MULTI-EXCITED RANDOM WALKS ON INTEGERS

By Martin P.W. Zerner

Abstract. We introduce a class of nearest-neighbor integer random walks in random and non-random media, which includes excited random walks considered in the literature. At each site the random walker has a drift to the right, the strength of which depends on the environment at that site and on how often the walker has visited that site before. We give exact criteria for recurrence and transience and consider the speed of the walk.

1. Introduction

The results of the present paper are best illustrated by the following example.

Example 1. We put two cookies on each integer and launch a nearest neighbor random walker at the origin. Whenever there is at least one cookie at the random walker's current position, the walker eats exactly one of these cookies, thus removing it from this site, and then jumps independently of its past to the right with probability p and to the left with probability $1-p$, where $p \in [1/2, 1]$ is a fixed parameter. Whenever there is no cookie left at the random walker's current position, the walker jumps independently of its past to the left or right with equal probability $1/2$.

We shall show a phase transition in the recurrence and transience behavior of the walk, see Theorem 12: If $1/2 \le p \le 3/4$ then the walker will visit its starting point 0 almost surely infinitely often. However, if $p > 3/4$ then the walker will visit 0 almost surely only finitely many times. Moreover, the probability that the walker will never return to 0 is $(2-1/(2p-1))_+$, see Figure 1 and Theorem 18. Finally, for all $p < 1$ the walk has zero speed, even if it is transient, see Theorem 19. $\square$

This example can be formalized and generalized as follows. A cookie environment is an element

$$\omega = (\omega(z))_{z\in\mathbb{Z}} = ((\omega(z,i))_{i\ge 1})_{z\in\mathbb{Z}} \quad \in \quad \Omega_+ := \left([1/2,1]^{\mathbb{N}}\right)^{\mathbb{Z}},$$

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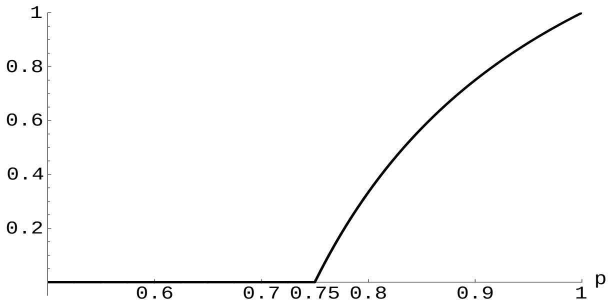

Figure 1. The probability that a random walker will never return to its starting point if there are two cookies with parameter p each at each integer.

where $\mathbb{N}$ denotes the strictly positive integers. We will refer to $\omega(z,i)$ as to the strength of the i-th cookie at z. This is the probability for the random walker to jump from z to $z+1$ if it is currently visiting z for the i-th time. Tomasz Komorowski suggested to consider the cookies as *bribes* which push an otherwise unbiased walker to the right. Noam Berger proposed the name *Brownie motion* for this walk.

More formally, given a starting point $x \in \mathbb{Z}$ and a cookie environment $\omega \in \Omega_+$, we consider an integer valued process $(X_n)_{n\geq 0}$ on some suitable probability space $(\Omega, \mathcal{F}, P_{x,\omega})$ for which the process of its history $(H_n)_{n\geq 0}$ defined by $H_n := (X_m)_{0\leq m\leq n} \in \mathbb{Z}^{n+1}$ is a Markov chain, which satisfies $P_{x,\omega}$-a.s.

$$\begin{aligned} P_{x,\omega}[X_0 = x] &= 1, \\ P_{x,\omega}[X_{n+1} = X_n + 1 \mid H_n] &= \omega\left(X_n, \#\{m \leq n \mid X_m = X_n\}\right), \\ P_{x,\omega}[X_{n+1} = X_n - 1 \mid H_n] &= 1 - \omega\left(X_n, \#\{m \leq n \mid X_m = X_n\}\right). \end{aligned}$$

Note that $(X_n)_n$ itself is in general not a Markov chain since its transition probabilities depend on the history of the process. In Example 1 we have chosen $x = 0$ as starting point and the cookie environment $\omega \in \Omega_+$ with $\omega(z) = (p, p, 1/2, 1/2, 1/2, \ldots)$ for all $z \in \mathbb{Z}$.

This model generalizes in part one-dimensional excited random walks (ERW) and random walk perturbed at its extrema, also called pq walks, see Benjamini-Wilson [1] and Davis [2] for results and references, regarding also continuous space and time analogues. For higher dimensional ERWs see [1], Kozma [6], and Volkov [11].

The intersection of our model, which we call multi-ERW, with one-dimensional ERW as defined in [1] and [2] deals, in our language, with cookie environments of the form $\omega(z) = (p, 1/2, 1/2, 1/2, \ldots)$ for all $z \in \mathbb{Z}$, where $p \in [1/2, 1]$ is fixed. In such an environment the walker is excited, i.e. biased to the right, only on the first visit to a site. We call such random walks once-excited.

The novelty of our model of multi-ERW is that it permits different levels of excitement for different visits to a site. Moreover, the excitement levels may vary randomly from site to site, see Section 4 for details. While once-ERW is recurrent for all $p < 1$ (see [1, p. 86]), multi-ERW exhibits a more complex recurrence and transience behavior, as highlighted in Example 1.

Our motivation for the study of multi-ERW on integers came from the problem posed at the end of [1], as to whether once-ERW on $\mathbb{Z}^2$ has positive speed. Although we do not see how to prove this, morally, this should be true if once-ERW on the strip $\mathbb{Z} \times \{0, \dots, k\}$ has positive speed for k large enough. By shrinking the width of the strip to one while piling the k cookies from $(z, 1), \dots, (z, k)$ onto the single site z, $z \in \mathbb{Z}$, we then arrive at the present model, hoping that this transformation did not completely change the character of the model.

A second source of motivation was to find a unifying model which includes both once-ERW and random walks in random environments (RWRE, see e.g. [8], [9], [12]) as special cases, see Remark 2 in Section 4 for details.

Let us now describe how the remainder of the present paper is organized. Section 2 provides basic lemmas which will be used throughout the paper. After some preparation we will also describe in Remark 1 the main idea behind the proof of the phase transition described in Example 1. In Section 3 we introduce the notion of recurrence and transience of states in fixed environments ω. This will be used in Section 4, which contains our main result Theorem 12, a sufficient and necessary criterion for recurrence in stationary and ergodic environments. In Section 5 we investigate random walks which one after the other live on the environment left over by the previous random walk. Section 6 is devoted to a strong law of large numbers for the walk in a stationary and ergodic environment. Section 7 deals with monotonicity of the return probability and the speed, two quantities, which are explicitly computed in the last section for the case in which the excitement is gone after the second visit.

2. NOTATION AND PRELIMINARIES

Let

$$T_k := \inf\{n \geq 0 \mid X_n = k\}$$

be the first passage time of k. The following lemma will be generalized in Lemma 15 by a different technique.

Lemma 1. *For all $x < y < z$ and all $\omega \in \Omega_+$,*

$$P_{y,\omega}[T_x < T_z] \leq \frac{z-y}{z-x}.$$

In particular, by letting $x \to -\infty$ we see that T_z is $P_{y,\omega}$-a.s. finite.

Proof. We couple $(X_n)_{n\geq 0}$ to a simple symmetric random walk $(Y_n)_{n\geq 0}$ starting at y such that almost surely $Y_n \leq X_n$ for all $n \geq 0$. To this end, we may assume that there is a sequence $(U_n)_{n\geq 0}$ of independent random variables on Ω which are uniformly distributed on $[0,1]$. If the walk $(X_n)_n$ visits at time n a site x for the j-th time ($j \geq 1$) then it moves to the right in the next step iff $U_n < \omega(x,j)$, whereas the walk $(Y_n)_n$ jumps to the right iff $U_n < 1/2$. Then $(X_n)_n$ is an ERW in the environment ω whereas $(Y_n)_n$ is a simple symmetric random walk. Since $\omega(x,j) \geq 1/2$ we get $Y_n \leq X_n$ almost surely by induction over n. Therefore, if $(X_n)_n$ exits the interval $]x,z[$ in x then so does $(Y_n)_n$, which has probability $(z-y)/(z-x)$. $\square$

The average displacement of the walk after having eaten a cookie of strength p is $2p-1$. Therefore,

$$\delta^x(\omega) := \sum_{i\geq 1}(2\omega(x,i)-1)$$

is the total drift stored in the cookies at site x in the environment ω. The drift contained in the cookies at site x which have been eaten before time $n \in [0,\infty]$ will be called

$$D_n^x := \sum_{i=1}^{\#\{m<n \mid X_m = x\}} (2\omega(x,i)-1).$$

To distinguish between recurrence and transience we will differentiate between cookies on non-negative and negative integers. Therefore, we introduce

$$D_n^+ := \sum_{x\geq 0} D_n^x, \qquad D_n^- := \sum_{x<0} D_n^x, \quad \text{and} \quad D_n := D_n^+ + D_n^-.$$

Lemma 2. *Let $\omega \in \Omega_+$ such that*

$$\liminf_{i\to\infty} \frac{1}{i} \sum_{y=-i}^{0} (2\omega(y,1)-1) > 0. \tag{1}$$

Then for all $x, k \in \mathbb{Z}$ with $k \geq x$,

$$E_{x,\omega}[D_{T_k}] = k - x. \tag{2}$$

Note that for simple symmetric random walk, i.e. for $\omega \equiv 1/2$, $D_{T_k} = 0$ $P_{x,\omega}$-a.s.. Hence assumption (1) is essential.

