INTEGRABILITY OF INFINITE WEIGHTED SUMS OF HEAVY-TAILED I.I.D. RANDOM VARIABLES

Abstract. We consider the sum X of i.i.d. random variables Y_n, $n \geq 0$, with weights a_n which decay exponentially fast to zero. For a smooth sublinear increasing function g, $g(|Y_0|)$ has finite expectation if and only if the expectation of $|X|g'(|X|)$ is finite. The proof uses characteristic functions. However, if g grows polynomially or exponentially fast, then the expectation of $g(|Y_0|)$ is finite if and only if the expectation of $g(|X|)$ is finite.

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1. Introduction and results

Let Y_n, $n \geq 0$, be a sequence of i.i.d. real valued random variables on some probability space with probability measure P and expectation operator E and let $a_n \in \mathbb{R}$, $n \geq 0$, such that

$$X := \sum_{n \geq 0} a_n Y_n \tag{1}$$

is well-defined as a P-almost surely absolutely convergent series.

We are interested in the tail of the distribution of X. The distribution of X is of interest because the marginal distribution of any stationary linear process

$$X_m = \sum_{n=-\infty}^{\infty} a_n Y_{m-n} \quad (m \in \mathbb{Z})$$

for two-sided sequences $(a_n)_n$ and $(Y_n)_n$ can be represented as the distribution of some X of the form (1). Linear processes, however, are basic in classical time series analysis. For example, every stationary

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causal ARMA process is linear with weights a_n which decay exponentially fast to zero, see e.g. Brockwell and Davis [3, Sections 3.1 and 13.3] and Embrechts, Klüppelberg and Mikosch [5, Section 7.1].

The purpose of the present paper is to investigate how the tails of $|Y_0|$ and $|X|$ are related to each other. More precisely, we ask how integrability of $|X|$ under some positive increasing function f corresponds to integrability of $|Y_0|$ under some other function g. In general, one expects the tail of X to be at least as heavy as the one of the Y_n's. For example, if all a_nY_n are P-a.s. nonnegative and $a_0 = 1$ then clearly $X \geq Y_0$ almost surely. The question is whether the tail of X can be really heavier than the one of Y_0 in the sense that there is some positive increasing function g for which $g(|Y_0|)$ has a finite expectation but not $g(|X|)$.

If X was the sum of only a finite number of independent random variables then this cannot happen for functions g which grow exponentially, polynomially or logarithmically: A finite sum of independent random variables has a finite expectation under one of these functions if and only if the same function of each of the summands has a finite expectation. (Cf. Lemma 4 and the proof of Proposition 1. This is false for superexponential functions, see Remark 2 below.)

However, the situation is different, if one considers infinite sums X as defined in (1). Roughly speaking, integrability of $|Y_0|$ is equivalent to integrability of $|X|$ only in the case of exponential and polynomial functions, but not for logarithmic functions. The first part of this statement is made precise in the following proposition.

Proposition 1. (Polynomial and exponential functions) *Assume that X in (1) is well-defined as an almost surely absolutely convergent series and let $\sum_{n\geq 0}|a_n| < \infty$ with $|a_n| \leq 1$ for all n. Then*

$$E[f(|X|)] < \infty \quad \text{if and only if} \quad E[f(|Y_0|)] < \infty, \tag{2}$$

provided one of the following two cases holds:

a) *$f(t) = h(t)t^p$, where $p > 0$ and $h \in \mathcal{C}([0,\infty),[0,\infty))$ is concave increasing with $h(t) = O(t^p)$ as $t \to \infty$.*

b) *$f(t) = t^p \exp(ct)$ with $c \geq 0$, $p \geq 0$ and $|a_0| = 1$.*

In both cases a) and b) we additionally assume $\sum_{n\geq 0}|a_n|^p < \infty$ if $0 < p < 1$.

For $f(t) = t^p$ with $p > 0$ this has been observed before by Vervaat [11, Theorem 5.1] in a more general context and Elton and Yan [4, Proposition 7 (ii)].

For smooth subpolynomial functions g, like logarithms, and exponentially decaying weights a_n we have a different picture. Here integrability of $g(|Y_0|)$ is not enough to ensure integrability of $g(|X|)$. Instead it is equivalent to integrability of $f(|X|)$ for a function f which in general grows slower than g. This is the content of our following main result.

