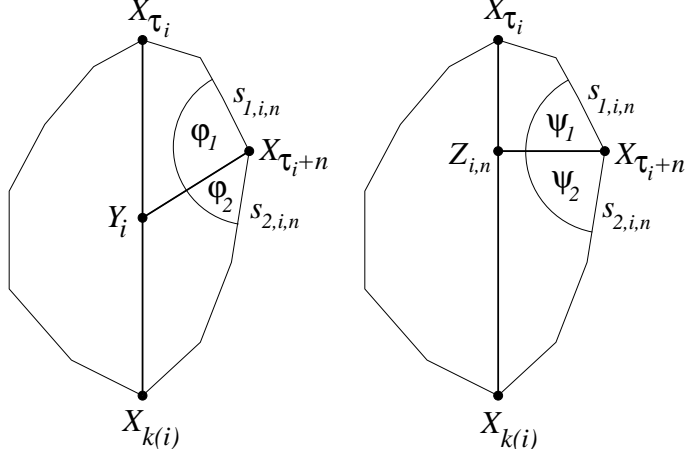


FIGURE 4. General notation

$s_{1,i,n}$, while the other line segment starting in X_{τ_i+n} is denoted by $s_{2,i,n}$, see Figure 4. The angle between $s_{j,i,n}$ and $\overline{X_{\tau_i+n}, Y_i}$ is called $\varphi_{j,i,n} \in [0, \pi]$ ($j = 1, 2$), see the left part of Figure 4. Similarly, the angle between $s_{j,i,n}$ and $\overline{X_{\tau_i+n}, Z_{i,n}}$ is denoted by $\psi_{j,i,n} \in [0, \pi]$ ($j = 1, 2$), see the right part of Figure 4. Occasionally, we will drop the subscripts i and n from φ and ψ . Since K_{τ_i+n} is convex,

$$(4) \quad \varphi_1 + \varphi_2 = \psi_1 + \psi_2 \leq \pi.$$

Furthermore, $|\varphi_1 - \psi_1|$ is one of the angles in a right angled triangle, namely the triangle with vertices X_{τ_i+n}, Y_i and $Z_{i,n}$. Hence,

$$(5) \quad |\varphi_1 - \psi_1| = |\varphi_2 - \psi_2| \leq \pi/2.$$

3. SOME SUB- AND SUPERMARTINGALES

The following result shows that for every $i \geq 1$, both $(R_{i,n})_n$ and $(D_{i,n})_n$ ($1 \leq n < \tau_{i+1} - \tau_i$) are submartingales.

Lemma 2. *For all $i, n \geq 1$, P -a.s. on $\{\tau_i + n < \tau_{i+1}\}$,*

$$(6) \quad E[R_{i,n+1} - R_{i,n} \mid \mathcal{F}_{\tau_i+n}] \geq \frac{\sin \varphi_{1,i,n} + \sin \varphi_{2,i,n}}{2\pi} \geq \frac{\sin \varphi_{1,i,n}}{2\pi} \geq 0,$$

$$(7) \quad E[D_{i,n+1} - D_{i,n} \mid \mathcal{F}_{\tau_i+n}] \geq \frac{\sin \psi_{1,i,n} + \sin \psi_{2,i,n}}{2\pi} \geq \frac{\sin \psi_{1,i,n}}{2\pi} \geq 0$$

and

$$(8) \quad E[D_{i,n+1} - D_{i,n} + R_{i,n+1} - R_{i,n} \mid \mathcal{F}_{\tau_i+n}] \geq c_2$$

for some constant $c_2 > 0$.

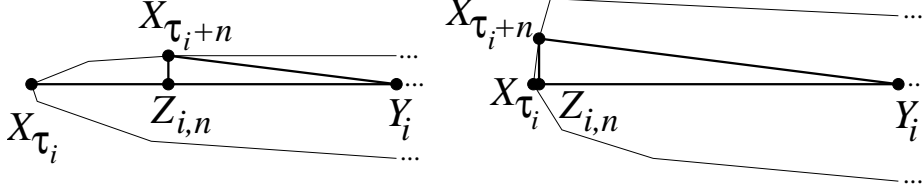


FIGURE 5. The expected increment of $R_{i,n}$ is small in the left figure and large in the right figure. For $D_{i,n}$ it is the other way round.

Figure 5 shows examples in which the expected increments of $R_{i,n}$ and $D_{i,n}$ are close to 0, thus explaining, why we are not able to bound in (6) and (7) these expected increments individually away from 0. Note however, that in both situation depicted in Figure 5, if the expected increment of $R_{i,n}$ or of $D_{i,n}$ is small then the expected increment of the other quantity is large. This confirms that the expected increments of $R_{i,n}$ and $D_{i,n}$ cannot both be small at the same time, see (8).

Proof of Lemma 2. We fix $i, n \geq 1$ and drop them as subscripts of $\varphi_{j,i,n}$ and $\psi_{j,i,n}$ ($j = 1, 2$). Then the following statements hold on the event $\{\tau_i + n < \tau_{i+1}\}$. Consider the angle between $\overline{Y_i, X_{\tau_i+n}}$ and $\overline{X_{\tau_i+n}, X_{\tau_i+n+1}}$ which includes $s_{2,i,n}$. This angle is chosen uniformly at random from the interval $[\varphi_2, 2\pi - \varphi_1]$. Hence we get by a change of basis argument using $\overline{Y_i, X_{\tau_i+n}}$ for the first basis vector,