Proof. By shifting ω by x to the left we may assume without loss of generality $k \geq x = 0$. Consider the process $M_n := X_n - D_n$ ($n \geq 0$). It is standard to check that $(M_n)_{n\geq 0}$ is a $P_{0,\omega}$-martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$

generated by $(X_n)_{n\geq 0}$. Therefore, by the Optional Stopping Theorem for all $n \geq 0$,

$$0 = E_{0,\omega}[M_{T_k\wedge n}] = E_{0,\omega}[X_{T_k\wedge n}, T_k \leq n] + E_{0,\omega}[X_{T_k\wedge n}, n < T_k] - E_{0,\omega}[D_{T_k\wedge n}]$$

and consequently,

$$E_{0,\omega}[D_{T_k\wedge n}] = kP_{0,\omega}[T_k \leq n] + E_{0,\omega}[X_n, n < T_k]. \tag{3}$$

(Here we use the notation $E[X, A]$ with X a random variable and A an event as a short form for $E[X\mathbf{1}_A]$.) Now consider (3) as $n \to \infty$. Since $\omega \in \Omega_+$, the left hand side of (3) tends by monotone convergence to $E_{0,\omega}[D_{T_k}]$. Moreover, the first term on the right-hand side goes to k. Consequently,

$$E_{0,\omega}[D_{T_k}] = k + \lim_{n\to\infty} E_{0,\omega}[X_n, n < T_k]. \tag{4}$$

Hence all that remains to be shown is

$$\lim_{n\to\infty} E_{0,\omega}[X_n, n < T_k] = 0. \tag{5}$$

Now observe that for all $n \geq 0$,

$$\min_{m\leq T_k} X_m \leq X_n\mathbf{1}\{n < T_k\} \leq k \qquad P_{0,\omega}\text{-a.s.} \tag{6}$$

Since $X_n\mathbf{1}\{n < T_k\}$ tends $P_{0,\omega}$-a.s. to 0 as $n \to \infty$, (5) will follow by dominated convergence once we show that the two bounds for $X_n\mathbf{1}\{n < T_k\}$ given in (6) are integrable with respect to $E_{0,\omega}$. For the upper bound, the constant k, this is trivial. Thus it suffices to show that the lower bound, the non-positive random variable $\min_{m\leq T_k} X_m$, has finite $E_{0,\omega}$-expectation. Denote by γ the left hand side of (1). Then

$$E_{0,\omega}\Big[-\min_{m\leq T_k} X_m\Big] \leq E_{0,\omega}[2D_{T_k}/\gamma] \tag{7}$$

$$+ E_{0,\omega}\Big[-\min_{m\leq T_k} X_m,\ -\min_{m\leq T_k} X_m > 2D_{T_k}/\gamma\Big]. \tag{8}$$

The term on the right hand side of (7) is finite since (4) and (6) imply $E_{0,\omega}[D_{T_k}] \leq 2k$. The term in (8) equals

$$\sum_{i\geq 1} iP_{0,\omega}\Big[-\min_{m\leq T_k} X_m = i, D_{T_k} < \gamma i/2\Big]. \tag{9}$$

Observe that on the event $\{T_{-i} < T_k\}$ the walk has eaten all the first cookies between $-i$ and 0 before reaching k. Therefore,

$$D_{T_k} \geq \sum_{y=-i}^{0} (2\omega(y,1) - 1).$$

Therefore, (9) is less than or equal to

$$\sum_{i\geq 1} i\mathbf{1}\left\{\frac{1}{i}\sum_{y=-i}^{0}(2\omega(y,1)-1)<\frac{\gamma}{2}\right\}, \tag{10}$$

which is finite since, due to the choice of γ, only finitely many indicator functions in (10) do not vanish. □

Remark 1. We are now ready to present the idea of the proof of the recurrence and transience behavior in the two-cookie case described in Example 1. This will be made rigorous and more general in Theorem 12. Roughly speaking, (2) states that an ERW starting at 0 should have eaten not more and not less than $k/(2p-1)$ cookies by the time it reaches k. Compare this number to the total number $2k$ of cookies available between 0 and $k-1$. If $2k < k/(2p-1)$ then the walker needs to visit once in a while negative integers in order to meet its cookie needs because there are not enough cookies available on the positive integers. This makes the walker recurrent.

On the other hand, if $2k > k/(2p-1)$ then the walker cannot afford to return to 0 too often because on its way back to 0 from its up-to-date maximum value, say $k-1$, the ERW will eat all the remaining cookies between 0 and $k-1$, thus consuming at least $2k$ cookies before it reaches k. This would be more than the $k/(2p-1)$ cookies the ERW should eat before reaching k. Therefore, the walker has to be transient. □

In the following we are concerned with the probabilities of the events

$$R_k := \{X_n = k \ i.o.\} = \limsup_{n\to\infty}\{X_n = k\} \quad (k\in\mathbb{Z})$$

that any given site k is visited infinitely often. For a precise statement of the following result we introduce the sequences $(\tau_{k,m})_{m\geq 1}$ $(k\in\mathbb{Z})$ defined by

$$\tau_{k,0} := -1 \quad \text{ and } \quad \tau_{k,m+1} := \inf\{n > \tau_{k,m} \mid X_n \geq k\}.$$

They enumerate the times n at which $X_n \geq k$. Note that these times are stopping times with respect to $(\mathcal{F}_n)_{n\geq 0}$. Moreover, they are $P_{x,\omega}$-a.s. finite $(x\in\mathbb{Z}, \omega\in\Omega_+)$ since $T_r < \infty$ for all $r \geq 0$.

The following result is an analogue of the so-called restriction principle for vertex-reinforced jump processes on integers, see [3, p.287]. A related result for Brownian motion perturbed at its extrema is [7, Proposition 1].

Lemma 3. *Let $x_1, x_2 \leq k$ and $\omega_1, \omega_2 \in \Omega_+$ such that $\omega_1(x) = \omega_2(x)$ for all $x \geq k$. Then $(X_{\tau_{k,m}})_{m\geq 0}$ has the same distribution under P_{x_1,ω_1} as under P_{x_2,ω_2}. In particular,*

$$P_{x_1,\omega_1}[R_k] = P_{x_2,\omega_2}[R_k]. \tag{11}$$

This lemma, which we shall give no formal proof for, says that the behavior of the walk to the right of k does not depend on the environment to the left of k nor on where to the left of k the walk started.

This is obvious for the following reason: While the walk is in $k_- := \{x \mid x < k\}$ it can change the environment only in k_-. Therefore, after entering the set $k_+ := \{x \mid x \geq k\}$, which will always happen through the site k, the walk has no recollection of what it did in k_-, because its steps inside k_+ are directed by the environment on k_+ only. It is true that the lengths $(\tau_{k,m+1} - \tau_{k,m})_{m\geq 0}$ of the excursions to k_-, and therefore $(\tau_{k,m})_{m\geq 0}$ itself, do depend on the behavior of the walk and the environment in k_-. However, not the times of visits to k_+ are recorded in $(X_{\tau_{k,m}})_{m\geq 0}$, but only the relative order in which the sites in k_+ are visited.

For future purposes let us introduce some notation for cookie environments partially visited by the walk. For any $\omega \in \Omega_+$ and any finite sequence $(x_n)_{n\leq m}$ of integers we define $\psi\left(\omega, (x_n)_{n\leq m}\right) \in \Omega_+$ by

$$\psi(\omega, (x_n)_{n\leq m})(x, i) := \omega\left(x, i + \#\{n < m \mid x_n = x\}\right). \tag{12}$$

This is the environment we obtain form ω by following the path $(x_n)_{n<m}$ and removing the top cookie each time we visit a site. Note that in the definition of ψ we do not remove the cookie from the final site x_m.

3. Recurrence and Transience in Deterministic Environments

In this section we establish recurrence and transience criteria for fixed environments.

Lemma 4. *Let $x, y, z \in \mathbb{Z}$ with $y < z$ and $\omega \in \Omega_+$. Then $P_{x,\omega}$-a.s. $R_y \subseteq R_z$.*

Proof. After each visit to y the conditional probability, given the past, that the walker will first visit z before returning to y is at least 2^{y-z}, since the walker can always choose the direct path from y to z and will never encounter a negative drift. The second Borel Cantelli lemma (e.g. [4, Ch.4 (3.2)]) then yields the statement. ☐

Proposition 5. *Let $\omega \in \Omega_+$ and $y \in \mathbb{Z}$. Then either for all $x \in \mathbb{Z}$, $P_{x,\omega}[R_y] = 0$ or for all $x \in \mathbb{Z}$, $P_{x,\omega}[R_y] = 1$.*

If $P_{x,\omega}[R_y] = 1$ for all $x \in \mathbb{Z}$ then we shall call y *ω-recurrent.* Otherwise, i.e. if $P_{x,\omega}[R_y] = 0$ for all $x \in \mathbb{Z}$, y is called *ω-transient.* A different characterization of ω-transience will be given in Lemma 8.

Proof of Proposition 5. Let $x \in \mathbb{Z}$ with $P_{x,\omega}[R_y] > 0$. All we have to show is that

$$\forall k \in \mathbb{Z} \quad P_{k,\omega}[R_y] = 1. \tag{13}$$

By Lemma 4, for all $z > y$,

$$0 < P_{x,\omega}[R_y] = P_{x,\omega}[R_y \cap R_z] \leq P_{x,\omega}[(X_n, X_{n+1}) = (z, z-1) \text{ i.o.}].$$

Therefore, $\sum_i (1 - \omega(z,i)) = \infty$ for all $z > y$, because otherwise the walk would jump $P_{x,\omega}$-a.s. only finitely many times from z to $z-1$ due to the convergence part of the Borel Cantelli lemma. However, since the decisions whether to jump from z to $z-1$ or to $z+1$ are made independently of each other under $P_{k,\omega}$ ($k \in \mathbb{Z}$), the divergence part of the Borel Cantelli lemma then implies that for all $k \in \mathbb{Z}$ and all $z > y$ we have $P_{k,\omega}$-a.s. $R_z \subseteq R_{z-1}$. Since the opposite inclusion holds anyway due to Lemma 4, we have

$$\forall k \in \mathbb{Z} \quad \forall z \geq y \quad R_z \overset{P_{k,\omega}}{=} R_y. \tag{14}$$

Since $R_y \in \sigma\left(\bigcup_{z \geq 0} \mathcal{F}_{T_z}\right)$ the martingale convergence theorem applied to the martingale $(P_{x,\omega}[R_y \mid \mathcal{F}_{T_z}])_{z\geq 0}$ yields that $P_{x,\omega}$-a.s.,

$$\mathbf{1}_{R_y} = \lim_{z\to\infty} P_{x,\omega}\left[R_y \mid \mathcal{F}_{T_z}\right] \overset{(14)}{=} \lim_{z\to\infty} P_{x,\omega}\left[R_z \mid \mathcal{F}_{T_z}\right].$$