Theorem 2. (Sublinear functions) *Assume that $(|a_n|)_{n\geq 0}$ decays exponentially in the sense that*

$$-\infty < \liminf_{n\to\infty} \frac{\log |a_n|}{n} \leq \limsup_{n\to\infty} \frac{\log |a_n|}{n} < 0 \tag{3}$$

and let

$$E[\log^+ |Y_0|] < \infty. \tag{4}$$

Then X converges absolutely almost surely. Moreover,

$$E\left[f(|X|)\right] < \infty \quad \text{if and only if} \quad E\left[g(|Y_0|)\right] < \infty, \tag{5}$$

provided there are $T > 0$ and $0 < \alpha < 1$ such that $f, g \in \mathcal{C}([0,\infty[, [0,\infty[)$ are increasing on $[0,\infty[$ and continuously differentiable on $]T,\infty[$ with

$$t \mapsto f(t)/t^\alpha \text{ decreasing on }]T,\infty[\text{ and} \tag{6}$$

$$f(t) = tg'(t) \text{ for } t > T. \tag{7}$$

Remarks. 1. There is some overlap between Proposition 1 a) and Theorem 2. Some functions f which grow polynomially fast but not faster than linear are covered by both results.

2. Relation (2) does not need to hold if f grows faster than exponential, not even if $X = Y_0 + Y_1$ is the sum of only two i.i.d. random variables Y_0 and Y_1. For instance, let $\exp(\exp(Y_0))$ have a finite first but infinite second moment and let $P[Y_1 \geq \ln 2] > 0$. Then

$$E[\exp(\exp(X))] \geq E[(\exp(\exp(Y_0)))^2] P[Y_1 \geq \ln 2] = \infty.$$

This raises the question if there is a statement similar to Proposition 1 and Theorem 2 for superexponential functions.

3. Assumption (6) is essential since Proposition 1 b) shows that the statement of Theorem 2 does not hold in the case of exponential functions. For instance, let $g(t) = \exp(t), f(t) = t\exp(t)$ and let Y_0 have the properties $E[g(|Y_0|)] < \infty$ and $E[f(|Y_0|)] = \infty$. Then, according to (2), also $E[f(|X|)] = \infty$, thus violating (5).

4. Since due to assumption (6), Theorem 2 does not cover functions growing at least linearly, Theorem 2 is meaningless if $|Y_0|$ has a finite expectation because in this case both expectations in (5) are a priori finite for all f under consideration. However, Theorem 2 does provide interesting information if Y_0 is heavy-tailed in the sense that $E[|Y_0|^\alpha] = \infty$ for some or even all $0 < \alpha < 1$.

To the best of our knowledge, there are only few results in the literature which correlate the tail behavior of $|Y_0|$ and $|X|$ in the case where all the moments of order $\alpha > 0$ are infinite. Elton and Yan [4, Proposition 7 (i)] show that $E[(\log^+ |Y_0|)^2] < \infty$ implies $E[\log^+ \log^+ |X|] < \infty$. This statement is improved by Theorem 2, which implies that $E[(\log^+ |Y_0|)^2] < \infty$ is in fact equivalent to $E[\log^+ |X|] < \infty$ and that $E[\log^+ \log^+ |X|] < \infty$ is equivalent to $E[\log^+ |Y_0| \log^+ \log^+ |Y_0|] < \infty$. Our proof of Theorem 2 relies on the approach used by Elton and Yan [4].

5. An alternative and much more popular way to describe the relation of the tails of Y_0 and X is to compare asymptotic formulas for $P[|Y_0| > t]$ and $P[|X| > t]$ as $t \to \infty$. For a summary of results in this direction see e.g. Embrechts, Klüppelberg and Mikosch [5, Appendix 3.3], Goldie [6, Section 4] and more recently Mikosch and Samorodnitsky [9]. Integrability as we used it and tail behavior are connected through the formula

$$E[h(Z)] = \int_0^\infty P\left[Z > h^{-1}(t)\right]\ dt \tag{8}$$

for nonnegative random variables Z and increasing positive functions h. For instance, in [5, Appendix 3.3] two classes of distributions of Y_0 are considered which have a finite α-moment for some $\alpha > 0$ and for which the tails of X and Y_0 are essentially the same up to a constant c in the sense that $P[Y_0 > t] \sim cP[X > t]$. For these two classes relation (2) follows from relation (8).

Let us now describe how the remainder of the present paper is organized. In the next section we provide examples for Theorem 2 and describe integrability for sums of two random variables as far as we need it in the sequel. In Section 3 we prove Proposition 1. Sections 4 to 6 are devoted to the proof of Theorem 2. In Section 4 we introduce characteristic functions and provide the tools for translating the statement of Theorem 2 into a statement about characteristic functions. Section 5 proves Theorem 2 in the special case where $a_n = a^n$ for some $0 < |a| < 1$. This is used in the last section to derive Theorem 2 in the general case.