$$\begin{aligned}
& E[R_{i,n+1} \mid \mathcal{F}_{\tau_i+n}] \\
&= \frac{1}{(2\pi - \varphi_1) - \varphi_2} \int_{\varphi_2}^{2\pi - \varphi_1} \|(R_{i,n}, 0) - (\cos \varphi, \sin \varphi)\| \, d\varphi \\
&\geq \frac{1}{2\pi - \varphi_1 - \varphi_2} \int_{\varphi_2}^{2\pi - \varphi_1} |R_{i,n} - \cos \varphi| \, d\varphi \\
&\geq R_{i,n} + \frac{1}{2\pi - \varphi_1 - \varphi_2} \int_{\varphi_2}^{2\pi - \varphi_1} -\cos \varphi \, d\varphi \\
&= R_{i,n} + \frac{\sin \varphi_1 + \sin \varphi_2}{2\pi - \varphi_1 - \varphi_2} \geq R_{i,n} + \frac{\sin \varphi_1 + \sin \varphi_2}{2\pi},
\end{aligned}$$

which shows (6). Similarly, (7) follows from

$$E[D_{i,n+1} \mid \mathcal{F}_{\tau_i+n}] = \frac{1}{(2\pi - \psi_1) - \psi_2} \int_{\psi_2}^{2\pi - \psi_1} |D_{i,n} - \cos \psi| \, d\psi.$$

For the proof of (8) we assume without loss of generality $\varphi_1 \leq \pi/2$. Indeed, otherwise $\varphi_2 \leq \pi/2$ because of $\varphi_1 + \varphi_2 \leq \pi$, see (4), and in the following proof one only has to replace the subscript 1 by the subscript 2 and swap

X_{τ_i} and $X_{k(i)}$. By (6) and (7),

$$(9) \quad E[D_{i,n+1} - D_{i,n} + R_{i,n+1} - R_{i,n} \mid \mathcal{F}_{\tau_i+n}] \geq \frac{\sin \varphi_1 + \sin \psi_1}{2\pi}.$$

We will show that the right side of (9) is always greater than $c_2 := (4\pi^2)^{-1}$. Assume that it is less than c_2 . Then

$$(10) \quad \sin \varphi_1, \sin \psi_1 \leq (2\pi)^{-1}$$

and hence $\varphi_1 \leq (\pi/2) \sin \varphi_1 \leq 1/4$ by concavity of \sin on $[0, \pi/2]$. Similarly, (10) implies that either $\psi_1 \leq 1/4$ or $\pi - \psi_1 \leq 1/4$. Due to $|\varphi_1 - \psi_1| \leq \pi/2$, see (5), the latter case is impossible. Therefore,

$$(11) \quad |\varphi_1 - \psi_1| \leq \max\{|\varphi_1|, |\psi_1|\} \leq 1/4.$$

The angle $\alpha \in [0, \pi/2]$ between $\overline{Z_{i,n}, X_{\tau_i+n}}$ and $\overline{X_{\tau_i+n}, X_{\tau_i}}$ is less than or equal to ψ_1 . Consequently,

$$(12) \quad \sin \psi_1 \geq \sin \alpha = \frac{\|X_{\tau_i} - Z_{i,n}\|}{\|X_{\tau_i} - X_{\tau_i+n}\|} \geq \frac{\|X_{\tau_i} - Y_i\| - \|Y_i - Z_{i,n}\|}{d_{\tau_i}}.$$

However, since $Y_i, X_{\tau_i+n} \in K_{\tau_i+n}$,

$$(13) \quad R_{i,n} = \|Y_i - X_{\tau_i+n}\| \leq d_{\tau_i+n} = d_{\tau_i}.$$

Therefore,

$$\sin \psi_1 \stackrel{(12)}{\geq} \frac{d_{\tau_i}/2}{d_{\tau_i}} - \frac{\|Y_i - Z_{i,n}\|}{\|Y_i - X_{\tau_i+n}\|} = \frac{1}{2} - \sin |\varphi_1 - \psi_1| \stackrel{(11)}{\geq} \frac{1}{2} - \frac{1}{4} = \frac{1}{4},$$

which contradicts (10). \square

We fix the constants

$$(14) \quad \beta := 1 + 4\pi\sqrt{8} > 30 \quad \text{and} \quad \gamma := \frac{1}{2\beta} < \frac{1}{60}.$$

Whenever $\tau_i < \infty$ we denote the first exit time after τ_i from $B(X_{\tau_i}, \gamma d_{\tau_i})$ by

$$\gamma_{i+1} := \inf\{n > \tau_i : \|X_n - X_{\tau_i}\| > \gamma d_{\tau_i}\} \quad (\leq \infty),$$

see Figure 2. If $i \geq 1$ and $n \geq 0$ then we shall call n *good for i* if $n = 0$ or if

$$(15) \quad \tau_i + n < \tau_{i+1} \wedge \gamma_{i+1} \quad \text{and} \quad E[R_{i,n+1} - R_{i,n} \mid \mathcal{F}_{\tau_i+n}] \geq \frac{1}{\pi\sqrt{8}} \quad P\text{-a.s.}$$

This means, $n \geq 1$ is good for i if at time $\tau_i + n$ the walker has not yet left the intersection of the small ball around X_{τ_i} and the lens shown in Figure 2 and, roughly speaking, feels a substantial centrifugal force pushing it away from the center Y_i . Good times help the walker to leave the lens shortly after τ_i and closely to the point X_{τ_i} .

Next we introduce an Azuma type inequality for a certain family of supermartingales, which will help make this idea more precise.