By the strong Markov property this is $P_{x,\omega}$-a.s. equal to $\lim_z P_{z,\psi(\omega, H_{T_z})}\left[R_z\right]$. We are now going to remove z in two steps from the expression we are taking the limit $z \to \infty$ of. First observe that the environments $\psi\left(\omega, H_{T_z}\right)$ and ω coincide on all sites greater than or equal to z. Hence, due to (11), this limit is equal to $\lim_z P_{k,\omega}\left[R_z\right]$ for all $k \in \mathbb{Z}$. However, by (14) this is the same as $\lim_z P_{k,\omega}\left[R_y\right] = P_{k,\omega}\left[R_y\right]$. Thus we have proven

$$\forall k \in \mathbb{Z} \quad \mathbf{1}_{R_y} = P_{k,\omega}\left[R_y\right] \quad P_{x,\omega}\text{-a.s.}. \tag{15}$$

Applying this first to $k = x$ gives that $\mathbf{1}_{R_y}$ is $P_{x,\omega}$-a.s. equal to $P_{x,\omega}\left[R_y\right]$. Therefore, $P_{x,\omega}\left[R_y\right]$ has to be equal to 1 since it is positive by assumption. Applying (15) once again for general k then yields (13). ☐

Example 2. Let $x \in \mathbb{Z}$ and define $\omega \in \Omega_+$ by $\omega(y,i) = 1/2$ if $y \neq 0$ and $\omega(0,i) = 1 - (i+1)^{-2}$ for all $i \geq 1$. Since $\sum_i (i+1)^{-2}$ converges, the Borel Cantelli lemma implies that negative integers are ω-transient. On the other hand, non-negative integers are ω-recurrent because simple symmetric random walk is recurrent. ☐

We consider the following as the heart of the idea outlined in Remark 1.

Lemma 6. *Let $\omega \in \Omega_+$ such that 0 is ω-transient. Then*

$$\lim_{K\to\infty} \frac{E_{0,\omega}\left[D^+_{T_K}\right]}{K} = 1.$$

Proof. Since $\mathbf{1}_{R_0}$ and $D^+_{T_K}$ are functions of $\left(X_{\tau_{0,m}}\right)_{m\geq 0}$ and $(\omega(x))_{x\geq 0}$ we may change ω due to Lemma 3 at negative sites without changing ω-transience of 0 and $E_{0,\omega}\left[D^+_{T_K}\right]$. Hence we may assume without loss of generality that ω satisfies (1). For $k \geq 1$ consider the possibly infinite stopping time

$$\sigma_k := \inf\{n > T_{k-1} \mid X_n = 0\} \quad \text{and the event} \quad A_k := \{\sigma_k < T_k\}$$

that the walk after hitting $k-1$ for the first time, returns to 0 before it reaches k. Note that $A_k \in \mathcal{F}_{T_k}$. Since 0 is ω-transient, A_k occurs $P_{0,\omega}$-a.s. only for finitely many k's. Hence by the second Borel Cantelli lemma,

$$\sum_{k \geq 1} P_{0,\omega}\left[A_k \mid \mathcal{F}_{T_{k-1}}\right] < \infty \quad P_{0,\omega}\text{-a.s.}. \tag{16}$$

Now let $\varepsilon > 0$. Omitting in (16) those k's which are not elements of the set $S_\varepsilon := \{k \geq 1 \mid Y_k > \varepsilon/k\}$, where $Y_k := P_{0,\omega}\left[A_k \mid \mathcal{F}_{T_{k-1}}\right]$, we obtain

$$\sum_{k \in S_\varepsilon} \frac{1}{k} < \infty \quad P_{0,\omega}\text{-a.s.}. \tag{17}$$

Consequently, S_ε has $P_{0,\omega}$-a.s. upper density 0, i.e. $r_K := \#(\{1, \dots, K\} \cap S_\varepsilon)/K \to 0$ as $K \to \infty$. Indeed, otherwise, if $\limsup_K r_K > 2\varepsilon > 0$ for some $\varepsilon > 0$ then there is $(K_m)_m$ with $r_{K_m} > 2\varepsilon$ and $K_{m+1} > K_m/\varepsilon$ for all m. Splitting S_ε into the sets $S_{\varepsilon,m} := \{k \in S_\varepsilon \mid K_m < k \leq K_{m+1}\}$ we see that $\#S_{\varepsilon,m} \geq \varepsilon K_{m+1}$. Therefore, $\sum_{S_{\varepsilon,m}} 1/k \geq \varepsilon$, which summed over m would be infinite, contradicting (17).

Thus, since $0 \leq r_K \leq 1$, we get by dominated convergence,

$$0 = \lim_{K \to \infty} E_{0,\omega}\left[r_K\right] \;=\; \lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} P_{0,\omega}\left[Y_k > \varepsilon/k\right]. \tag{18}$$

By the strong Markov property, $P_{0,\omega}$-a.s.,

$$Y_k = P_{k-1,\psi\left(\omega, H_{T_{k-1}}\right)}[T_0 < T_k] \;\leq\; \frac{1}{k}$$

due to Lemma 1. Therefore,

$$E_{0,\omega}[Y_k] = E_{0,\omega}[Y_k,\ Y_k > \varepsilon/k] + E_{0,\omega}[Y_k,\ Y_k \leq \varepsilon/k] \leq \frac{1}{k} P_{0,\omega}[Y_k > \varepsilon/k] + \frac{\varepsilon}{k}$$

and hence

$$P_{0,\omega}[Y_k > \varepsilon/k] \geq k E_{0,\omega}[Y_k] - \varepsilon.$$

Substituting this into (18) yields

$$0 \geq \limsup_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} (k E_{0,\omega}[Y_k] - \varepsilon) \;=\; -\varepsilon + \limsup_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} k P_{0,\omega}[A_k].$$

Letting $\varepsilon \searrow 0$ gives

$$0 = \lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{K} kP_{0,\omega}[A_k]. \tag{19}$$

For abbreviation set $\Delta_k^- := D_{T_k}^- - D_{T_{k-1}}^-$ for $k \geq 1$. This is the total drift of the cookies on negative sites which have been eaten between T_{k-1} and T_k. Since $\Delta_k^- = 0$ on A_k^c and since $A_k \in \mathcal{F}_{\sigma_k}$ we have

$$E_{0,\omega}[\Delta_k^-] = E_{0,\omega}[\Delta_k^-, A_k] = E_{0,\omega}\left[E_{0,\omega}[\Delta_k^- \mid \mathcal{F}_{\sigma_k}], A_k\right]. \tag{20}$$

By the strong Markov property this is equal to

$$E_{0,\omega}\left[E_{0,\psi\left(\omega,H_{\sigma_k}\right)}\left[D_{T_k}^-\right], A_k\right], \tag{21}$$

where we note that $\psi\left(\omega, H_{\sigma_k}\right)$ is well-defined on A_k since $\sigma_k < \infty$ on A_k. Also observe that on A_k, $\psi\left(\omega, H_{\sigma_k}\right)$ differs only at finitely many sites from ω and therefore satisfies (1) since ω does so. Hence we may use Lemma 2 and $D_{T_k}^- \leq D_{T_k}$ to conclude from (20) and (21) that $E_{0,\omega}[\Delta_k^-] \leq kP_{0,\omega}[A_k]$. Therefore, due to (19),

$$0 = \lim_{K\to\infty}\frac{1}{K}\sum_{k=1}^{K} E_{0,\omega}[\Delta_k^-] = \lim_{K\to\infty}\frac{E_{0,\omega}\left[D_{T_K}^-\right]}{K}.$$

The claim now follows from $E_{0,\omega}\left[D_{T_K}\right] = K$, see Lemma 2, and $D_{T_K} = D_{T_K}^+ + D_{T_K}^-$. ☐

Lemma 6 and $D_{T_K}^+ \leq \sum_{x=0}^{K-1}\delta^x$ imply the following sufficient criterion for recurrence.

Corollary 7. *0 is ω-recurrent if*

$$\liminf_{K\to\infty}\frac{1}{K}\sum_{x=0}^{K-1}\delta^x(\omega) < 1.$$

We conclude this section by showing that the probability of never returning to the starting point is positive whenever the starting point is ω-transient. In Section 8 we shall explicitly compute this probability in some cases.

Lemma 8. *If 0 is ω-transient then $P_{0,\omega}[\forall n > 0 : \ X_n > 0] > 0$.*

Proof. Since 0 is ω-transient and since all positive integers are $P_{0,\omega}$-a.s. eventually hit by the walk, we have $P_{0,\omega}$-a.s. $X_n > 0$ for n large. Now we distinguish two cases.

If 1 is ω-recurrent then it follows from the divergence part of the Borel Cantelli lemma that $\sum_i(1-\omega(1,i)) < \infty$. Since 0 is ω-transient this implies that

$$\begin{aligned} P_{0,\omega}[\forall n > 0 : \ X_n > 0] &= \omega(0,1)\ P_{1,\omega}\left[\forall n : \ (X_n = 1 \Rightarrow X_{n+1} = 2)\right] \\ &\geq \omega(0,1) \prod_{i\geq 0} \omega(1,i) \ > \ 0 \end{aligned}$$

as required.