2. Sublinear Examples and Preliminaries

Some examples of functions f and g satisfying the assumptions of Theorem 2 are given in Table 1. We list only the leading order functions $\tilde{f}$ and $\tilde{g}$, that is we omit constant coefficients and lower order additive functions because they do not matter in (5). The definitions of $\tilde{f}$ and

$\tilde{g}(t)$	$\tilde{f}(t)$	parameters
$t^{\beta}(\log t)^{\gamma}$	$\tilde{g}(t)$	$0 < \beta, 0 \leq \gamma$
$t^{\beta/(\log\log t)^{\gamma}}$	$\dfrac{\tilde{g}(t)}{(\log\log t)^{\gamma}}$	$0 < \beta, 0 < \gamma$
$\exp\left(\beta(\log t)^{\gamma}\right)$	$\dfrac{\tilde{g}(t)}{(\log t)^{1-\gamma}}$	$0 < \beta, 0 < \gamma < 1$
$\exp\left(\beta(\log\log t)^{\gamma}\right)$	$\dfrac{\tilde{g}(t)(\log\log t)^{\gamma-1}}{\log t}$	$0 < \beta, 1 < \gamma$
$(\log t)^{\beta}(\log\log t)^{\gamma}$	$\dfrac{\tilde{g}(t)}{\log t}$	$1 \leq \beta, 0 \leq \gamma$

TABLE 1. Pairs of $\tilde{f}$ and $\tilde{g}$ (valid only for t large) for which $E[\tilde{f}[|X|]] < \infty$ if and only if $E[\tilde{g}[|Y_0|]] < \infty$.

$\tilde{g}$ are only valid for t large enough. The examples are ordered from fast increasing functions to slowly increasing ones.

Observe in Table 1 that the slower $\tilde{g}$ increases, the more deviates the corresponding $\tilde{f}$ from $\tilde{g}$. The extent to which this may happen is described in the following result.

Proposition 3. *For any f, g, T and α satisfying the assumptions of Theorem 2 there is some $c_1 \in \mathbb{R}$ with*

$$f(t) - c_1 \leq g(t) \leq f(t)\log(t) + c_1 \quad \textit{for } t > T. \tag{9}$$

In the proof and the rest of the paper we will use the observation that assumption (6) is equivalent to

$$tf'(t) \leq \alpha f(t) \text{ for all } t > T, \tag{10}$$

which follows from differentiating the decreasing function $f(t)t^{-\alpha}$.

Proof of Proposition 3. The lower bound follows from (7) and (10) by

$$g(t) = g(T) + \int_T^t \frac{f(s)}{s}\, ds \geq g(T) + \int_T^t \frac{sf'(s)}{s}\, ds = g(T) + f(t) - f(T). \tag{11}$$

For the upper bound we partially integrate (11) to get

$$g(t) = g(T) + f(t)\log t - f(T)\log T - \int_T^t f'(s)\log s\; ds$$

and observe that $f'(s)\log s$ is non-negative for $s \geq 1 \vee T$ since f is increasing. □

The first and the last group in Table 1 provide examples which show that the bounds given in (9) are sharp: In the first example $\tilde{f} = \tilde{g}$ whereas we lose a factor of $\log t$ in the last example when deriving $\tilde{f}$ from $\tilde{g}$. An intermediate behavior is exhibited by the other families.

For $\gamma > 0$ in the first or the last group we get examples of a product $\tilde{g}$ of two functions which show that in general the dominant factor determines how $\tilde{f}$ is obtained from $\tilde{g}$.

The following lemma is needed at various places in the proofs of Proposition 1 and Theorem 2.

Lemma 4. (Finite sums) *Let Z_1, Z_2 be random variables on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let f_1 and f_2 satisfy the assumptions for f of Proposition 1 and Theorem 2 (except (7)), respectively. Then $\mathbb{E}[f_2(|Z_i|)] < \infty$ for $i = 1, 2$ implies $\mathbb{E}[f_2(|Z_1 + Z_2|)] < \infty$. Moreover, if $0 < c < \infty$ then $\mathbb{E}[f_2(|Z_1|)] < \infty$ if and only if $\mathbb{E}[f_2(c|Z_1|)] < \infty$. If Z_1 and Z_2 are additionally independent then for $j = 1, 2$, $\mathbb{E}[f_j(|Z_1+Z_2|)] < \infty$ implies $\mathbb{E}[f_j(|Z_i|)] < \infty$ for $i = 1, 2$.*

Proof. For the first statement observe that $t \mapsto f_2(t)/t$ is decreasing for $t > T$ thanks to (6). Therefore, $f_2(x+y)/(x+y) \leq f_2(x)/x, f_2(y)/y$ for $x, y > T$, which implies the triangle inequality $f_2(x+y) \leq f_2(x)+f_2(y)$ for $x, y \geq T$. Hence, since f_2 is monotone,

$$\mathbb{E}[f_2(|Z_1 + Z_2|)] \leq \mathbb{E}[f_2(|Z_1| \vee T)] + \mathbb{E}[f_2(|Z_2| \vee T)].$$

