# Lyapounov Exponents and Quenched Large Deviations for Multidimensional Random Walk in Random Environment

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Assign to the lattice sites $z \in \mathbb{Z}^d$ i.i.d. random $2d$-dimensional vectors $(\omega(z, z+e))_{|e|=1}$ whose entries take values in the open unit interval and add up to one. Given a realization $\omega$ of this environment, let $(X_n)_{n\geq 0}$ be a Markov chain on $\mathbb{Z}^d$ which, when at $z$, moves one step to its neighbor $z+e$ with transition probability $\omega(z, z+e)$. We derive a large deviation principle for $X_n/n$ by means of a result similar to the shape theorem of first-passage percolation and related models. This result produces certain constants that are the analogues of the Lyapounov exponents known from Brownian motion in Poissonian potential or random walk in random potential. We follow a similar strategy as in Sznitman [18], see also Sznitman [20] and Zerner [21].

**0. Introduction, notations and main results.** In the present work we obtain a large deviation principle for the position of a nearest neighbor random walk on the hypercubic lattice $\mathbb{Z}^d$ ($d \geq 1$) with site-dependent random transition probabilities. The precise model is as follows.

We attach to each vertex $z \in \mathbb{Z}^d$ a $2d$-dimensional vector $(\omega(z, z+e))_{|e|=1}$ where $e$ runs over all signed canonical unit vectors of the lattice. The entries $\omega(z, z+e)$ are strictly positive and fulfill $\sum_e \omega(z, z+e) = 1$. We assume throughout this paper that the vectors $(\omega(z, z+e))_{|e|=1}$, $z \in \mathbb{Z}^d$, are independent and identically distributed random vectors on some probability space with sample space $\Omega$ and probability measure $\mathbb{P}$. For the sake of simplicity we denote the elements of $\Omega$ by $\omega$, too. Each such $\omega$ serves as environment for a Markov chain $(X_n)_{n\geq 0}$ with start in $x \in \mathbb{Z}^d$ defined on another probability space with probability measure $P_{x,\omega}$ such that $\omega(z, z+e)$ is the transition probability from $z$ to its neighbor $z+e$, i.e.

$$P_{x,\omega}[X_0 = x] = 1$$

$$P_{x,\omega}[X_{n+1} = z+e \mid X_n = z] = \begin{cases} \omega(z, z+e) & \text{if } |e| = 1 \\ 0 & \text{otherwise} \end{cases}$$

For an illustration see Figure 1. This discrete time Markov chain $(X_n)_{n\geq 0}$ on $\mathbb{Z}^d$ is called random walk in random environment (RWIRE). If $d = 1$ we recover

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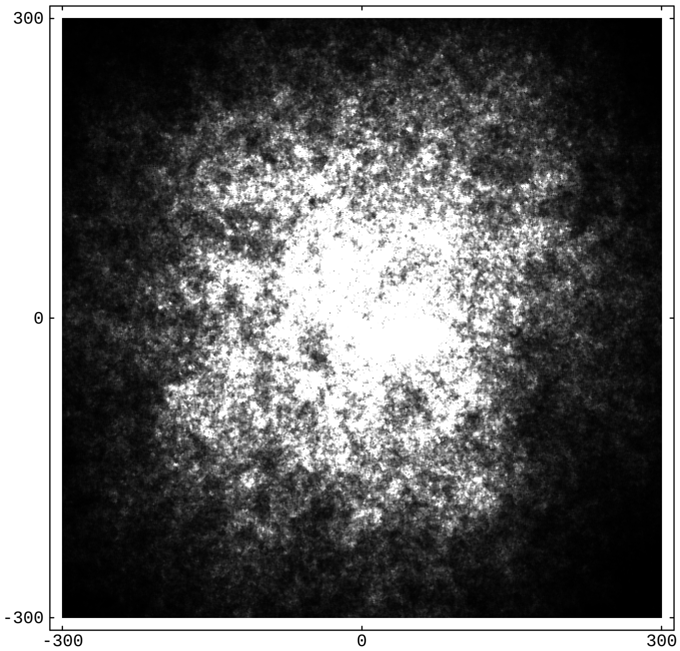