If 1 is ω-transient then there is some $K > 0$ and a nearest neighbor path $(x_n)_{n=0}^K$ of integers with $x_0 = 0$ and $x_K = 2$ such that

$$0 < P_{0,\omega}[H_K = (x_n)_{n=0}^K, \ \forall n \geq K : \ X_n \geq 2] = UV,$$

where

$$U := P_{0,\omega}[H_K = (x_n)_{n=0}^K] \quad \text{and} \quad V := P_{2,\psi\left(\omega,(x_n)_{n\leq K}\right)}[\forall n \geq 0 : \ X_n \geq 2].$$

Hence $U > 0$ and $V > 0$. Now we are doing some surgery on $(x_n)_{n=0}^K$ by cutting out the excursions from 1 toward 0. To this end, we let $n_1, \dots, n_{k-1}$ be the enumeration of the times $1 \leq n < K$ for which $x_n \geq 2$ or $x_{n+1} \geq 2$ and set $n_k := K$. Then $y_0 := 0$ and $y_i := x_{n_i}$ for $1 \leq i \leq k$ defines a nearest neighbor path $(y_n)_{n=0}^k$, which starts at 0, ends at 2, and is strictly positive in between. Therefore,

$$P_{0,\omega}[\forall n > 0 : \ X_n > 0] \ \geq \ P_{0,\omega}[H_k = (y_n)_{n=0}^k, \ \forall n \geq k : \ X_n \geq 2] = uv,$$

where

$$u := P_{0,\omega}[H_k = (y_n)_{n=0}^k] \quad \text{and} \quad v := P_{2,\psi\left(\omega,(y_n)_{n\leq k}\right)}[\forall n \geq 0 : \ X_n \geq 2].$$

Consequently, it suffices to show that $u > 0$ and $v > 0$. By Lemma 3, $V = v$ because $\psi\left(\omega, (x_n)_{n\leq K}\right)(z) = \psi\left(\omega, (y_n)_{n\leq k}\right)(z)$ for $z \geq 2$ since $(y_n)_{n\leq k}$ visits each number ≥ 2 as often as $(x_n)_{n\leq K}$ does. Hence, $v > 0$ since $V > 0$.

As for U and u, both are products of finitely many factors of the form $\omega(x,i)$ and $1 - \omega(x,i)$. We have to make sure that none of the factors involved in u is 0. Since $\omega(x,i) \geq 1/2$, only terms of the form $1 - \omega(x,i)$ are critical. Having a factor $1 - \omega(x,i) = 0$ in u, which is not present in U, means that the path $(x_n)_n$ jumps to $x + 1$ after the i-th visit to x whereas $(y_n)_n$ jumps to $x - 1$ after the i-th visit to x. Since there are no steps from 1 to 0 in $(y_n)_n$, any such x must be at least 2. However, for any $x \geq 2$ all the steps from x to $x + 1$ and from x to $x - 1$ happen in the same order for $(x_n)_n$ as for $(y_n)_n$, thus giving rise to the same factors $\omega(x,i)$ and $1 - \omega(x,i)$ in U and u. Consequently, $u > 0$ since $U > 0$. □

4. Recurrence and Transience in Random Environments

For $\mathbb{P}$ a probability measure on Ω_+, equipped with its canonical σ-field, and for $x \in \mathbb{Z}$ we define the semi-direct product $P_x := \mathbb{P} \times P_{x,\omega}$ on $\Omega_+ \times \Omega$ by $P_x[\cdot] := \mathbb{E}\left[P_{x,\omega}[\cdot]\right]$. This is the so-called *annealed* measure which we get after averaging the *quenched* measure $P_{x,\omega}$ over $\mathbb{P}$. Here the expectation operators for $\mathbb{P}$ and P_x are denoted by $\mathbb{E}$ and E_x, respectively.

Not much can be said about recurrence and transience for general $\mathbb{P}$. A conclusive answer can be given if $(\omega(x))_{x\geq 0}$ is stationary and ergodic under $\mathbb{P}$ with respect to the shift on $\mathbb{Z}$. Stationarity of $(\omega(x))_{x\geq 0}$ means that the distribution of $f\left(\theta^x(\omega)\right)$ under $\mathbb{P}$ for $x \geq 0$ does not depend on x, where $f : \Omega_+ \to \Omega_+$ is defined by

$$(f(\omega))(x) := \begin{cases} \omega(x) & \text{if } x \geq 0 \\ 1/2 & \text{if } x < 0 \end{cases}$$

and $\theta^x : \Omega_+ \to \Omega_+$ is the canonical shift of ω to the left by x steps ($x \in \mathbb{Z}$) as defined by $(\theta^x(\omega))(z) := \omega(z + x)$.

Remark 2. In the special case where $\omega(x, i)$ is $\mathbb{P}$-a.s. for all $x \in \mathbb{Z}$ constant in i (but not necessarily constant in x), we get a one-dimensional *random walk in random environment* (RWRE) with a non-negative drift. The general model of RWRE for $d = 1$, which allows positive and negative drifts, has been studied e.g. by Solomon [8], see also [9] and [12] for results and references. For a unifying model which includes RWRE and ERW we would have to replace Ω_+ by $\Omega_\pm := \left([-1, 1]^{\mathbb{N}}\right)^{\mathbb{Z}}$. Our methods do not immediately work in this case. □

Theorem 9. *If $\omega = (\omega(x))_{x\geq 0}$ is stationary and ergodic under $\mathbb{P}$ then either every $x \geq 0$ is ω-recurrent for $\mathbb{P}$-almost all realizations of ω or every $x \geq 0$ is ω-transient for $\mathbb{P}$-almost all realizations of ω.*

In the first case mentioned above, i.e. when every $x \geq 0$ is $\mathbb{P}$-a.s. ω-recurrent, we shall call $(X_n)_n$ *recurrent*, in the second case $(X_n)_n$ is called *transient*.

Proof. For all $x \geq 0$ and all $\omega \in \Omega_+$ by Lemma 4,

$$P_{0,\omega}[R_0] \leq P_{0,\omega}[R_x] = P_{-x,\theta^x(\omega)}[R_0] \overset{(11)}{=} P_{0,f(\theta^x(\omega))}[R_0]. \tag{22}$$

Consequently, taking $\mathbb{E}$-expectations in (22) and using stationarity yields

$$P_0[R_0] \leq \mathbb{E}\left[P_{0,f(\theta^x(\omega))}[R_0]\right] = \mathbb{E}\left[P_{0,f(\omega)}[R_0]\right] \overset{(11)}{=} \mathbb{E}\left[P_{0,\omega}[R_0]\right] = P_0[R_0].$$

Therefore, the inequality in (22) is in fact $\mathbb{P}$-a.s. an equality. Hence, $P_{0,f(\theta^x(\omega))}[R_0]$ does $\mathbb{P}$-a.s. not depend on x. Moreover, the sequence $P_{0,f(\theta^x(\omega))}[R_0]$ $(x \geq 0)$ is ergodic because it is of the form $g\left((\omega(y + x))_{y\geq 0}\right)$ $(x \geq 0)$. Consequently, this sequence is $\mathbb{P}$-a.s. equal to a deterministic constant, which is either 0 or 1 by Proposition 5. □

The following lemma shows how the path inherits stationarity and/or ergodicity from the environment. Here $\mathbb{N}_0$ denotes the non-negative integers.

Lemma 10. *If* $(\omega(x))_{x\geq 0}$ *is stationary (resp. ergodic) under* $\mathbb{P}$ *then*

$$\xi := (\xi_k)_{k\geq 0} := \Big((\omega(x+k))_{x\geq 0}\, , \left(X_{\tau_{k,m}} - k\right)_{m\geq 0}\Big)_{k\geq 0}$$

is stationary (resp. ergodic) under P_0. *In particular, if* g *is a measurable function on* $([1/2,1]^{\mathbb{N}})^{\mathbb{N}_0} \times \mathbb{Z}^{\mathbb{N}_0}$ *with values in a measurable space then the sequence* $(g(\xi_k))_{k\geq 0}$ *is stationary (resp. ergodic) under* P_0 *if the sequence* $(\omega(x))_{x\geq 0}$ *is so under* $\mathbb{P}$.

Here ξ_k consists of the environment to the right of k and of the part of the trajectory to the right of k.

Proof. To prove stationarity of ξ we shall show that for all measurable subsets B of the codomain of ξ, $P_0\left[(\xi_k)_{k\geq K} \in B\right]$ is the same for all $K \geq 0$. For the proof of ergodicity we need to show that $P_0[A] \in \{0,1\}$ whenever there is a B as above such that

$$A = \{(\xi_k)_{k\geq K} \in B\} \qquad \text{for all } K \geq 0. \tag{23}$$

In both proofs the following identities will be used. For all $\omega \in \Omega_+$, $K \geq 0$, and B as above we have by the strong Markov property $P_{0,\omega}$-a.s.

$$\begin{aligned} P_{0,\omega}\left[(\xi_k)_{k\geq K} \in B \mid \mathcal{F}_{T_K}\right] &= P_{K,\psi\left(\omega,H_{T_K}\right)}\left[(\xi_k)_{k\geq K} \in B\right] \\ &= P_{0,\theta^K\left(\psi\left(\omega,H_{T_K}\right)\right)}\left[(\xi_k)_{k\geq 0} \in B\right]. \end{aligned} \tag{24}$$

Since $\theta^K\left(\psi\left(\omega, H_{T_K}\right)\right)(x)$ and $f\left(\theta^K(\omega)\right)(x)$ coincide $P_{0,\omega}$-a.s. for $x \geq 0$ we can apply Lemma 3 to see that (24) equals

$$P_{0,f(\theta^K(\omega))}\left[(\xi_k)_{k\geq 0} \in B\right] =: \eta_K(\omega). \tag{25}$$

Hence taking $E_{0,\omega}$-expectations in (24) and (25) yields

$$P_{0,\omega}\left[(\xi_k)_{k\geq K} \in B\right] = \eta_K(\omega). \tag{26}$$

If we now take $\mathbb{E}$-expectations on both side of (26) we get

$$\begin{aligned} P_0\left[(\xi_k)_{k\geq K} \in B\right] &= \mathbb{E}\left[P_{0,f(\theta^K(\omega))}\left[(\xi_k)_{k\geq 0} \in B\right]\right] \\ &= \mathbb{E}\left[P_{0,f(\omega)}\left[(\xi_k)_{k\geq 0} \in B\right]\right], \end{aligned} \tag{27}$$

if $(\omega(x))_{x\geq 0}$ is stationary under $\mathbb{P}$. Hence in this case the left hand side of (27) does not depend on K, which proves stationarity of $(\xi_x)_{x\geq 0}$.