This proves the first statement. The second claim of the lemma follows from the monotonicity of f_2 and from iterated application of the first statement. For the last assertion, we get from the monotonicity of f_j that for any $\gamma > 0$,

$$\begin{aligned} \infty &> \mathbb{E}\left[f_j(|Z_1 + Z_2|)\right] \geq \mathbb{E}\left[f_j\left(||Z_1| - |Z_2||\right) 1_{|Z_2| \leq \gamma \leq |Z_1|}\right] \\ &\geq \mathbb{E}\left[f_j\left(||Z_1| - \gamma|\right) 1_{\gamma \leq |Z_1|}\right] \mathbb{P}[|Z_2| \leq \gamma], \end{aligned}$$

where we used independence in the last step. Therefore $\mathbb{E}[f_j(||Z_1| - \gamma|)] < \infty$ for some large γ. In the case $j = 2$, we apply the first part of the proof to Z_1 and Z_2 replaced by $\tilde{Z}_1 := |Z_1| - \gamma$ and $\tilde{Z}_2 := \gamma$ respectively, to see that also $\mathbb{E}[f_2(|Z_1|)] < \infty$ as required. By symmetry the same holds for Z_2. For $j = 1$ we observe that there is some $c_2 > 0$ such that $f_1(x - \gamma) \geq c_2 f_1(x)$ for x large, which gives $\mathbb{E}[f_1(|Z_1|)] < \infty$ and analogously $\mathbb{E}[f_1(|Z_2|)] < \infty$. □

3. Polynomial and exponential functions

Proof of Proposition 1. The only-if-part of (2) follows from the last statement of Lemma 4 with $j = 1, i = 1, Z_1 = a_0 Y_0$ and $Z_2 = X - Z_1$. Note that we used here concavity in part a) and $|a_0| = 1$ in part b).

For the if-part, we may assume without loss of generality that $a_n \geq 0$ and $Y_n \geq 0$ P-a.s. because f is increasing and $|X| \leq \sum_n |a_n||Y_n|$ which is P-a.s. finite due to the assumption of absolute convergence. We first settle the case $f(t) = \exp(ct)$. In this case, due to independence,

$$\log E[f(X)] = \sum_{n\geq 0} \log\left(1 + E[\exp(ca_n Y_n) - 1]\right)$$

$$\leq \sum_{n\geq 0} E[\exp(ca_n Y_0) - 1] \;=\; \sum_{m\geq 1} E\left[\frac{(cY_0)^m}{m!}\right] \sum_{n\geq 0} a_n^m.$$

Since the a_n are not larger than 1 and summable with a finite total sum, say c_3, the above is less than $c_3 E[f(Y_0)]$ which is finite by assumption.

The functions f which remain to be considered are of the form $f(t) = t^p s(t)$ with $p > 0$ and some increasing positive function s. For $p \geq 1$ by monotone convergence and Minkowski's inequality,

$$E[f(X)]^{1/p} = \lim_{k\to\infty} \left\| \sum_{n=0}^{k} a_n Y_n \left(s\left(X\right)\right)^{1/p} \right\|_p$$

$$\leq \lim_{k\to\infty} \sum_{n=0}^{k} a_n E\left[Y_n^p s(X)\right]^{1/p}, \tag{12}$$

where $\|\cdot\|_p$ is the L^p-norm. Now let $(\tilde{Y}_n)_n$ be an independent copy of $(Y_n)_n$ and set $X_n := a_n \tilde{Y}_n + \sum_{m\neq n} a_m Y_m$ for $n \geq 0$. Note that Y_n and X_n are independent. Then by monotonicity of s and $a_n \leq 1$, (12) is less than

$$\lim_{k\to\infty} \sum_{n=0}^{k} a_n E\left[Y_n^p s(Y_n + X_n)\right]^{1/p} = c_3 E\left[Y_0^p s(Y_0 + X_0)\right]^{1/p}.$$

If however $0 < p < 1$, then if follows directly from the concavity of t^p that

$$E[f(X)] \le \sum_{n\ge 0} a_n^p E[Y_n^p s(X)] \le E[Y_0^p s(Y_0 + X_0)] \sum_{n\ge 0} a_n^p.$$

Thus in either case, $p \ge 1$ or $p < 1$, it remains to show that

$$E[Y_0^p s(Y_0 + X_0)] \tag{13}$$

is finite.

In the case $s(t) = \exp(ct)$, (13) equals $E[f(Y_0)]E[\exp(cX)]$. The first factor is finite by assumption and the second one is finite due to the previously considered case $f(t) = \exp(ct)$.

In the case $s = h$, we assume without loss of generality $h(0) = 0$ which implies $h(x+y) \le h(x)+h(y)$ for all $x, y \ge 0$. Moreover we may assume that h does not vanish identically. Therefore the assumption $E[f(Y_0)] < \infty$ implies $E[Y_0^p] < \infty$. By independence of Y_0 and X_0, (13) can be estimated from above by

$$E\left[Y_0^p h(Y_0)\right] + E[Y_0^p]E[h(X_0)] \le E\left[f(Y_0)\right] + c_4 E[Y_0^p]E[X^p] \tag{14}$$

for some finite c_4 where we used $h(t) = O(t^p)$. For $h \equiv 1$, the left hand side of (14) is finite if $E[Y_0^p] < \infty$. Thus for $h \equiv 1$ the above calculation up to the left side of (14), inclusive, reproduces the known result that $E[X^p] < \infty$ if $E[Y_0^p] < \infty$. Consequently, also for the original f and h, the right hand side of (14) is finite, if $E[f(Y_0)] < \infty$, thus proving finiteness of $E[f(X)]$. □

4. Integrability and characteristic functions

For a probability measure σ on $\mathbb{R}$ we denote its characteristic function by $\widehat{\sigma}(t) = \int \exp(itx)\, d\sigma(x)$ $(t \in \mathbb{R})$. The following lemma, which may be of independent interest, relates the local behavior of $\widehat{\sigma}$ at 0 to integrability under σ.