FIG. 1. *Density plot of the distribution of the displacement vector* $X_n$ *after* $n = 15000$ *steps in two dimensions. The environment* $\omega$ *is a realization of* $\omega(x, x+e) = \xi(x, x+e)/\sum_{e'} \xi(x, x+e')$, *where the random variables* $\xi(x, x+e)$ $(x \in \mathbb{Z}^d, |e| = 1)$ *are i.i.d. and exponentially distributed with mean 1.*

the classical one-dimensional RWIRE that has been extensively studied. Some of the fundamental results concerning recurrence properties and limit theorems for $d = 1$ are due to Solomon [17]. For $d \geq 2$ this model has been introduced by Kalikow [7]. However up to now, compared with the one-dimensional situation, not much is known about the higher-dimensional case. For a survey on both cases see e.g. [6, Ch. 6] and [16, Ch. III]. We are interested in large deviations of $X_n/n$ as $n \to \infty$ that hold for $\mathbb{P}$-almost all fixed environments $\omega$, the so-called *quenched* case. Greven and den Hollander [5] studied in $d = 1$ quenched large deviations of $X_n/n$ under $P_{0,\omega}$. They characterized the rate function as a solution to a variational problem involving specific relative entropy with respect to a certain stationary Markov process and solve this variational problem in terms of random continued fractions and Lyapounov exponents of products of certain infinite random matrices. The works of Gantert and Zeitouni [4] and Pisztora and Povel [14] have completed the one-dimensional quenched large deviation picture. In the one-dimensional *annealed* case, i.e. after averaging over the environment, large deviations have been obtained by Dembo, Peres and Zeitouni [1] and Pisztora, Povel and Zeitouni [15].

Our approach to quenched large deviations is different. Most of the success in the study of the one-dimensional RWIRE comes from the fact that due to the simple geometry of $\mathbb{Z}$ one is able to construct explicit solutions for certain desired quantities or distributions. In higher dimensions this is not possible any more. Instead we use and modify more indirect but nevertheless powerful methods which have been developed for the study of Brownian motion in a Poissonian potential by Sznitman ([18], [20]), and random walk in a nonnegative random potential, see [21]. The rough idea is to derive suitable shape theorems analogous to those of first-passage percolation (see e.g. [8]) and use these shape theorems to deduce the large deviation principle. First of all we construct certain two-point functions which are independent of time but still contain enough information about the properties of $X_n$ at a fixed time $n$. These functions are defined as

$$e_\lambda(x,y,\omega) := E_{x,\omega}\left[e^{-\lambda H(y)} \cdot 1_{\{H(y)<\infty\}}\right] \qquad (x,y \in \mathbb{Z}^d, \omega \in \Omega)$$

where $\lambda$ is a real parameter, $E_{x,\omega}$ denotes the expectation with respect to $P_{x,\omega}$ and

$$H(y) := \inf\{n \geq 0 : \ X_n = y\} \quad (\leq \infty) \tag{1}$$

is the first-passage time through $y$. For $\lambda < 0$ the two-point function $e_\lambda(x,y,\omega)$ might be infinite. If $\lambda$ is strictly positive $e_\lambda(x,y,\omega)$ is simply the Laplace transform of $H(y)$ under the measure $P_{x,\omega}$ at point $\lambda$. For $\lambda \geq 0$ there is also an interesting interpretation of this quantity as the probability that a "mortal" walker, who survives each step only with probability $e^{-\lambda}$, ever reaches $y$ before dying. We are interested in the $\mathbb{P}$-almost sure decay rates of $e_\lambda(x,y,\omega)$ when $|x-y|$ tends to infinity. For this reason we introduce

$$a_\lambda(x,y,\omega) := -\ln e_\lambda(x,y,\omega).$$

Figure 2 shows a contour plot of $a_\lambda(x,0,\omega)$ for some specific realization and some positive $\lambda$. One might see in this figure that $a_\lambda(nx,0)$ grows roughly linear as $n \to \infty$ with a slope depending on the direction $x$. Furthermore one might guess that the random contours converge after scaling to an asymptotic shape. In fact, our first main result is

THEOREM A. (Lyapounov exponents and shape theorems) *Suppose that $-\ln\omega(0,e)$ has finite $d$th moment for all nearest neighbors $e$ of the origin $0$. Then there exist a continuous function $\alpha : [0,\infty) \times \mathbb{R}^d \to [0,\infty), (\lambda,x) \mapsto \alpha_\lambda(x)$, which is concave increasing in $\lambda$ and homogeneous and convex in $x$, and a set $\Omega_1$ of full $\mathbb{P}$-measure with the following properties:*