For the proof of ergodicity of ξ we assume (23). Then η_K does not depend on K since the left-hand side of (26) does not. Moreover, since η_K is a function of the form $g((\omega(x+K))_{x\geq 0})$, the process $(\eta_K)_{K\geq 0}$ is ergodic if $(\omega(x))_{x\geq 0}$ is so. Therefore, in this case, being independent of K, η_K is $\mathbb{P}$-a.s. equal to a deterministic constant c. Going back from (26) via (25) to (24) we obtain that $\mathbb{P}$-a.s. $P_{0,\omega}[A \mid \mathcal{F}_{T_K}] = c$. Therefore, we have for $\mathbb{P}$-almost all ω by the martingale convergence theorem, $P_{0,\omega}$-a.s.

$$P_{0,\omega}[A \mid \mathcal{F}_{T_K}] \to P_{0,\omega}\Big[A \;\Big|\; \sigma\Big(\bigcup_K \mathcal{F}_{T_K}\Big)\Big] = \mathbf{1}_A \quad \text{as } K \to \infty.$$

Hence, either P_0-a.s. $c = P_{0,\omega}[A \mid \mathcal{F}_{T_K}] = 0$ or P_0-a.s. $c = P_{0,\omega}[A \mid \mathcal{F}_{T_K}] = 1$. Integration with respect to P_0 gives $P_0[A] \in \{0, 1\}$. □

The next result deals with the maximal cookie consumption per site.

Lemma 11. *If $(\omega(x))_{x\geq 0}$ is stationary under $\mathbb{P}$ then $E_0[D^x_\infty] \leq 1$ for all $x \geq 0$.*

Proof. Let $x \geq 0$. For ω given, the distribution of D^x_∞ depends only on the distribution of $(X_{\tau_{0,m}})_{m\geq 0}$. Therefore, we may assume due to Lemma 3 without loss of generality, that (1) is $\mathbb{P}$-a.s. fulfilled. Let $0 \leq k < K$. Then P_0-a.s.

$$D_{T_K} \;\geq\; D^+_{T_K} = \sum_{y=0}^{K-1} D^y_{T_K} \;\geq\; \sum_{y=0}^{K-1-k} D^y_{T_K} \;\geq\; \sum_{y=0}^{K-1-k} D^y_{T_{y+k}}$$

since $T_{y+k} < T_K$ for $y < K - k$. Consequently, we obtain from Lemma 2,

$$K = E_0[D_{T_K}] \;\geq\; \sum_{y=0}^{K-1-k} E_0\left[D^y_{T_{y+k}}\right] \;=\; (K-k)E_0\left[D^x_{T_{x+k}}\right]$$

due to stationarity of the sequences $(D^y_{T_{y+k}})_{y\geq 0}$ $(k \geq 0)$, which we get from Lemma 10 applied for all $k \geq 0$ to

$$g\left((\omega(x))_{x\geq 0}, (x_m)_{m\geq 0}\right) := \sum_{i=1}^{\#\{m<T_k((x_n)_n) \mid x_m=0\}} (2\omega(0,i) - 1).$$

Therefore, $E_0[D^x_{T_{x+k}}] \leq K/(K-k)$. Letting $K \to \infty$ gives $E_0[D^x_{T_{x+k}}] \leq 1$ for all $k \geq 0$. Monotone convergence as $k \to \infty$ then yields the claim. □

The second part of the following theorem classifies recurrent and transient walks.

Theorem 12. *Assume that $(\omega(x))_{x\geq 0}$ is stationary and ergodic under $\mathbb{P}$. Then*

$$E_0[D^x_\infty] = \min\left\{1, \mathbb{E}[\delta^0]\right\} \qquad \text{for all } x \geq 0. \tag{28}$$

Moreover, if

$$\mathbb{P}[\omega(0) = (1, 1/2, 1/2, 1/2, \ldots)] < 1 \tag{29}$$

then

$$(X_n)_{n\geq 0} \text{ is recurrent if and only if } \mathbb{E}\left[\delta^0\right] \leq 1. \tag{30}$$

Obviously, if (29) fails then $X_n = n$ P_0-a.s. for all n, which makes the walk transient although $\mathbb{E}[\delta^0] = 1$.

Proof. Lemma 10 applied to

$$g\left((\omega(x))_{x\geq 0}, (x_m)_{m\geq 0}\right) := \sum_{i=1}^{\#\{m|x_m=0\}} (2\omega(0,i)-1)$$

yields that $(D_\infty^k)_{k\geq 0}$ is stationary. Therefore, we may assume for the proof of (28) that $x = 0$. Moreover, since $\mathbf{1}_{R_0}$ and D_∞^0 are functions of $(\omega(x))_{x\geq 0}$ and $(X_{\tau_0,m})_{m\geq 0}$ we may assume thanks to Lemma 3 without loss of generality that the assumption (1) is satisfied for all $\omega \in \Omega_+$.

Due to Theorem 9, $(X_n)_n$ is either recurrent or transient. If it is recurrent then the walker will eat P_0-a.s. all the cookies at 0, which results in $D_\infty^0 = \delta^0$, thus showing $E_0[D_\infty^0] = E_0[\delta^0]$. Lemma 11 then yields $\mathbb{E}\left[\delta^0\right] \leq 1$ and (28) and the only-if direction of (30).

Now we assume that the walk is transient. Then by stationarity of $(D_\infty^k)_{k\geq 0}$, see above,

$$E_0[D_\infty^0] = \frac{1}{K}\sum_{k=0}^{K-1} E_0[D_\infty^k] \;=\; \mathbb{E}\left[\frac{1}{K}E_{0,\omega}\left[\sum_{k=0}^{K-1} D_\infty^k\right]\right] \;\geq\; \mathbb{E}\left[\frac{E_{0,\omega}\left[D_{T_K}^+\right]}{K}\right].$$

Since $E_{0,\omega}\left[D_{T_K}^+\right]/K \leq E_{0,\omega}\left[D_{T_K}\right]/K = 1$, see Lemma 2, we get by dominated convergence and Lemma 6 that $E_0[D_\infty^0] \geq 1$. Since the opposite inequality holds due to Lemma 11 we conclude

$$E_0[D_\infty^0] = 1. \tag{31}$$

Now consider the event

$$S := \left\{\sum_{i\geq 2}(2\omega(0,i)-1) > 0\right\}$$

that 0 has not all its drift stored in its first cookie. We claim $\mathbb{P}[S] > 0$. Indeed, otherwise $1 > \mathbb{E}[\delta^0]$ since we excluded the degenerate case in which the first cookie has $\mathbb{P}$-a.s. parameter 1. However, because of $\delta^0 \geq D_\infty^0$ this would contradict (31).

By Lemma 8, $\mathbb{P}$-a.s. $P_{0,\omega}[\forall n > 0 :\ X_n > 0] > 0$. Therefore, since $\mathbb{P}[S] > 0$ we have

$$\begin{aligned} 0 \;&<\; \mathbb{E}\left[P_{0,\omega}[\forall n > 0 :\ X_n > 0], S\right] = P_0\left[\{D_\infty^0 = 2\omega(0,1)-1\} \cap S\right] \\ &\leq\; P_0[D_\infty^0 < \delta^0]. \end{aligned}$$

Since $D_\infty^0 \leq \delta^0$ this implies $E_0[D_\infty^0] < E_0[\delta^0]$. From this we obtain by (31) that $1 < \mathbb{E}[\delta^0]$ and (28) and therefore also the if-direction of (30). ☐

5. Eating left-overs

Assume that $(\omega(x))_{x\ge 0}$ is stationary and ergodic. Then by Theorem 9 $(X_n)_{n\ge 0}$ is either recurrent or transient and Theorem 12 tells us which is the case.

Let us assume that $(X_n)_{n\ge 0}$ is transient. Then the walk will visit each site $x \in \mathbb{Z}$ P_0-a.s. only a finite number of times. Hence $\omega_2 := \psi\left(\omega, (X_n)_{n<\infty}\right)$, with a straightforward extension of definition (12) to infinite sequences, is P_0-a.s. well defined and consists of the cookies left over by the random walk. So we may start a second ERW $(X_n^{(2)})_{n\ge 0}$ in the environment ω_2. Since $(\omega(x))_{x\ge 0}$ was stationary and ergodic, so is $(\omega_2(x))_{x\ge 0}$ due to Lemma 10 applied to

$$g\left((\omega(x))_{x\ge 0}, (x_m)_{m\ge 0}\right) := \left(\omega\left(0, i + \#\{m \ge 0 \mid x_m = 0\}\right)\right)_{i\ge 1}.$$

Consequently, also $(X_n^{(2)})_{n\ge 0}$ is either recurrent or transient. Moreover, due to (28) and (30) the first random walk has reduced the expected total drift stored in the cookies at any site $x \ge 0$ by 1. If the total drift stored in the cookies, which were left over by the first walk, is less than 1 then $(X_n^{(2)})_{n\ge 0}$ will be recurrent due to Theorem 12. If it is larger than 1 then it will be transient and will leave behind another stationary and ergodic environment $\omega_3 := \psi\big(\omega_2, (X_n^{(2)})_{n<\infty}\big)$, in which we can start a third ERW $(X_n^{(3)})_{n\ge 0}$.

This can be iterated. E.g. if $\mathbb{E}[\delta^0]$ is finite but not an integer (to avoid exceptions related to the one ruled out in (29)) then the first $\lfloor \mathbb{E}[\delta^0] \rfloor$ ERWs will almost surely be transient and the next one will almost surely be recurrent and will eventually eat all the cookies on $\mathbb{N}_0$.

6. Strong law of large numbers

Theorem 13. *If $(\omega(x))_{x\ge 0}$ is stationary and ergodic under $\mathbb{P}$ then P_0-a.s.*

$$\lim_{n\to\infty} \frac{X_n}{n} = v := \frac{1}{u} \ge 0, \quad \textit{where} \quad u := \sum_{j\ge 1} P_0\left[T_{j+1} - T_j \ge j\right] \in [1, \infty].$$

Roughly speaking, u is the expected time it takes a walker who has just arrived at ∞ to reach level $\infty + 1$. One could phrase Theorem 13 and its proof in terms of a limiting distribution of the environment viewed from the particle. The following proof is a bit more elementary.