Lemma 3. *There are constants $c_3 > 0, c_4 > 0$ and $1 \leq c_5 < \infty$ such that P -a.s. for all $i \geq 1, n \geq 0$,*

$$E[M_{i,n} \mid \mathcal{F}_{\tau_i}] \leq c_5 \exp(-c_4 n),$$

where

$$M_{i,n} := \mathbf{1}\{\tau_i + n < \tau_{i+1}\} \exp\left(c_3 \sum_{m=1}^n W_{i,m}\right) \quad \text{and}$$

$$W_{i,m} := D_{i,m-1} - D_{i,m} + \beta(R_{i,m-1} - R_{i,m}) + 4 \mathbf{1}\{m-1 \text{ is good for } i\}.$$

Proof. Fix $i \geq 1$. Firstly, we shall prove that for suitable $c_4 > 0$,

$$(16) \quad E[M_{i,n+1} \mid \mathcal{F}_{\tau_i+n}] \leq \exp(-c_4) M_{i,n} \quad P\text{-a.s. for all } n \geq 1,$$

thus showing that $(M_{i,n})_{n \geq 1}$ is an exponentially fast decreasing submartingale. Since $W_{i,m}$ is \mathcal{F}_{τ_i+m} -measurable ($m \geq 1$), we have

$$\begin{aligned} E[M_{i,n+1} \mid \mathcal{F}_{\tau_i+n}] &= M_{i,n} E[\exp(c_3 W_{i,n+1}), \tau_i + n + 1 < \tau_{i+1} \mid \mathcal{F}_{\tau_i+n}] \\ &\leq M_{i,n} E[\exp(c_3 W_{i,n+1}) \mid \mathcal{F}_{\tau_i+n}]. \end{aligned}$$

Therefore it suffices to show for the proof of (16) that on the event $\{\tau_i + n < \tau_{i+1}\}$, $E[\exp(c_3 W_{i,n+1}) \mid \mathcal{F}_{\tau_i+n}] < 1$ for some suitable $c_3 > 0$. However, since the $W_{i,m}$'s ($m \geq 1$) are uniformly bounded by a constant this follows from the fact that on the event $\{\tau_i + n < \tau_{i+1}\}$,

$$\begin{aligned} E[W_{i,n+1} \mid \mathcal{F}_{\tau_i+n}] &= E[D_{i,n} - D_{i,n+1} + R_{i,n} - R_{i,n+1} \mid \mathcal{F}_{\tau_i+n}] \\ &\quad + 4 \left(\pi \sqrt{8} E[R_{i,n} - R_{i,n+1} \mid \mathcal{F}_{\tau_i+n}] + \mathbf{1}\{n \text{ is good for } i\} \right) \\ &\leq -c_2 < 0 \end{aligned}$$

P -a.s. by virtue of definition (15) and Lemma 2 (6), (8). Using (16) for induction over n we obtain by the tower property,

$$E[M_{i,n} \mid \mathcal{F}_{\tau_i}] \leq \exp(-c_4(n-1)) E[M_{i,1} \mid \mathcal{F}_{\tau_i}] \quad P\text{-a.s.}$$

for all $n \geq 1$. Since $M_{i,1}$ is bounded by a constant this finishes the proof. \square

4. EXPONENTIAL MOMENTS OF $\tau_{i+1} - \tau_i$.

The next two lemmas provide tools needed to bound in Proposition 6 the times Δ_i during which the diameter does not increase. The first lemma shows that the walk cannot spend too much time inside the lens and the small ball around X_{τ_i} shown in Figure 2.

Lemma 4. *For all $i \geq 1, n \geq 0$, P -a.s.*

$$P[\tau_i + n < \tau_{i+1} \wedge \gamma_{i+1} \mid \mathcal{F}_{\tau_i}] \leq c_5 \exp(-c_4 n),$$

where c_4 and c_5 are as in Lemma 3.

The second lemma gives a first crude upper bound for Δ_i .

Lemma 5. *There is a finite constant c_6 such that for all $i \geq 1, n \geq 0$, P -a.s.*

$$P[\Delta_i > n \mid \mathcal{F}_{\tau_i}] \leq c_5 \exp(c_4(c_6 d_{\tau_i} - n)),$$

where c_4 and c_5 are as in Lemma 3.

In the proof of both lemmas we will use the fact that the sum in the definition of $M_{i,n}$ is in part telescopic, i.e.

$$(17) \quad \sum_{m=1}^n W_{i,m} = -D_{i,n} + \beta(R_{i,0} - R_{i,n}) + 4 \sum_{m=1}^n \mathbf{1}\{m-1 \text{ is good for } i\}$$

since $D_{i,0} = 0$.

Proof of Lemma 4. For $n = 0$ the statement is true since $c_5 \geq 1$. Fix $n \geq 1$. The statement follows from Lemma 3 and (17) once we have shown that on the event $\{\tau_i + n < \tau_{i+1} \wedge \gamma_{i+1}\}$, the right hand side of (17) is nonnegative. First we will show that on $\{\tau_i + n < \tau_{i+1} \wedge \gamma_{i+1}\}$,

$$(18) \quad D_{i,n} + \beta(R_{i,n} - R_{i,0}) \leq 2(D_{i,n} - \|X_{\tau_i} - Z_{i,n}\|).$$