Lemma 5. *Let σ be a probability measure on $\mathbb{R}$, let $0 < \theta < 1$ and let T, α and f satisfy the assumptions of Theorem 2 (except (7)). Then the following two assertions are equivalent.*

$$\int f(|x|)\, d\sigma(x) \;<\; \infty. \tag{15}$$

$$\int_0^{1/T} \frac{f'(1/t)}{t^2} \int_0^1 |1 - \widehat{\sigma}(\theta^y t)|\, dy\, dt \;<\; \infty. \tag{16}$$

If additionally

$$t \mapsto t^2 f'(t) \quad \textit{is increasing for } t > T \tag{17}$$

then (15) and (16) are equivalent to

$$\int_0^{1/T} \frac{f'(1/t)}{t^2} |1 - \widehat{\sigma}(t)| \, dt < \infty. \tag{18}$$

The equivalence of (15) and (18) is contained in Boas [2, Theorem 3] for $f(t) = t^\gamma$ with $0 < \gamma < 1$ and $f(t) = \log^+ t$ and has been shown for $f(t) = \log^+ \log^+ t$ and $f(t) = (\log^+ t)^2$ by Elton and Yan [4, Lemma 2].

Proof of Lemma 5. We first show that (15) implies (16) for all $\theta > 0$. Setting $\theta = 1$ then also yields the implication (15) $\Rightarrow$ (18). We use

$$|1 - \widehat{\sigma}(t)| \leq \int |1 - \exp(itx)| \, d\sigma(x) = 2 \int \left| \sin \frac{t|x|}{2} \right| \, d\sigma(x)$$

and Fubini's theorem to estimate the left hand side of (16) from above by $\int_0^1 \int r(x, y) \, d\sigma(x) \, dy$ where

$$\begin{aligned} r(x, y) \quad &:= \quad 2 \int_0^{1/T} \frac{f'(1/t)}{t^2} \left| \sin \frac{\theta^y t|x|}{2} \right| \, dt \\ &\leq \quad \theta^y |x| \int_0^{1/(T \vee |x|)} \frac{f'(1/t)}{t} \, dt + 2 \int_{1/(T \vee |x|)}^{1/T} \frac{f'(1/t)}{t^2} \, dt. \end{aligned} \tag{19}$$

Here we used $|\sin y| \leq y \wedge 1$ for $y \geq 0$. To bound the first term in (19) we prove that for all $\gamma \in]0, 1/T[$,

$$\int_0^{\gamma} \frac{f'(1/t)}{t} \, dt \leq \frac{\alpha}{1 - \alpha} \gamma f(1/\gamma). \tag{20}$$

To this end, first observe that the integral in (20) is finite because due to (10) and (6) there is some finite c_5 such that for t small, $f'(1/t)/t \leq f(1/t) \leq c_5 t^{-\alpha}$, which is integrable. Therefore for $\gamma \searrow 0$, the left hand side (20) vanishes. Hence it suffices to show that the derivative with respect to γ of the left hand side of (20) is dominated by the derivative of the right hand side. This means that we have to check whether

$$f'(1/\gamma)/\gamma \leq \alpha/(1 - \alpha)(f(1/\gamma) - f'(1/\gamma)/\gamma)$$

which is equivalent to $f'(1/\gamma)/\gamma \leq \alpha f(1/\gamma)$. This is true due to (10) and proves (20).

Consequently, the first term in (19) is less than $c_6 \theta^y |x|/(T \vee |x|) f(T \vee |x|) \leq c_6 \theta^y f(T \vee |x|)$ for some finite c_6. Since the second term in (19) equals

$$2[-f(1/t)]_{1/(T \vee |x|)}^{1/T} = 2(f(T \vee |x|) - f(T)) \leq 2f(T \vee |x|)$$

we can therefore estimate the left side of (16) by

$$\left(\int_0^1 2 + c_6\theta^y \, dy\right) \int f(T \vee |x|) \, d\sigma(x)$$

which is finite by assumption (15).