Proof. We shall first show that

$$\lim_{k\to\infty} \frac{T_k}{k} = u \quad P_0\text{-a.s..} \tag{32}$$

By telescopic summation

$$T_k = \sum_{i=0}^{k-1} T_{i+1} - T_i. \tag{33}$$

Consequently,

$$\liminf_{k\to\infty} \frac{T_k}{k} \geq \sup_{t\geq 0} \liminf_{k\to\infty} \frac{1}{k} \sum_{i=t}^{k-1} \left((T_{i+1} - T_i) \wedge t\right). \tag{34}$$

Applying Lemma 10 for all $t \geq 0$ to

$$g\left((\omega(x))_{x\geq 0}, (x_m)_{m\geq 0}\right) := (T_{t+1} - T_t)\left((x_m)_{m\geq 0}\right) \wedge t$$

yields that

$$\left((T_{i+1} - T_i) \wedge t\right)_{i\geq t} \quad \text{is stationary and ergodic for all } t \geq 0. \tag{35}$$

Therefore, by the ergodic theorem, the right-hand side of (34) is equal to

$$\sup_{t\geq 0} E_0\left[(T_{t+1} - T_t) \wedge t\right] \;=\; \sup_{t\geq 0} \sum_{j=1}^{t} P_0\left[(T_{t+1} - T_t) \wedge t \geq j\right]. \tag{36}$$

We are now going to remove t in two steps from the summands in (36). Firstly, note that for $j \leq t$, $P_0[(T_{t+1} - T_t) \wedge t \geq j] = P_0\left[(T_{t+1} - T_t) \wedge j \geq j\right]$. Secondly, recall from (35) that $\left((T_{i+1} - T_i) \wedge j\right)_{i\geq j}$ is stationary for all j. Consequently, $P_0\left[(T_{t+1} - T_t) \wedge j \geq j\right] = P_0\left[T_{j+1} - T_j \geq j\right]$. Therefore, (36) is equal to

$$\sup_{t\geq 0} \sum_{j=1}^{t} P_0\left[T_{j+1} - T_j \geq j\right] = u,$$

which establishes one of the two inequalities needed. For the other inequality note that (33) implies

$$\begin{aligned} \limsup_{k\to\infty} \frac{T_k}{k} &= \limsup_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \sum_{j\geq 1} \mathbf{1}_{T_{i+1} - T_i \geq j} \\ &\leq \sum_{j\geq 1} \limsup_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbf{1}_{T_{i+1} - T_i \geq j}. \end{aligned} \tag{37}$$

Due to Lemma 10 applied to

$$g\left((\omega(x))_{x\geq 0}, (x_m)_{m\geq 0}\right) := \mathbf{1}\left\{(T_{j+1} - T_j)\left((x_m)_{m\geq 0}\right) \geq j\right\}$$

the sequences $\left(\mathbf{1}_{T_{i+1} - T_i \geq j}\right)_{i\geq j}$, $j \geq 1$, are stationary and ergodic. Consequently, by the ergodic theorem, the right-hand side of (37) is P_0-a.s. equal to u, too. This completes the proof of (32).

The statement of the Theorem now follows from (32) by the following standard argument (see e.g. [12, Lemma 2.1.17]). Let $(k_n)_{n\geq 0}$ be the increasing sequence of non-negative integers with $T_{k_n} \leq n < T_{k_n+1}$. Then by (32), $\lim_n n/k_n = u$. Moreover, by definition of T_k and since the walk is between nearest neighbors, $k_n - (n - T_{k_n}) \leq X_n \leq k_n$. This immediately yields $\limsup_n X_n/n \leq 1/u = v$ and also, under the additional assumption $u < \infty$, that $\liminf_n X_n/n \geq v$. If $u = \infty$, then the lower bound $\liminf_n X_n/n \geq 0$ can be obtained by coupling $(X_n)_n$, like in the proof of Lemma 1, to a simple symmetric random walk $(Y_n)_n$ with $Y_n \leq X_n$, and then using $\lim_n Y_n/n = 0$. □

Remark 3. The one-dimensional model under consideration can be extended to higher dimensions $d \geq 1$ by letting $\omega(x,i) \in [0,1]^{2d}$ $(x \in \mathbb{Z}^d, i \geq 1)$ be a vector of transition probabilities to the $2d$ neighbors of x in $\mathbb{Z}^d$. In the case where $\omega(x)$, $x \in \mathbb{Z}^d$, are i.i.d. under $\mathbb{P}$, a straightforward adaptation of a renewal structure technique introduced by Sznitman and Zerner for random walks in random environments (RWRE) gives the P_0-a.s. convergence of $(X_n \cdot \ell)/n$ towards a deterministic limit on the event $\{\lim_{n\to\infty} X_n \cdot \ell = +\infty\}$, where ℓ is any direction in $\mathbb{R}^d$, see [10] and [12, Theorem 3.2.2]. Here we assume that none of the transition probabilities is equal to 0. In this case Lemma 8 and its higher dimensional analogue (see [10, (1.16)] for RWRE) are easy to obtain. □

7. Monotonicity

Monotonicity results are often difficult to obtain for processes in random media since standard coupling techniques, similar to the one used in the proof of Lemma 1 tend to fail, see Example 3 below.

The following result shows that, roughly speaking, starting further to the right, helps to reach a goal located to the right sooner.

Lemma 14. (Monotonicity w.r.t. initial point) *Let* $\omega \in \Omega_+$, $-\infty \leq x \leq y_1 \leq y_2 \leq z \leq \infty$, $y_1, y_2 \in \mathbb{Z}$, *and* $t \in [0,\infty]$. *Then*

$$P_{y_1,\omega}[T_z \leq T_x \wedge t] \leq P_{y_2,\omega}[T_z \leq T_x \wedge t]. \tag{38}$$

Here we define $T_\infty = T_{-\infty} = \infty$.

Proof. By continuity it is enough to show the claim for $z < \infty$. Moreover, by induction it suffices to show the statement for $y_2 = y_1 + 1$. So assume $y_2 = y_1 + 1$. For $y_1 < z$ denote by $\Pi_{y_1}^z$ the set of all finite nearest-neighbor paths $\pi = (x_n)_{n\leq m}$, $m > 0$, which start at $x_0 = y_1$, end at $x_m = z$ and do not hit z in between. Any such path π can be uniquely written as the concatenation $\pi = (B_1, A_1, B_2, A_2, \ldots, B_{j(\pi)}, A_{j(\pi)})$ for some $j(\pi) \geq 1$, where A_i and B_i $(i \leq j(\pi))$ are nonempty nearest-neighbor paths such that the

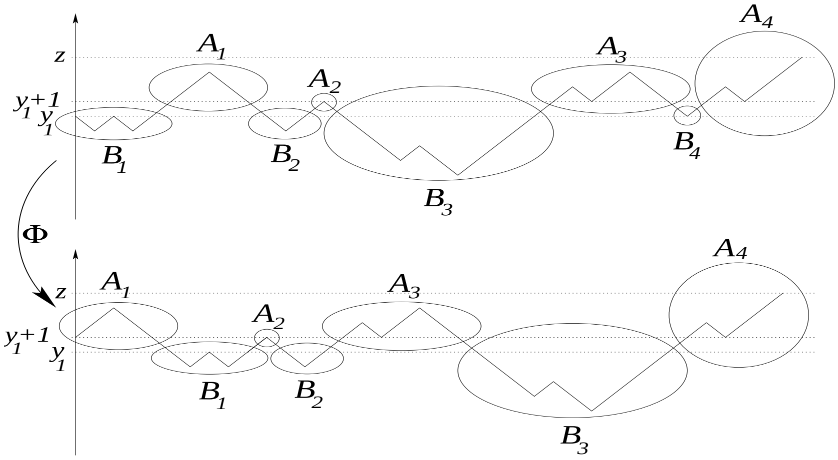

FIGURE 2. Transforming paths from y_1 to z into paths from y_1+1 to z.

A_i's contain only points $> y_1$ ("**A**bove y_1") and the B_i's contain only points $\le y_1$ ("**B**elow y_1"). Then the function $\Phi : \Pi^z_{y_1} \to \Pi^z_{y_1+1}$ defined by

$$\Phi(B_1, A_1, B_2, A_2, \ldots, B_j, A_j) := (A_1, B_1, A_2, B_2, \ldots, A_{j-1}, B_{j-1}, A_j),$$

see Figure 2, is well-defined and surjective. This function cuts out the last excursion B_j from y_1 downward but otherwise only rearranges pieces of π without changing the relative order in which the points above y_1 are visited nor the relative order in which the points below y_1 are visited. Therefore, for any $\pi \in \Pi^z_{y_1}$,

$$P_{y_1,\omega}\left[H_{T_z} = \pi\right] = P_{y_1+1,\omega}\left[H_{T_z} = \Phi(\pi)\right] \cdot P_{y_1,\omega'}\left[H_{T_{y_1+1}} = (B_{j(\pi)}, y_1+1)\right],$$

where $\omega' := \psi(\omega, (B_1, A_1, \ldots, B_{j(\pi)-1}, A_{j(\pi)-1}, y_1))$ is the environment π faces before it starts its excursion $B_{j(\pi)}$. By summing over all possible excursions $B_{j(\pi)}$ we get for all $\pi \in \Pi^z_{y_1}$,

$$P_{y_1,\omega}\left[H_{T_z} \in \Phi^{-1}(\{\Phi(\pi)\})\right] = P_{y_1+1,\omega}\left[H_{T_z} = \Phi(\pi)\right].$$

Since Φ is surjective this means that for all $\pi \in \Pi^z_{y_1+1}$,

$$P_{y_1,\omega}\left[H_{T_z} \in \Phi^{-1}(\{\pi\})\right] = P_{y_1+1,\omega}\left[H_{T_z} = \pi\right]. \tag{39}$$

Now denote by $\Pi^z_{y_1}(x,t)$ the set of paths π in $\Pi^z_{y_1}$ which do not visit x before z and which make at most t steps. Then the right-hand side of (38) can be written as

$$P_{y_1+1,\omega}\left[H_{T_z} \in \Pi^z_{y_1+1}(x,t)\right] = P_{y_1,\omega}\left[H_{T_z} \in \Phi^{-1}\left(\Pi^z_{y_1+1}(x,t)\right)\right] \tag{40}$$

due to (39). Cutting out an excursion does not make a path longer nor does it make a path visit x if the path did not do so before. Therefore,

$$\Phi^{-1}\left(\Pi^z_{y_1+1}(x,t)\right) \supseteq \Phi^{-1}\left(\Phi\left(\Pi^z_{y_1}(x,t)\right)\right) \supseteq \Pi^z_{y_1}(x,t).$$

Consequently, the right-hand side of (40) is greater than or equal to the left-hand side of (38). □

Roughly speaking, the following result states that increasing the strength of some cookies does not slow down the walk. Here we denote by $\leq$ the canonical partial order on Ω_+, i.e. $\omega_1 \leq \omega_2$ if and only if $\omega_1(x,i) \leq \omega_2(x,i)$ for all $x \in \mathbb{Z}$ and all $i \geq 1$.