This is done by brute force. For abbreviation we set $d := d_{\tau_i}, y := D_{i,n}$ and $x := \|X_{\tau_i} - Z_{i,n}\|$ and note that on $\{\tau_i + n < \tau_{i+1} \wedge \gamma_{i+1}\}$ we have $x, y \in [0, \gamma d]$. Observe that x and y play the role of cartesian coordinates of X_{τ_i+n} , see Figure 6. Using $R_{i,0} = d/2$ and $R_{i,n} = \sqrt{(d/2 - x)^2 + y^2}$ we see that (18) is equivalent to

$$\beta \sqrt{(d/2 - x)^2 + y^2} \leq y - 2x + \beta d/2.$$

Both sides of this inequality are nonnegative since x is less than γd , which is tiny compared to βd . Taking the square and rearranging shows that (18) is equivalent to

$$(19) \quad x(4x - 4y - 2\beta d - \beta^2 x + \beta^2 d) + y(y + \beta d - \beta^2 y) \geq 0.$$

Since $x, y \in [0, \gamma d]$ and $\beta\gamma = 1/2$, see (14), the terms $\beta^2 d$ in the first bracket and βd in the second bracket are the dominant terms, respectively, which shows that (19) and thus (18) holds. For the proof that the right hand side of (17) is indeed nonnegative on $\{\tau_i + n < \tau_{i+1} \wedge \gamma_{i+1}\}$ it therefore suffices to show that,

$$(20) \quad D_{i,n} - \|X_{\tau_i} - Z_{i,n}\| \leq 2 \sum_{j=0}^{n-1} \mathbf{1}\{j \text{ is good for } i\}.$$

Both $D_{i,j}$ and $-\|X_{\tau_i} - Z_{i,j}\|$ can increase by at most 1 if j increases by 1. Therefore, the left hand side of (20) is less than or equal to $2(\#J_{i,n})$ where $J_{i,n} := \{0 \leq j < n \mid \forall 0 \leq m < j : D_{i,m} - \|X_{\tau_i} - Z_{i,m}\| \leq D_{i,j} - \|X_{\tau_i} - Z_{i,j}\|\}$.

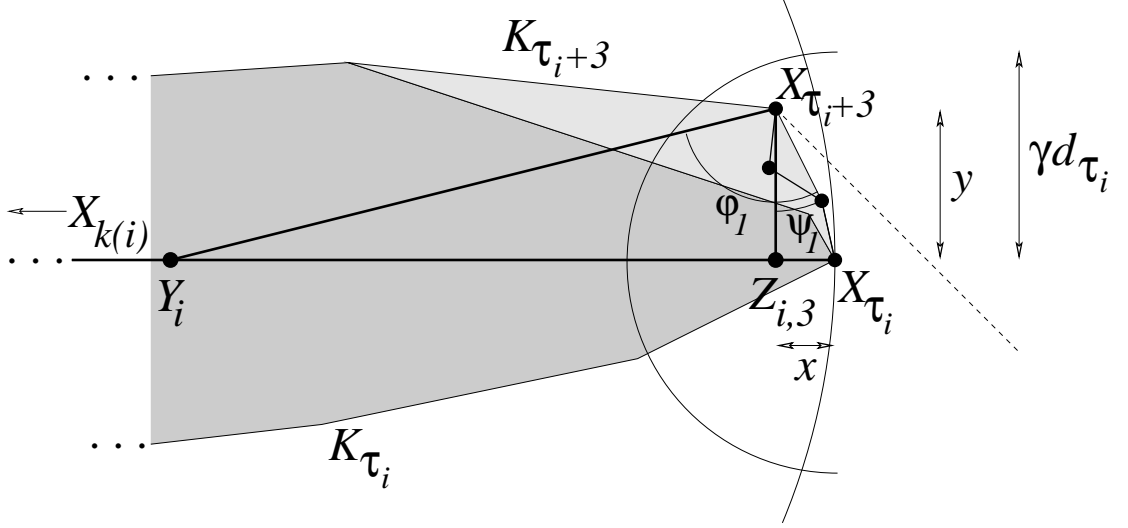


FIGURE 6. More realistic and detailed than Figure 3. The darkly shaded convex hull K_{τ_i} at time τ_i has been enlarged after three steps by the lightly shaded part. $j = 3$ is good for i since it satisfies the sufficient criterion $\psi_{1,i,3} \leq \pi/4$, see (22), which corresponds to the fact that the dashed line, which intersects the horizontal axis at an angle of $\pi/4$, does not intersect $K_{\tau_{i+3}}^o$. $j = 1$ is also good for i for the same reason, while $j = 2$ might be good for i but fails to satisfy the sufficient condition (22), since the corresponding dashed line starting in $X_{\tau_{i+2}}$ would have intersected $K_{\tau_{i+2}}^o$.

Hence it suffices to show that the elements of $J_{i,n}$ are good for i . Note that $j = 0 \in J_{i,n}$ is good for i by definition of being good. So fix $1 \leq j \in J_{i,n}$. By Lemma 2 (6) it is enough to show that $\sin \varphi_{1,i,j} \geq 2^{-1/2}$, that is

$$(21) \quad \varphi_{1,i,j} \in [\pi/4, 3\pi/4].$$

On the one hand, $\varphi_1 - \psi_1$ is close to $\pi/2$, as can be seen in Figure 6. More precisely,

$$\begin{aligned} \sin(\varphi_1 - \psi_1) &= \frac{\|Y_i - Z_{i,j}\|}{\|Y_i - X_{\tau_{i+j}}\|} \geq \frac{\|Y_i - X_{\tau_i}\| - \|X_{\tau_i} - Z_{i,j}\|}{\|Y_i - X_{\tau_i}\| + \|X_{\tau_i} - X_{\tau_{i+j}}\|} \\ &\geq \frac{d_{\tau_i}/2 - \gamma d_{\tau_i}}{d_{\tau_i}/2 + \gamma d_{\tau_i}} = \frac{1 - 2\gamma}{1 + 2\gamma} \geq \frac{1}{\sqrt{2}} = \sin \frac{\pi}{4}. \end{aligned}$$