To show that (16) implies (15) when $0 < \theta < 1$, we insert the estimate

$$|1 - \widehat{\sigma}(t)| \geq |1 - \mathrm{Re}\, \widehat{\sigma}(t)| = \int 1 - \cos t|x| \, d\sigma(x)$$

into the assumption (16) to get by Fubini's theorem

$$\begin{aligned} \infty &> \int \int_0^{1/T} \frac{f'(1/t)}{t^2} \phi(t|x|) \, dt \, d\sigma(x) \\ &= \int \frac{1}{|x|} \int_0^{|x|/T} f'(|x|/s)(|x|/s)^2 \phi(s) \, ds \, d\sigma(x), \end{aligned} \tag{21}$$

where

$$\phi(s) := \int_0^1 1 - \cos \theta^y s \, dy = 1 + (\mathrm{ci}\,(s) - \mathrm{ci}\,(\theta s))/\log\theta$$

and $\mathrm{ci}\,(s) := -\int_s^\infty \cos(t)/t \, dt$ is the cosine integral function. Since $\mathrm{ci}\,(s) \to 0$ as $s \to \infty$ there is a positive constants c_7 such that $\phi(s) \geq 1/2$ for $s > c_7$. Hence (21) can be estimated from below by

$$\begin{aligned} & \frac{1}{2} \int_{|x|>c_7 T} \int_{c_7}^{|x|/T} f'(|x|/s)|x|/s^2 \, ds \, d\sigma(x) \\ = & \frac{1}{2} \int_{|x|>c_7 T} [-f(|x|/s)]_{s=c_7}^{s=|x|/T} \, d\sigma(x) \\ = & \frac{1}{2} \int_{|x|>c_7 T} f(|x|/c_7) - f(T) \, d\sigma(x). \end{aligned}$$

By the second statement of Lemma 4 with $c = 1/c_7$ we conclude that (15) holds.

Now we additionally assume (17). The proof of (18) $\Rightarrow$ (15) is similar to the one of (16) $\Rightarrow$ (15). Indeed, we proceed as above and arrive with $\phi(s) := 1 - \cos s$ at (21), which we estimate from below by

$$\int_{|x|>T} \frac{1}{|x|} \int_1^{|x|/T} f'(|x|/s)(|x|/s)^2 (1 - \cos s) \, ds \, d\sigma(x). \tag{22}$$

Observe, that there is a constant $c_8 > 0$ such that

$$\int_1^y h(s)(1 - \cos s) \, ds \geq c_8 \int_1^y h(s) \, ds$$

for all nonnegative decreasing functions h and all $y \geq 1$. Therefore, since $h_x(s) := f'(|x|/s)(|x|/s)^2$ is decreasing in $s \in [1, |x|/T[$ for any $|x| > T$ by assumption (17), (22) is greater than

$$c_8 \int_{|x|>T} \frac{1}{|x|} \int_1^{|x|/T} h_x(s)\, ds\, d\sigma(x) = c_8 \int_{|x|>T} f(|x|) - f(T)\, d\sigma(x),$$

which implies (15). □

5. The AR(1) case for sublinear functions

Throughout this section we assume that there is some $a \in \mathbb{R}$ with $0 < |a| < 1$ such that $a_n = a^n$ for all $n \geq 0$. In this case X is the random power series

$$X = \sum_{n \geq 0} a^n Y_n. \tag{23}$$

It is well known that X converges absolutely P-a.s., if and only if

$$E[\log^+ |Y_0|] < \infty, \tag{24}$$

see for instance Billingsley [1, Exercise 22.11] and Kawata [8, Theorem 14.4.1]. We denote by ν the common distribution of the Y_n, $n \in \mathbb{N}$, and by μ the distribution of X. Then given a random variable Y which is independent of X and distributed according to ν it is immediate from (23) that X and $aX + Y$ have the same distribution μ. This is the simplest example where μ arises as stationary distribution of a linear process, namely of the AR(1)-process which satisfies the recursion

$$X_{n+1} = aX_n + Y_n \quad (n \geq 0).$$

In terms of the characteristic functions of ν and μ, stationarity of this process means

$$\widehat{\mu}(t) = \widehat{\mu}(at)\widehat{\nu}(t) \quad (t \in \mathbb{R}). \tag{25}$$

The next lemma is the heart of the proof of Theorem 2. It relates the local behaviors of $\widehat{\nu}$ and $\widehat{\mu}$ at the origin.

Lemma 6. *Assume (24) and the existence of some $0 < |a| < 1$ such that $a_n = a^n$ for all $n \geq 0$. Let $\tau > 0$ and let $h \in \mathcal{C}^1(]0, \tau], [0, \infty[)$ such that $h'(t) \leq 0$ for all $t \in]0, \tau[$. Furthermore assume that ν is symmetric around 0. Then*

$$\int_0^\tau \frac{h(t)}{t} |1 - \widehat{\nu}(t)|\, dt < \infty \tag{26}$$

if and only if

$$\int_0^\tau -h'(t) \int_0^1 |1 - \widehat{\mu}(|a|^y t)| \; dy \; dt < \infty. \tag{27}$$

Note that the integrands in (26) and (27) are critical only at $t = 0$.