Lemma 15. (Monotonicity w.r.t. environment) *Let* $\omega_1, \omega_2 \in \Omega_+$ *with* $\omega_1 \leq \omega_2$ *and* $-\infty \leq x \leq y \leq z \leq +\infty$, $y \in \mathbb{Z}$, *and* $t \in \mathbb{N} \cup \{\infty\}$. *Then*

$$P_{y,\omega_1}[T_z \leq T_x \wedge t] \leq P_{y,\omega_2}[T_z \leq T_x \wedge t]. \tag{41}$$

Intuitively, this result seems to be clear. However, the following example shows that the naive coupling approach to prove it fails.

Example 3. Let $\omega_1, \omega_2 \in \Omega_+$ with $\omega_j(x,1) = p_j$ and $\omega_j(x,i) = 1/2$ for $x \in \mathbb{Z}, i \geq 2$ and $j = 1,2$, where $1/2 < p_1 < p_2 < 1$. Thus $\omega_1 \leq \omega_2$. There does not seem to be a simple way to couple like in the proof of Lemma 1 two ERWs $(X_n^{(1)})_n$ and $(X_n^{(2)})_n$ in the environments ω_1 and ω_2, respectively, such that $X_n^{(1)} \leq X_n^{(2)}$ for all $n \geq 0$ almost surely, as we shall show now.

Again, let U_n, $n \geq 0$, be a sequence of independent random variables uniformly distributed on $[0,1]$. If the walk $(X_n^{(j)})_n$ $(j = 1,2)$ visits at time n a site for the first time then it moves to the right in the next step iff $U_n < p_j$. If it has visited the site it currently occupies at least once before then it moves to the right iff $U_n < 1/2$. Clearly, this defines two ERWs in the environments ω_1 and ω_2. However, $X_6^{(1)} > X_6^{(2)}$ on the event

$$\left\{(U_n)_{n=0}^5 \in \left]p_1,p_2\right[\times \left]p_2,1\right] \times \left]1/2,p_1\right[\times \left[0,1/2\right[^2 \times \left]1/2,p_1\right[\right\},$$

which has positive probability, see Figure 3. □

Proof of Lemma 15. We assume that t is finite. The case of $t = \infty$ follows then by continuity from the finite case.

By time t the walker can eat only cookies which are among the first t cookies at sites which are within distance t from the starting point y. Hence we may assume that ω_1 and ω_2 differ in the strength of only a finite number r of cookies. By induction it suffices to consider the case $r = 1$. So let us assume that $\omega_1(u,i) = \omega_2(u,i)$ for all $(u,i) \in (\mathbb{Z} \times \mathbb{N}) \backslash \{(v,j)\}$, where (v,j) denotes location and number of the only cookie which might be stronger in ω_2 than in ω_1. We shall refer to this cookie as to the crucial cookie. Since

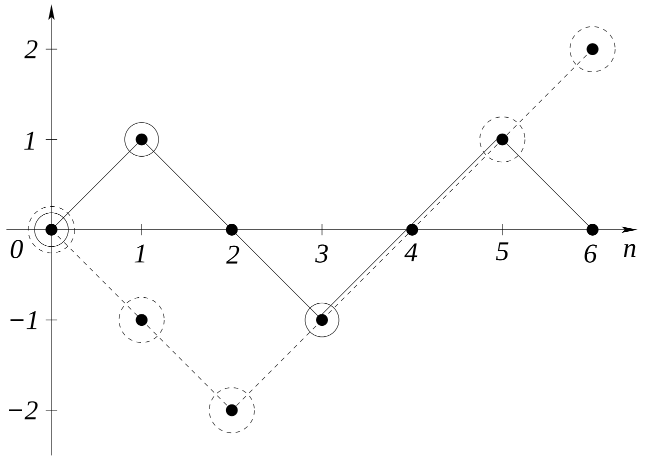

FIGURE 3. How $(X_n^{(1)})_n$ (dashed), which eats weak cookies, might overtake $(X_n^{(2)})_n$ (solid), which eats strong cookies, see Example 3. Sites which are visited for the first time are encircled accordingly.

for the event $\{T_z < T_x \wedge t\}$ only cookies between x and z matter we may additionally assume $x < v < z$.

Denote by S the time of the j-th visit to v. This is the time at which the walk reaches the crucial cookie. Then for $i = 1, 2$,

$$P_{y,\omega_i}[T_z < T_x \wedge t] = P_{y,\omega_i}[T_z < T_x \wedge t \wedge S] + P_{y,\omega_i}[S < T_z < T_x \wedge t]. \tag{42}$$

Note that the first term on the right-hand side of (42) does not depend on i. It therefore suffices to show that the second term is non-decreasing in i. By the Markov property the second term equals

$$\sum_{s\geq 0}^{t-2} E_{y,\omega_i}\left[P_{v,\psi(\omega_i,H_s)}[T_z < T_x \wedge (t-s)],\ s = S < T_z \wedge T_x\right]. \tag{43}$$

Decomposition after the next step, in which the strength of the crucial cookie comes into play, yields P_{y,ω_i}-a.s. $(i = 1, 2)$

$$\begin{aligned} &P_{v,\psi(\omega_i,H_s)}[T_z < T_x \wedge (t-s)] \\ &\quad= (1-\omega_i(v,j))P_{v-1,\psi(\omega_i,H_{s+1})}[T_z < T_x \wedge (t-s-1)] \\ &\qquad+ \omega_i(v,j)P_{v+1,\psi(\omega_i,H_{s+1})}[T_z < T_x \wedge (t-s-1)]. \end{aligned} \tag{44}$$

However, on $\{S = s\}$,

$$\psi(\omega_1, H_{s+1}) = \psi(\omega_2, H_{s+1}) \tag{45}$$

P_{y,ω_1}-a.s. because at time S the walker reaches and then eats the crucial cookie, thus erasing the only difference between the two environments. Moreover, we may apply Lemma 14 with $y_1 = v - 1$ and $y_2 = v + 1$ to deduce that P_{y,ω_1}-a.s.

$$P_{v-1,\psi(\omega_1,H_{s+1})}[T_z < T_x \wedge (t-s-1)] \leq P_{v+1,\psi(\omega_1,H_{s+1})}[T_z < T_x \wedge (t-s-1)].$$

Combined with (45) and $\omega_1(v,j) \leq \omega_2(v,j)$ this implies that (44) for $i = 1$ is P_{y,ω_1}-a.s. less than or equal to (44) for $i = 2$. Consequently, (43) for $i = 1$ is less than or equal to

$$\sum_{s\geq 0}^{t-2} E_{y,\omega_1}\left[P_{v,\psi(\omega_2,H_s)}[T_z < T_x \wedge (t-s)],\ s = S < T_z \wedge T_x\right]. \tag{46}$$

However, the distribution of the above integrand under E_{y,ω_1} does not depend on $\omega_1(v,j)$ any more. Therefore, we can replace E_{y,ω_1} in expression (46) by E_{y,ω_2} without changing its value, thus getting (43) with $i = 2$. □

The following corollaries show that the probability of return to the origin and the speed of the walk are monotone increasing in ω.

Theorem 16. *The probability $P_{0,\omega}[\forall n > 0\ X_n > 0]$ never to return to the initial point is monotone increasing in ω.*

Proof. By the simple Markov property for all $\omega \in \Omega_+$,

$$\begin{aligned} P_{0,\omega}[\forall n > 0\ X_n > 0] &= \omega(0,1) P_{1,\psi(\omega,(0,1))}[\forall n > 0\ X_n > 0] \\ &= \omega(0,1) P_{1,\omega}[\forall n > 0\ X_n > 0], \end{aligned} \tag{47}$$

which is monotone increasing in ω due to Lemma 15 applied to $x = 0, y = 1$ and $z = t = \infty$. □

Theorem 17. *Let $\overline{\mathbb{P}}$ be a probability measure on Ω_+^2 such that*

$$\overline{\mathbb{P}}\left[\{(\omega_1,\omega_2) \in \Omega_+^2 \mid \omega_1 \leq \omega_2\}\right] = 1$$

and for which $(\omega_i(x))_{x\geq 0}$ is stationary and ergodic for $i = 1, 2$. Then $v_1 \leq v_2$ if we denote by v_i ($i = 1, 2$) the $\overline{\mathbb{P}} \times P_{0,\omega_i}$-a.s. constant limit of X_n/n as $n \to \infty$.

Proof. By dominated convergence for $i = 1, 2$,

$$\begin{aligned} v_i &= \lim_{k\to\infty} \overline{\mathbb{E}}\left[E_{0,\omega_i}\left[X_{T_k}/T_k\right]\right] = \lim_{k\to\infty} \overline{\mathbb{E}}\left[E_{0,\omega_i}\left[k/T_k\right]\right] \\ &= \lim_{k\to\infty} \overline{\mathbb{E}}\left[\int_0^1 P_{0,\omega_i}\left[T_k \leq k/t\right]\ dt\right]. \end{aligned}$$

The statement now follows from Lemma 15 with $x = -\infty$, $y = 0$, and $z = k$. □

Open Problem. Are the return probability and the velocity in an appropriate sense continuous in ω?

8. No excitement after the second visit

In some cases when the random walk behaves on the third and any later visit to a site like a simple symmetric random walk one can determine the probability that the walk will never return to its starting point and can show that the walk has zero speed.