Since $0 \leq \psi_1 \leq \varphi_1$ this implies $\varphi_1 \geq \pi/4$, thus proving the first part of (21). On the other hand, $\varphi_1 - \psi_1 \leq \pi/2$, see (5). Hence all that remains to be

shown for the completion of the proof of (21) is that

$$(22) \quad \psi_{1,i,j} \leq \pi/4.$$

Consider the half line (dashed in Figure 6) starting at X_{τ_i+j} which includes an angle of $\pi/4$ with $\overline{X_{\tau_i+j}, Z_{i,j}}$ that contains $s_{1,i,j}$. We claim that this line does not intersect $K_{\tau_i+j}^o$. This would imply (22). To prove this claim observe that for any $c > 0$ the set of possible values for X_{τ_i+m} with $D_{i,m} - \|X_{\tau_i} - Z_{i,m}\| = c$ is a line parallel to the half line just described. Since $j \in J_{i,n}$ the walker did not cross between time τ_i and time $\tau_i + j - 1$ the dashed line passing through X_{τ_i+j} . Consequently, it suffices to show that the dashed line does not intersect $K_{\tau_i}^o$. If it did intersect $K_{\tau_i}^o$ then this would force the walker on its way from X_{τ_i} to X_{τ_i+j} to cross the dashed line strictly before time $\tau_i + j$, which is impossible as we just saw. \square

Proof of Lemma 5. Again, the statement is true for $n = 0$ since $c_5 \geq 1$. For $n \geq 1$, by the Pythagorean theorem and (13), $D_{i,n} \leq R_{i,n} \leq d_{\tau_i}$ on the event $\{\tau_i + n < \tau_{i+1}\}$. Thus on this event the right hand side of (17) is greater than or equal to $-d_{\tau_i} + \beta(0 - d_{\tau_i}) + 0$. Hence due to Lemma 3 and (17), P -a.s. for all $n \geq 1$,

$$\exp(-c_3(1 + \beta)d_{\tau_i})P[\tau_i + n < \tau_{i+1} \mid \mathcal{F}_{\tau_i}] \leq c_5 \exp(-c_4 n),$$

which is equivalent to the claim of the lemma with $c_6 := c_3(1 + \beta)/c_4$. \square

The following result is stronger than Lemma 5.

Proposition 6. $\tau_i < \infty$ P -a.s. for all $i \geq 0$. Moreover, there are positive constants c_7, c_8 and finite constants c_9 and $c_{10} \geq 1$ such that for all $i \geq 0$ and $n \geq 0$, P -a.s.,

$$(23) \quad P[\Delta_i \geq n \mid \mathcal{F}_{\tau_i}] \leq c_9 \exp(-c_7 n) \quad \text{and}$$

$$(24) \quad E[\exp(c_8 \Delta_i) \mid \mathcal{F}_{\tau_i}] \leq c_{10}.$$

Proof. (24) is an immediate consequence of (23). For the proof of (23) choose

$$(25) \quad c_7 := \frac{\gamma c_4}{c_6 + \gamma} \quad \text{and} \quad c_9 := c_5 e^{c_4}$$

with c_4, c_5 and c_6 according to Lemma 5. We only need to show that (23) holds for all $i \geq 1$ with $\tau_i < \infty$. Indeed, the case $i = 0$ is trivial since $\tau_0 = 0, \tau_1 = 1$ and hence $\Delta_0 = 1$. Moreover, since (23) implies $\Delta_i < \infty$ P -a.s., we then have $\tau_i = \Delta_0 + \dots + \Delta_{i-1} < \infty$ as well.

Fix $i \geq 1$ and $n \geq 0$. We distinguish three cases:

$$\{n < \gamma d_{\tau_i}\}, \quad \{\gamma d_{\tau_i} \leq n < (c_6 + \gamma)d_{\tau_i}\} \quad \text{and} \quad \{(c_6 + \gamma)d_{\tau_i} \leq n\}.$$

Note that these events are elements of \mathcal{F}_{τ_i} and partition Ω . By definition of γ_{i+1} ,

$$(26) \quad \gamma_{i+1} \geq \tau_i + \lceil \gamma d_{\tau_i} \rceil$$

since the walker takes steps of length one. Therefore, on $\{n < \gamma d_{\tau_i}\}$,

$$\begin{aligned} P[\Delta_i > n \mid \mathcal{F}_{\tau_i}] &\stackrel{(26)}{=} P[\tau_i + n < \tau_{i+1} \wedge \gamma_{i+1} \mid \mathcal{F}_{\tau_i}] \\ &\stackrel{\text{Lemma 4}}{\leq} c_5 \exp(-c_4 n) \stackrel{(25)}{\leq} c_9 \exp(-c_7 n). \end{aligned}$$

On $\{\gamma d_{\tau_i} \leq n < (c_6 + \gamma)d_{\tau_i}\}$,

$$\begin{aligned} P[\Delta_i > n \mid \mathcal{F}_{\tau_i}] &\leq P[\tau_i + \lceil \gamma d_{\tau_i} \rceil < \tau_{i+1} \mid \mathcal{F}_{\tau_i}] \stackrel{(26)}{\leq} P[\tau_i + \lceil \gamma d_{\tau_i} \rceil - 1 < \tau_{i+1} \wedge \gamma_{i+1} \mid \mathcal{F}_{\tau_i}] \\ &\stackrel{\text{Lemma 4}}{\leq} c_5 \exp(-c_4(\gamma \lceil d_{\tau_i} \rceil - 1)) \stackrel{(25)}{\leq} c_9 \exp(-c_7 n). \end{aligned}$$

Finally, on $\{(c_6 + \gamma)d_{\tau_i} \leq n\}$,

$$P[\Delta_i > n \mid \mathcal{F}_{\tau_i}] \stackrel{\text{Lemma 5}}{\leq} c_5 \exp(c_4(c_6 d_{\tau_i} - n)) \stackrel{(25)}{\leq} c_9 \exp(-c_7 n),$$

where the last inequality can easily be checked. \square

5. LINEAR GROWTH OF THE DIAMETER AND PROOF OF THEOREM 1

The following result (with $m = 0$) implies that $(d_n)_n$ has a positive lim inf speed.