(Shape Theorem:) *For all $\lambda \geq 0$ and all sequences $y_n$ with $|y_n| \to \infty$*

$$\lim_{n\to\infty} \frac{a_\lambda(0,y_n,\omega) - \alpha_\lambda(y_n)}{|y_n|} = 0 \qquad \text{for all } \omega \in \Omega_1 \text{ and in } L^1(\mathbb{P}). \tag{2}$$

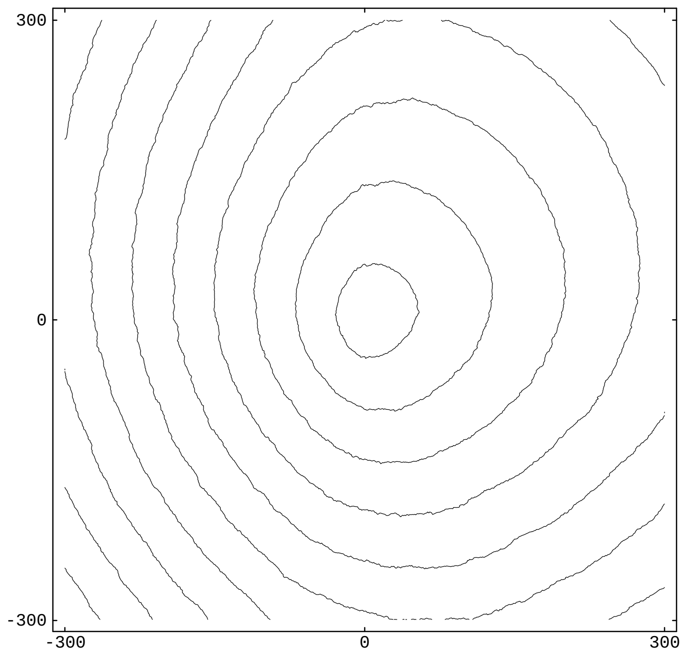


FIG. 2. *Contour plot of $a_\lambda(x,0,\omega)$ with $\lambda = 0.5$. The environment $\omega$ is of the same type as in Figure 1 with random variables $\xi(x,x+e)$ $(x \in \mathbb{Z}^d, |e| = 1)$ that are uniformly distributed on the interval $(0, m_e)$ with $m_{e_1} = 1, m_{e_2} = 2, m_{-e_1} = 3$ and $m_{-e_2} = 4$.*

(Uniform Shape Theorem:) *For all $\lambda > 0$ and all sequences $x_n, y_n$ such that $c(|x_n| \vee |y_n|) \leq |y_n - x_n| \to \infty$ for some $c > 0$*

$$(3) \quad \lim_{n\to\infty} \frac{a_\lambda(x_n, y_n, \omega) - \alpha_\lambda(y_n - x_n)}{|y_n - x_n|} = 0 \qquad \text{for all } \omega \in \Omega_1 \text{ and in } L^1(\mathbb{P}).$$

*Note that the sequences $x_n, y_n$ can be taken function of $\omega$.*

These Lyapounov exponents $\alpha_\lambda(x)$ are the counterparts of the time constant of first-passage percolation and the Lyapounov exponents appearing in Brownian motion in Poissonian potential and random walk in random potential. Their existence is a consequence of the subadditive ergodic theorem and a supermultiplicative property of $e_\lambda(x,y)$ for $\lambda \geq 0$. The Lyapounov exponents show up in the rate function of the large deviation principle we are going to state next. It seems that our method only works if there are arbitrary large regions where the local drifts point approximately to the origin. These regions allow for slowing down the walk without paying an exponential cost. The right condition to guarantee the existence of such regions is the following.