Proof. Without loss of generality we may assume that there is no neighborhood of 0 in which h is constant because otherwise the statement is true since (27) is fulfilled trivially and (26) follows from Lemma 5 (15) $\Leftrightarrow$ (18) with $f = \log^+$ and assumption (24).
It follows directly from the symmetry of ν and (23) that μ is symmetric around 0, too. Therefore both $\widehat{\nu}$ and $\widehat{\mu}$ are real valued continuous and even functions which are at most 1 and achieve the value 1 at 0. Consequently, there is some $T \in]0, \tau]$ such that $\widehat{\nu}(t) > 0$ and $\widehat{\mu}(t) > 0$ for $t \in [-T, T]$. Since $\widehat{\mu}(x) \to 1$ as $x \to 0$ we can represent $1 - \widehat{\mu}(|a|^y t)$ for $y \geq 0$ as the telescopic sum

$$1 - \widehat{\mu}(|a|^y t) = \sum_{k \geq 0} \widehat{\mu}(|a|^{k+y+1} t) - \widehat{\mu}(|a|^{k+y} t).$$

For $|t| < T$, all the above summands are nonnegative since due to (25),

$$\widehat{\mu}(|a|^{k+y} t) = \widehat{\mu}(a|a|^{k+y} t) \widehat{\nu}(|a|^{k+y} t) \leq \widehat{\mu}(|a|^{k+y+1} t),$$

where we used that $\widehat{\mu}$ is even and that $\widehat{\mu}(|a|^{k+y+1} t) \geq 0$ and $\widehat{\nu}(|a|^{k+y} t) \leq 1$. Consequently, for $|t| < T$ by Fubini's theorem,

$$\begin{aligned} \int_0^1 1 - \widehat{\mu}(|a|^y t) \; dy \;&=\; \sum_{k \geq 0} \int_k^{k+1} \widehat{\mu}(|a|^{y+1} t) - \widehat{\mu}(|a|^y t) \; dy \\ = \int_0^\infty \widehat{\mu}(|a|^{y+1} t) - \widehat{\mu}(|a|^y t) \; dy \;&=\; \frac{1}{-\ln |a|} \int_0^t \frac{\widehat{\mu}(|a| s) - \widehat{\mu}(s)}{s} \; ds \\ = \frac{1}{-\ln |a|} \int_0^t \frac{\widehat{\mu}(as)(1 - \widehat{\nu}(s))}{s} \; ds, \end{aligned}$$

where we used the symmetry of $\widehat{\mu}$ again in the last step. Therefore, (27) is equivalent to convergence of

$$\begin{aligned} &\int_0^T -h'(t) \int_0^t \frac{\widehat{\mu}(as)(1 - \widehat{\nu}(s))}{s} \; ds \; dt \\ &= \int_0^T \frac{\widehat{\mu}(as)(1 - \widehat{\nu}(s))}{s} \int_s^T -h'(t) \; dt \; ds \\ &= \int_0^T \frac{h(s)}{s} (1 - \widehat{\nu}(s)) \phi(s) \; ds, \end{aligned} \tag{28}$$

where $\phi(s) := \widehat{\mu}(as)(1 - h(T)/h(s))$. Since h is not constant on $]0, T]$ but decreasing, $\phi(s)$ tends to some positive and finite number as $s \searrow 0$. Hence we can omit $\phi(s)$ from (28) without changing the convergence of this integral and arrive at the expression in (26). □

Lemma 7. *Theorem 2 holds under the additional assumption that there is some $a \in \mathbb{R}$ with $0 < |a| < 1$ such that $a_n = a^n$ for all $n \geq 0$.*

Proof. 1. Due to the comments about convergence of X at the beginning of this section it only remains to show (5). Observe that, given f, g is determined by (7) only up to an additive constant and that adding a constant c_9 to g means to add c_9 to $E[g(|Y_0|)]$. Hence by choosing $c_9 \geq 0$ large enough we may assume without loss of generality that

$$f(T) \leq \alpha g(T). \tag{29}$$

2. Now we check the assumptions of Lemma 5 for g instead of f. It only remains to show (6) and (17) for g.

For (6) for g, which is equivalent to (10) for g, we check that

$$tg'(t) = f(t) \leq \alpha g(t) \text{ for } t \geq T. \tag{30}$$

By (29), this is true for $t = T$. Therefore, the inequality in (30) holds if it holds after differentiating and multiplying both of its sides by t, which gives $tf'(t) \leq \alpha t g'(t) = \alpha f(t)$, which is true due to (10).

Finally, (17) for g is a consequence of $t^2 g'(t) = tf(t)$ which is increasing due to $f, f' \geq 0$.