Lemma 7. *There are constants $c_{11} > 0$ and $c_{12} < \infty$ such that for all $0 \leq m \leq n$,*

$$E[\exp(d_m - d_n)] \leq c_{12}(n + 1) \exp(c_{11}(m - n)).$$

For the proof of this lemma and of Theorem 1 we need the following definition: Given $n \geq 0$ let $i_n := \sup\{i \geq 0 \mid \tau_i \leq n\}$. Note that

$$(27) \quad d_{\tau_{i_n}} = d_n \quad \text{and} \quad i_n \leq \tau_{i_n} \leq n < \tau_{i_n+1}.$$

Proof of Lemma 7. The case $n = m$ is trivial. So let $0 \leq m < n$ and set

$$(28) \quad f(m, n) := \frac{n - m}{2n} > 0 \quad \text{and} \quad g(m, n) := \frac{c_8 f(m, n)}{2 \ln c_{10}} > 0,$$

where c_8 and c_{10} are according to Proposition 6. A simple union bound yields

$$\begin{aligned} E[e^{d_m - d_n}] &\leq \text{I} + \text{II} + \text{III}, \quad \text{where} \\ \text{I} &:= P[\tau_{i_m+1} - m \geq f(m, n)n], \\ \text{II} &:= P[\tau_{i_m+1} - m < f(m, n)n, i_n < i_m + \lceil g(m, n)n \rceil] \quad \text{and} \\ \text{III} &:= E[\exp(d_m - d_n), i_n \geq i_m + \lceil g(m, n)n \rceil], \end{aligned}$$

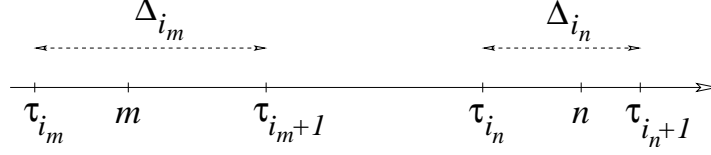


FIGURE 7.

see also Figure 7. Here term I corresponds to the situation in which after time m the diameter does not increase for an untypical long while. Term II handles the case in which the diameter does increase shortly after time m , as it should, but not often enough in the remaining time until n . The third term III considers the original random variable on the typical event that the number of times at which the diameter increases is at least proportional to n with a constant of proportionality not too small.

It suffices to show that each of these three terms decays as $n \rightarrow \infty$ in the way claimed in Lemma 7 with constants c_{11} and c_{12} independent of n and m . As for the first term,

$$\begin{aligned} \text{I} &\leq P[\Delta_{i_m} \geq \lceil f(m, n)n \rceil] \stackrel{(27)}{=} \sum_{i=0}^m P[i_m = i, \Delta_i \geq \lceil f(m, n)n \rceil] \\ &\stackrel{(23)}{\leq} c_9(m+1)e^{-c_7 \lceil f(m, n)n \rceil} \leq c_9(n+1)e^{c_7(m-n)/2}, \end{aligned}$$

which is an upper bound like the one requested in Lemma 7. The second term is estimated as follows.

$$\begin{aligned} \text{II} &\stackrel{(27)}{=} P[\tau_{i_{m+1}} < m + f(m, n)n, n < \tau_{i_n+1} \leq \tau_{i_m+\lceil g(m, n)n \rceil}] \\ &\stackrel{(28)}{\leq} P[\tau_{i_m+\lceil g(m, n)n \rceil} - \tau_{i_m+1} \geq n - m - f(m, n)n = f(m, n)n] \\ &\leq E[\exp(c_8(\tau_{i_m+\lceil g(m, n)n \rceil} - \tau_{i_m+1} - f(m, n)n))] \\ &= e^{-c_8 f(m, n)n} \sum_{k \geq 1} E[\exp(c_8(\tau_{k+\lceil g(m, n)n \rceil-1} - \tau_k)), i_m + 1 = k] \\ (29) \quad &= e^{-c_8 f(m, n)n} \sum_{k \geq 1} E \left[\prod_{i=0}^{\lceil g(m, n)n \rceil-2} \exp(c_8 \Delta_{k+i}), i_m + 1 = k \right]. \end{aligned}$$

Note that $\{i_m + 1 = k\}$ is the event that τ_k is the first time after time m at which the diameter increases. Therefore,

$$(30) \quad \{i_m + 1 = k\} \in \mathcal{F}_{\tau_k}.$$

Moreover, the increments Δ_{k+i} are measurable with respect to $\mathcal{F}_{\tau_{k+i+1}}$. Consequently, by conditioning in (29) on $\mathcal{F}_{\tau_{k+\lceil g(m, n)n \rceil-2}}$ and applying Proposition