DEFINITION. *The* **nestling property (NP)** *is said to hold if the convex hull of the support of the law of* $\sum_{|e|=1} \omega(0,e)e$ *contains the origin, i.e.*

$$0 \in \operatorname{conv}\left(\operatorname{supp}\left(\operatorname{law}\left(\sum_{|e|=1} \omega(0,e)e\right)\right)\right).$$

*Note that this convex hull is closed.*

THEOREM B. (Large deviation principle) *Suppose that* $-\ln \omega(0,e)$ *has finite dth moment for all nearest neighbors e of the origin 0. Then* $X_n/n$ *obeys the following large deviation principle with rate function*

$$I(z) := \sup_{\lambda \geq 0} \left(\alpha_\lambda(z) - \lambda\right).$$

(Upper bound:) *On a set* $\Omega_2$ *of full* $\mathbb{P}$*-measure for any* $x \in \mathbb{R}^d$ *and any closed subset* $A \subseteq \mathbb{R}^d$

$$\limsup_{n\to\infty} \frac{1}{n} \ln P_{nx,\omega}[X_n \in [nA]] \leq -\inf_{y\in A} I(y-x). \tag{4}$$

(Lower bound:) *The nestling property holds if and only if on a set* $\Omega_3$ *of full* $\mathbb{P}$*-measure for any* $x \in \mathbb{R}^d$ *and any open subset* $B \subseteq \mathbb{R}^d$

$$\liminf_{n\to\infty} \frac{1}{n} \ln P_{nx,\omega}[X_n \in nB] \geq -\inf_{y\in B} I(y-x). \tag{5}$$

*Here the RWIRE is supposed to start at some neighboring lattice site* $[nx]$ *of* $nx$ *if* $nx \notin \mathbb{Z}^d$*, i.e.* $P_{nx,\omega} := P_{[nx],\omega}$*. Furthermore the set* $[nA]$ *denotes the set of all* $[ny]$ *with* $y \in A$*.*
*The rate function I vanishes at the origin, is nonnegative, convex, continuous, and finite on the closed* $|\cdot|$*-unit ball* $\bar{B}(0,1) = \{x \in \mathbb{R}^d : |x_1| + \ldots + |x_d| \leq 1\}$*, and infinite on its complement.*

If the nestling property does not hold then the appropriate rate function is supposed to depend also on Lyapounov exponents $\alpha_\lambda$ with negative $\lambda$. This can be shown in the one-dimensional situation, see [5]. However in $d \geq 2$ our method breaks down without the nestling property since we loose in general the supermultiplicativity of $e_\lambda(x,y)$ for $\lambda < 0$ and thus cannot even apply the subadditive ergodic theorem to construct $\alpha_\lambda$ for negative $\lambda$.

Let us now describe how the present article is organized. In Section 1 we use the subadditive ergodic theorem to prove the existence of the Lyapounov exponents $\alpha_\lambda(x)$ as limits of $a_\lambda(0,nx)/n$ as $n \to \infty$. Section 2 is independent of Section 1. It uses a martingale method to produce an upper bound on the variance of $a_\lambda(x,y,\omega)$ with respect to the environment. This bound is used in Section 3 to extend the existence of limits of $a_\lambda(nx,ny)/n$ from the case $x = 0$, which is provided by the subadditive ergodic theorem, to the case $x \neq 0$. These limits along parallel shifts are one ingredient for the proof of the uniform shape theorem (3) in Section 3. The other ingredient is the maximal lemma of Section 3 that enables us to patch up the Lyapounov exponents for different directions. In Section 4 we derive three other equivalent formulations of the nestling property that will be useful in Section 5 where we prove Theorem B. We want to remark that the uniform shape theorem and Section 2 could be avoided if one is interested only in large deviations with fixed starting point $x = 0$ (see Remark 1 after the proof of Theorem B). In the last section we show how part of the results obtained in [5] in the one-dimensional setting follow from Theorem B.