Theorem 18. *If* $(\omega(x))_{x\geq 0}$ *is an i.i.d. sequence under* $\mathbb{P}$ *such that* $\mathbb{P}$*-a.s.* $\omega(0,i)=1/2$ *for all* $i\geq 3$ *and* $\mathbb{P}[\omega(0,2)=1/2]<1$ *then*

$$P_0[\forall n>0\ X_n>0]=\frac{\mathbb{E}[\omega(0,1)](\mathbb{E}[\delta^0]-1)_+}{\mathbb{E}[\omega(0,1)(2\omega(0,2)-1)]}. \tag{48}$$

Proof. Consider $\delta^0-D^0_\infty$, the total drift stored in the cookies at 0 which will never be eaten by the random walk. On one hand, by (28)

$$E_0[\delta^0-D^0_\infty]=(\mathbb{E}[\delta^0]-1)_+.$$

On the other hand, since the first cookie at 0 is eaten P_0-a.s. right at the beginning of the walk and since only the first two cookies at 0 contribute to δ^0 we have P_0-a.s.

$$\delta^0-D^0_\infty=(2\omega(0,2)-1)\mathbf{1}\{\forall n>0\ X_n>0\}.$$

Combining these two facts we get

$$(\mathbb{E}[\delta^0]-1)_+=\mathbb{E}\left[(2\omega(0,2)-1)P_{0,\omega}[\forall n>0\ X_n>0]\right]. \tag{49}$$

Recall (47) and note that $P_{1,\omega}[\forall n>0\ X_n>0]$ is a function of $(\omega(x))_{x\geq 1}$. Therefore, it is independent of $\omega(0)$ under $\mathbb{P}$ by assumption. This has two consequences. Firstly, taking $\mathbb{E}$-expectations in (47) yields

$$P_0[\forall n>0\ X_n>0]=\mathbb{E}[\omega(0,1)]P_1[\forall n>0\ X_n>0]. \tag{50}$$

Secondly, substituting (47) into (49) gives

$$(\mathbb{E}[\delta^0]-1)_+=\mathbb{E}\left[(2\omega(0,2)-1)\omega(0,1)\right]P_1[\forall n>0\ X_n>0].$$

Combined with (50) this proves the claim. □

Theorem 19. *Let* $(\omega(x))_{x\geq 0}$ *be stationary and ergodic with* $\omega(0,i)=1/2$ $\mathbb{P}$*-a.s. for all* $i\geq 3$ *and* $\mathbb{P}[\omega(0,1)<1,\omega(1,1)<1]>0$*. Then*

$$\lim_{n\to\infty}\frac{X_n}{n}=0\quad P_0\text{-a.s.}.$$

Proof. By assumption, we can fix $\varepsilon > 0$ such that $\mathbb{P}[A_j] =: \alpha$ is strictly positive and independent of j, where

$$A_j := \{\omega(j-1,1) < 1-\varepsilon,\ \omega(j,1) < 1-\varepsilon\} \quad (j \geq 1).$$

Due to Theorem 13 we need to show that $u = \infty$. To simplify calculations we will do a worst case analysis by maximizing the strength of selected cookies as follows. Define $\bar{\omega}_j \in \Omega_+$ for $j \in \mathbb{Z}$ by

$$\bar{\omega}_j(x) := \begin{cases} (1, & 1/2, \quad 1/2, 1/2, \ldots) \text{ if } x < j-1, \\ (1/2, & 1/2, \quad 1/2, 1/2, \ldots) \text{ if } x = j-1, \\ (1-\varepsilon, & 1, \quad 1/2, 1/2, \ldots) \text{ if } x = j, \text{ and} \\ (1, & 1, \quad 1/2, 1/2, \ldots) \text{ if } x > j. \end{cases}$$

Now let $j \geq 1$. Then

$$\begin{aligned} & P[T_{j+1} - T_j \geq j] \\ \geq \ & \mathbb{E}\left[P_{0,\omega}\left[T_{j+1} - T_j \geq j,\ X_{T_{j-1}+1} = j-2\right], A_j\right] \\ = \ & \mathbb{E}\left[E_{0,\omega}\left[P_{j,\psi\left(\omega,H_{T_j}\right)}\left[T_{j+1} \geq j\right],\ X_{T_{j-1}+1} = j-2\right], A_j\right]. \end{aligned} \tag{51}$$

Observe that

$$P_{j,\psi\left(\omega,H_{T_j}\right)}\left[T_{j+1} \geq j\right] = P_{j,\omega'_j}\left[T_{j+1} \geq j\right], \tag{52}$$

where $\omega'_j(x) := \psi\left(\omega, H_{T_j}\right)(x)$ for $x \geq 0$ and $\omega'_j(x) := \bar{\omega}_j(x)$ for $x < 0$. Moreover,

$$\omega'_j \leq \bar{\omega}_j \quad P_0\text{-a.s. on the event } \left\{\omega(j,1) < 1-\varepsilon, X_{T_{j-1}+1} = j-2\right\}. \tag{53}$$

Indeed, $\omega'_j(x) \leq \bar{\omega}_j(x)$ is obvious for $x < 0$ and $x > j$. For $x = j$ we use the additional assumption $\omega(j,1) < 1-\varepsilon$. For $0 \leq x < j-1$ note that when traveling from 0 to j the walk removes the first cookies from all the sites between 0 and $j-1$ inclusively, leaving over at most one cookie with strength $> 1/2$, namely the next cookie. Finally, for the case $x = j-1$ note that even this second cookie on $j-1$ is gone by time T_j, if we ask the walker to take one step back after reaching $j-1$, since then $j-1$ must be visited for a second time before reaching j. This proves (53).

By (52), (53) and Lemma 15, (51) is greater than or equal to

$$P_{j,\bar{\omega}_j}[T_{j+1} \geq j]\ \mathbb{E}\left[P_{0,\omega}\left[X_{T_{j-1}+1} = j-2\right], A_j\right] \ \geq \ P_{0,\bar{\omega}_0}[T_1 \geq j]\ \varepsilon\ \alpha.$$

Hence it suffices to show that $\sum_{j\geq 1} P_{0,\bar{\omega}_0}[T_1 \geq j] = E_{0,\bar{\omega}_0}[T_1] = \infty$. Since

$$T_1 = \sum_{k\geq 0}(T_{-k-1} \wedge T_1 - T_{-k})\mathbf{1}\{T_{-k} < T_1\}$$

we have

$$E_{0,\bar\omega_0}[T_1]=\sum_{k\ge 0}E_{0,\bar\omega_0}\left[E_{0,\bar\omega_0}\left[T_{-k-1}\wedge T_1-T_{-k}\mid \mathcal{F}_{T_{-k}}\right],\ T_{-k}<T_1\right]. \tag{54}$$

For $k\ge 2$, the conditional expectation in (54) is on $\{T_{-k}<T_1\}$ by the strong Markov property $P_{0,\bar\omega}$-a.s. equal to

$$\begin{aligned} E_{-k,\psi(\bar\omega_0,H_{T_{-k}})}[T_{-k-1}\wedge T_1] &= 1+E_{-k+1,\psi(\bar\omega_0,H_{T_{-k}+1})}[T_{-k-1}\wedge T_1] && (55)\\ &\ge E_{-k+1,\psi(\bar\omega_0,H_{T_{-k}+1})}[T_{-k-1}\wedge T_0] \\ &= 2(k-1). && (56)\end{aligned}$$

Here (55) holds because of $\bar\omega_0(-k,1)=1$. (56) is true since the walker eats while traveling from 0 to $-k$ and back to $-k+1$ all the cookies between $-k$ and -1, which have strength $>1/2$, so that the formula for the expected exit time of a simple symmetric random walk from an interval (e.g. [5, Ch. 14.3 (3.5)]) can be applied. Substituting this into (54) yields

$$\begin{aligned} E_{0,\bar\omega_0}[T_1] &\ge 2\sum_{k\ge 2}(k-1)P_{0,\bar\omega_0}[T_{-k}<T_1] \\ &= 2(1-\varepsilon)\sum_{k\ge 2}(k-1)P_{-1,\bar\omega_0}[T_{-k}<T_0]. \end{aligned}$$

Therefore, since the harmonic series diverges it suffices to show for the proof of $E_{0,\bar\omega_0}[T_1]=\infty$ that for all $k\ge 2$,

$$P_{-1,\bar\omega_0}[T_{-k}<T_0]=\frac{1}{(k-1)k}. \tag{57}$$

This is done by induction over k. For $k=2$, the left-hand side of (57) is $\bar\omega_0(-1,1)=1/2$ by definition of $\bar\omega_0$. Now assume that (57) has been proven for k. Then by the strong Markov property and due to $\omega(-k,1)=1$,

$$P_{-1,\bar\omega_0}[T_{-k-1}<T_0] = E_{-1,\bar\omega_0}\left[T_{-k}<T_0, P_{-k+1,\psi(\bar\omega_0,H_{T_{-k}+1})}[T_{-k-1}<T_0]\right].$$

As above, on $\{T_{-k}<T_0\}$ all the cookies with strength $>1/2$ have been removed by time $T_{-k}+1$ from the interval between $-k$ and -1. Therefore, the last expression equals

$$P_{-1,\bar\omega_0}[T_{-k}<T_0]\frac{k-1}{k+1}=\frac{1}{(k-1)k}\,\frac{k-1}{k+1}=\frac{1}{k(k+1)}$$

by induction hypothesis. □

The following example shows that the assumption $\mathbb{P}[\omega(0,1)<1,\omega(1,1)<1]>0$ of Theorem 19 is essential.

Example 4. Let $\omega(x), x \in \mathbb{Z}$, alternate between $(1, 1, 1/2, 1/2, \ldots)$ and $(p, 1, 1/2, 1/2, \ldots)$ where $1/2 \leq p < 1$ is fixed. Then P_0-a.s. $T_{k+1} - T_k \leq 3$ for all $k \geq 0$, which generates a strictly positive speed. $\square$

Open Problem. Of course, it is possible to generate a strictly positive speed v by choosing $\omega(x, i) \geq 1/2 + \varepsilon$ $\mathbb{P}$-a.s. for all $x \in \mathbb{Z}^d, i \geq 1$, where $\varepsilon > 0$ is fixed. However, are a finite number of cookies with strength $> 1/2$ (and < 1) per site already sufficient for positive speed? More precisely, for which integers $m \geq 3$, if any, is there some $1/2 < p < 1$ such that $v > 0$ if $\mathbb{P}$-a.s. for all $x \in \mathbb{Z}$, $\omega(x, i) = p$ for $i \leq m$ and $\omega(x, i) = 1/2$ for $i > m$?

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DEPARTMENT OF MATHEMATICS
STANFORD UNIVERSITY
STANFORD, CA 94305, U.S.A.
E-MAIL: zerner@math.stanford.edu