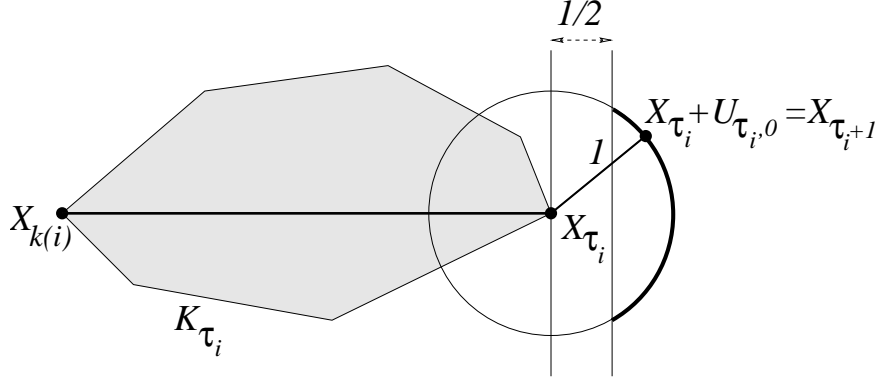


FIGURE 8. The event A_i occurs if the first trial point sampled lies on the bold arc.

6 (24) with $i = k + \lceil g(m, n)n \rceil - 2$ we conclude

$$\text{II} \leq e^{-c_8 f(m, n)n} c_{10} \sum_{k \geq 1} E \left[\prod_{i=0}^{\lceil g(m, n)n \rceil - 3} \exp(c_8 (\Delta_{k+i})), i_m + 1 = k \right].$$

Continuing in this way we obtain by induction after $\lceil g(m, n)n \rceil - 1$ steps,

$$\text{II} \leq e^{-c_8 f(m, n)n} c_{10}^{\lceil g(m, n)n \rceil - 1} \leq e^{-c_8 f(m, n)n} c_{10}^{g(m, n)n} \stackrel{(28)}{=} e^{(c_8/4)(m-n)},$$

which is again of the form required in Lemma 7.

In order to demonstrate that also the third term III behaves properly we will show that the increments $d_{\tau_{i+1}} - d_{\tau_i}$, $i \geq 1$, have a uniformly positive chance of being larger than a fixed constant, say $1/2$, independently of the past. To this end, we may assume that the process $(X_n)_n$ is generated in the following way: There are i.i.d. random variables $U_{n,k}$, $n \geq 0, k \geq 0$, uniformly distributed on the unit circle centered in 0 such that $X_{n+1} = X_n + U_{n,k}$, where k is the smallest integer such that $\overline{X_n, X_n + U_{n,k}}$ does not intersect K_n . Then for any $i \geq 1$, by definition of τ_i ,

$$(31) \quad \begin{aligned} \{d_{\tau_{i+1}} \geq d_{\tau_i} + 1/2\} &\supseteq \{d_{\tau_{i+1}} \geq d_{\tau_i} + 1/2\} \\ &\supseteq \left\{ U_{\tau_i,0} \cdot \frac{X_{\tau_i} - X_{k(i)}}{d_{\tau_i}} \geq \frac{1}{2} \right\} =: A_i. \end{aligned}$$

The reason for inclusion (31) is illustrated in Figure 8: If A_i occurs then K_{τ_i} and $X_{\tau_i} + U_{\tau_i,0}$ are separated by a slab of width $1/2$. In particular, the line connecting X_{τ_i} and $X_{\tau_i} + U_{\tau_i,0}$ does not intersect K_{τ_i} . Hence $\tau_{i+1} = \tau_i + 1$ and $X_{\tau_{i+1}} = X_{\tau_i+1} = X_{\tau_i} + U_{\tau_i,0}$. Therefore,

$$d_{\tau_{i+1}} \geq \|X_{\tau_{i+1}} - X_{k(i)}\| \geq \|X_{\tau_i} - X_{k(i)}\| + 1/2 = d_{\tau_i} + 1/2.$$

It follows from (31) that for all $1 \leq j_1 \leq j_2$,

$$(32) \quad d_{\tau_{j_2}} - d_{\tau_{j_1}} \geq \frac{1}{2} \sum_{i=j_1}^{j_2-1} \mathbf{1}\{d_{\tau_{i+1}} \geq d_{\tau_i} + 1/2\} \geq \frac{1}{2} \sum_{i=j_1}^{j_2-1} \mathbf{1}\{A_i\}.$$

This estimate will be useful since the random variables

$$(33) \quad \mathbf{1}\{A_i\} \ (i \geq 1) \quad \text{are i.i.d. with } P[A_i] > 0.$$

Indeed, let $\tilde{\mathcal{F}}_n$ ($n \geq 0$) be the σ -field generated by $U_{m,k}$, $0 \leq m < n, 0 \leq k$. Because of $\mathcal{F}_n \subseteq \tilde{\mathcal{F}}_n$ we have $A_j \in \tilde{\mathcal{F}}_{\tau_i}$ for all $1 \leq j < i$. Moreover, since the uniform distribution on the unit circle is invariant under rotations,

$$(34) \quad A_i \text{ is independent of } \tilde{\mathcal{F}}_{\tau_i} \ (i \geq 1)$$

and $P[A_i | \tilde{\mathcal{F}}_{\tau_i}] = P[A_i]$ is just the length of the bold circle segment shown in Figure 8 divided by 2π . This implies (33). Now we estimate III by

$$(35) \quad \begin{aligned} \text{III} &\stackrel{(27)}{=} E \left[\exp(d_{\tau_{i_m}} - d_{\tau_{i_n}}), i_n \geq i_m + \lceil g(m,n)n \rceil \right] \\ &\leq E \left[\exp(d_{\tau_{i_m+1}} - d_{\tau_{i_n}}), i_n \geq (i_m + 1) + \lceil g(m,n)n \rceil - 1 \right] \\ &\leq \sum_{k \geq 1} E \left[\exp(d_{\tau_k} - d_{\tau_{k+\lceil g(m,n)n \rceil - 1}}), i_m + 1 = k \right] \\ &\stackrel{(32)}{\leq} \sum_{k \geq 1} E \left[\exp \left(-\frac{1}{2} \sum_{i=k}^{k+\lceil g(m,n)n \rceil - 2} \mathbf{1}\{A_i\} \right), i_m + 1 = k \right]. \end{aligned}$$