3. In this part of the proof we show (5) under the additional assumption that Y_n is symmetrically distributed around 0. By Lemma 5, $E[f(|X|)] < \infty$ is equivalent to (16) with $(\sigma, \theta) = (\mu, |a|)$. This in turn is equivalent to (27) with $\tau = 1/T$ and $-h'(t) = f'(1/t)t^{-2}$, that is $h(t) = f(1/t)$. The assumptions of Lemma 6 for this function h are fulfilled since f is increasing and differentiable on $]T, \infty[$. Consequently, by Lemma 6, $E[f(|X|)] < \infty$ is equivalent to

$$\int_0^{1/T} \frac{f(1/t)}{t} |1 - \widehat{\nu}(t)| \, dt < \infty. \tag{31}$$

However, due to $f(t) = tg'(t)$, (31) is equivalent to (18) with (g, ν) instead of (f, σ). By the second part of the proof, Lemma 5 is applicable again and shows that (31) is equivalent to $E[g(|Y_0|)] < \infty$.

4. We now prove (5) for ν not necessarily symmetric. Let $(\tilde{X}, (\tilde{Y}_n)_n)$ be an independent copy of $(X, (Y_n)_n)$. Due to the second part of the proof we may apply Lemma 4 with $f_2 = g$ to see that $E[g(|Y_0|)] < \infty$ is equivalent to $E[g(|Y_0 - \tilde{Y}_0|)] < \infty$. However, $Y_0 - \tilde{Y}_0$ is distributed symmetrically around 0 and by absolute convergence in (23), $X - \tilde{X} =$

$\sum_n a^n(Y_n - \tilde{Y}_n)$. Therefore due to the third part of the proof, $E[g(|Y_0 - \tilde{Y}_0|)] < \infty$ is equivalent to $E[f(|X - \tilde{X}|)] < \infty$. Finally, again by Lemma 4 with $f_2 = f$, this is equivalent to $E[f(|X|)] < \infty$. □

6. The general sublinear case

Proof of Theorem 2. Observe that due to (3) there are constants $c_{10} < \infty$ and $0 < a \le A < 1$ such that $a^n \le |a_n| \le c_{10} A^n$ for all n with $a_n \neq 0$. The absolute convergence of X then follows from

$$\sum_{n\ge 0} |a_n Y_n| \le c_{10} \sum_{n\ge 0} A^n |Y_n|,$$

which is due to (24) P-a.s. finite as mentioned at the beginning of Section 5.

In the proof of (5) we first handle the case in which $a_n \ge 0$ and $Y_n \ge 0$ P-a.s. for all $n \ge 0$. In this case since f is increasing,

$$E\left[f\left(\sum_{n\ge 0} a^n Y_n\right)\right] \le E[f(X)] \le E\left[f\left(c_{10}\sum_{n\ge 0} A^n Y_n\right)\right]. \tag{32}$$

Due to the second part of Lemma 4 one can omit the constant c_{10} in (32) without changing the property of the right-most term in (32) to be finite or infinite. Lemma 7 therefore shows that the three terms in (32) are either all finite or all infinite, depending on whether $E[g(|Y_0|)]$ is finite or not. This proves the statement for nonnegative a_n and Y_n. Hence, the general statement will follow once we have shown the equivalence

$$E[f(|X|)] < \infty \quad \text{if and only if} \quad E\left[f\left(\sum_{n\ge 0} |a_n||Y_n|\right)\right] < \infty, \tag{33}$$

which we shall prove now. The if-part is immediate from the triangle inequality and the monotonicity of f. For the converse we assume $E[f(|X|)] < \infty$ and proceed similarly as in the proof of Lemma 4 as follows. We abbreviate

$$U := \sum_{n\ge 0} (a_n Y_n)^+ \quad \text{and} \quad V := \sum_{n\ge 0} (a_n Y_n)^-,$$

where $Z^+ := \max\{Z, 0\}$ and $Z^- := -\min\{Z, 0\}$. Then, using absolute convergence of X and the monotonicity of f, we get that for any $\gamma > 0$,

$$\infty > E[f(|X|)] = E[f(|U - V|)] \ge E\left[f(U-\gamma) 1_{V \le \gamma \le U}\right]. \tag{34}$$

Now note that both $f(U - \gamma)1_{\gamma \le U}$ and $1_{V\le\gamma}$ are increasing functions of the independent random variables $(a_n Y_n)_n$. Therefore, by the FKG inequality for product measures (see e.g. [10, Theorem 5.2.2.(d)] for finite products and [7, Theorem 2.4] for the generalization to infinite products) the right most term in (34) is greater than or equal to $E\,[f(U-\gamma)1_{\gamma\le U}]\,P[V\le\gamma]$. Consequently, $E[f(|U-\gamma|)]<\infty$ for some large γ. Applying the first part of Lemma 4 to $Z_1=U-\gamma$ and $Z_2=\gamma$, we get $E[f(U)]<\infty$. Analogously, $E[f(V)]<\infty$. Another application of the first part of Lemma 4, this time with $Z_1=U$ and $Z_2=V$, yields $E[f(U+V)]<\infty$, which is the statement on the right hand side of (33). This completes the proof of (33). □

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