As seen in (30), $\{i_m + 1 = k\} \in \mathcal{F}_{\tau_k} \subseteq \tilde{\mathcal{F}}_{\tau_k}$. Therefore, after conditioning in (35) on \mathcal{F}_{τ_k} , we see with the help of (34) for $i \geq k$ and (33) that the right hand side of (35) equals

$$E \left[\exp \left(-\frac{1}{2} A_1 \right) \right]^{\lceil g(m,n)n \rceil - 1},$$

which decays as required in Lemma 7, see (28). \square

Lemma 7 directly implies a weaker version of Theorem 1 in which $\|X_n\|$ is replaced by d_n . For the full statement we need the following additional argument.

Proof of Theorem 1. (2) follows from (1) by the Borel-Cantelli lemma. For the proof of (1) pick c_{11} and c_{12} according to Lemma 7 and choose $c_{13} > 0$ and $c_1 > 0$ small enough such that

$$(36) \quad 2c_{13} - c_{11} < 0 \quad \text{and} \quad 2c_1 - c_{11}(c_{13} - c_1) < 0.$$

We denote by $M_n := \max\{\|X_m\| \mid m \leq n\}$ the walker's maximal distance from the origin by time n . Note that M_n and d_n are related via

$$(37) \quad M_n \leq d_n \leq 2M_n \quad \text{for all } n \geq 0$$

because of $X_0 = 0$. By a union bound for any $n \geq 0$,

$$(38) \quad P[\|X_n\| \leq c_1 n] \leq P[\Delta_{i_n} \geq c_1 n] + P[M_n \leq c_{13} n] + P[B_n], \text{ where} \\ B_n := \{\Delta_{i_n} < c_1 n, M_n > c_{13} n, \|X_n\| \leq c_1 n\}.$$

It suffices to show that each one of the three terms on the right hand side of (38) decays exponentially fast in n . As for the first term,

$$P[\Delta_{i_n} \geq c_1 n] \stackrel{(27)}{=} \sum_{i=0}^n P[i_n = i, \Delta_i \geq \lceil c_1 n \rceil] \stackrel{(23)}{\leq} c_9(n+1)e^{-c_7 c_1 n},$$

which decays exponentially fast in n indeed. So does the second term in (38) since by Chebyshev's inequality,

$$P[M_n \leq c_{13} n] \stackrel{(37)}{\leq} P[d_n \leq 2c_{13} n] \leq e^{2c_{13} n} E[e^{-d_n}] \stackrel{\text{Lemma 7}}{\leq} c_{12}(n+1)e^{(2c_{13}-c_{11})n},$$

which decays exponentially fast due to the choice of c_{13} in (36). Finally, we are going to bound the third term in (38), $P[B_n]$. Define the ladder times $(\mu_j)_j$ of the process $(M_n)_{n \geq 0}$ recursively by

$$\mu_0 := 0 \quad \text{and} \quad \mu_{j+1} := \inf\{n > \mu_j \mid M_n > M_{\mu_j}\}.$$

In analogy to $(i_n)_n$ for $(\tau_i)_i$ we define for $(\mu_j)_j$ the increasing sequence $(j_n)_n$ by $j_n := \sup\{j \geq 0 \mid \mu_j \leq n\}$ and note that $\mu_{j_n} \leq n < \mu_{j_n+1}$ and $M_n = \|X_{\mu_{j_n}}\|$. Hence on the event B_n ,

$$\|X_{\mu_{j_n}} - X_n\| \geq \|X_{\mu_{j_n}}\| - \|X_n\| = M_n - \|X_n\| \geq (c_{13} - c_1)n.$$

Since the walker takes steps of length one, this implies $n - \mu_{j_n} \geq (c_{13} - c_1)n$ and therefore, on the event B_n ,

$$(39) \quad \mu_{j_n} \leq \lfloor (1 - c_{13} + c_1)n \rfloor.$$

On the other hand, on B_n ,

$$(40) \quad d_n \stackrel{(27)}{=} d_{\tau_{i_n}} = \|X_{\tau_{i_n}} - X_{k(i_n)}\| \leq \|X_n\| + \|X_n - X_{\tau_{i_n}}\| + \|X_{k(i_n)}\| \\ \leq c_1 n + \Delta_{i_n} + M_n,$$

where we used again the fact that the steps have length one to estimate the second term and $k(i_n) \leq i_n \leq n$ to bound the third term. Therefore (40) can be estimated on B_n from above by

$$c_1 n + c_1 n + M_{\mu_{j_n}} \stackrel{(37)}{\leq} 2c_1 n + d_{\mu_{j_n}} \stackrel{(39)}{\leq} 2c_1 n + d_{\lfloor (1-c_{13}+c_1)n \rfloor}.$$

Consequently, by Chebyshev's inequality and Lemma 7,

$$P[B_n] \leq P[d_n - d_{\lfloor (1-c_{13}+c_1)n \rfloor} \leq 2c_1n] \leq c_{12}(n+1)e^{(2c_1-c_{11}(c_{13}-c_1))n},$$

which decays exponentially in n due to the choice of c_{13} and c_1 in (36). \square

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