We close this section with some general notation. By $|x|$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we always mean the $\ell_1$-norm of $x$, i.e. $|x_1| + \ldots + |x_d|$. The $|\cdot|$-unit sphere will be called $S^{d-1}$. Open $|\cdot|$-balls with center $x$ and radius $r$ are designated by $B(x, r)$, closed balls by $\bar{B}(x, r)$. For any sites $x, y \in \mathbb{Z}^d$ the relation $x \sim y$ means that $x$ and $y$ are adjacent, i.e. $|x - y| = 1$. The euclidean norm is called $\|\cdot\|_2$. By $[x]$ we mean the lattice site with minimal $|\cdot|$-distance from $x$ with some deterministic rule for breaking ties. Note that always $|[x] - x| \leq d/2$. If we apply a function $f(x, y, \ldots)$ which has originally been defined solely for $x, y, \ldots \in \mathbb{Z}^d$ to $x, y, \ldots \in \mathbb{R}^d$ we always mean $f([x], [y], \ldots)$. $[A]$ is the set of all $[x]$ with $x \in A$. $e_1, \ldots, e_d$ are the canonical unit vectors of $\mathbb{R}^d$. To simplify the notation of expectations we use the abbreviation $E[Z, A] := E[Z \cdot 1_A]$. We use $c_1, c_2, \ldots$ to denote arbitrary positive constants, which depend only on dimension $d$ and the distribution of $(\omega(0, e))_{e \sim 0}$. If a constant $c_i$ is to depend on some other quantity $x$ , this will be made explicit by $c_i(x)$.

## 1. Lyapounov exponents.

We start with two basic observations about $a_\lambda(x, y)$: the triangle inequality and conditions for integrability.

LEMMA 1. (Triangle inequality) *For any $x, y, z \in \mathbb{Z}^d$, $\lambda \geq 0$ and $\omega \in \Omega$*

$$a_\lambda(x, y, \omega) \leq a_\lambda(x, z, \omega) + a_\lambda(z, y, \omega). \tag{6}$$

*If $d = 1$ and $z$ is located between $x$ and $y$ then equality holds in (6).*

PROOF. Let $\widetilde{H}_z(y) := \inf\{n \geq H(z) : X_n = y\} \leq \infty$ be the first time after $H(z)$ at which the walk reaches $y$. Note that $\widetilde{H}_z(y) \geq H(y)$. Consequently we get for $\lambda \geq 0$ by the strong Markov property applied to $H(z)$

$$\begin{aligned} e_\lambda(x, y, \omega) &\geq E_{x,\omega}\left[\exp\left(-\lambda \widetilde{H}_z(y)\right), \widetilde{H}_z(y) < \infty\right] \\ &= e_\lambda(x, z, \omega) \cdot e_\lambda(z, y, \omega) \end{aligned} \tag{7}$$

and thus (6). If $d = 1$ and $x \leq z \leq y$ or $y \leq z \leq x$ then any path from $x$ to $y$ must pass through $z$ such that $\widetilde{H}_z(y) = H(y)$. Thus in this case we have equality in (7) and (6). $\square$

LEMMA 2. (Integrability of $a_\lambda(x,y)$) *Let $\lambda \geq 0$ and $p \geq 1$. Then $a_\lambda(x,y,\omega)$ has finite pth moment for any $x,y \in \mathbb{Z}^d$ if $-\ln\omega(0,e)$ has finite pth moment for all $e \sim 0$. If $-\ln\omega(0,e)$ is integrable for all $e \sim 0$ then the family $a_\lambda(x,y,\omega)/|y-x|$ $(x,y \in \mathbb{Z}^d, x \neq y)$ is uniformly integrable and*

$$|y-x|\lambda \leq \mathbb{E}[a_\lambda(x,y,\omega)] \;\leq\; |y-x| \left(\lambda + \max_{e\sim 0} \mathbb{E}[-\ln\omega(0,e)]\right) \tag{8}$$

*for all $x,y \in \mathbb{Z}^d$. Here $\mathbb{E}$ denotes the expectation operator corresponding to $\mathbb{P}$.*

PROOF. Let $x,y \in \mathbb{Z}^d$ with $x \neq y$ and $\lambda \geq 0$. Then for any nearest-neighbor path $(x = x_0, x_1, \ldots, x_n = y)$ from $x$ to $y$ with minimal length $n = |y-x|$

$$\begin{aligned} e_\lambda(x,y,\omega) &\geq E_{x,\omega}\left[e^{-\lambda H(y)}, X_m = x_m \ (m = 0,\ldots,n)\right] \\ &= e^{-\lambda n} \prod_{m=1}^{n} \omega(x_{m-1}, x_m) \end{aligned}$$

and hence

$$\frac{a_\lambda(x,y,\omega)}{|y-x|} \leq\; \lambda - \frac{1}{n}\sum_{m=1}^{n} \ln\omega(x_{m-1},x_m). \tag{9}$$

This shows that $a_\lambda(x,y) \in L^p(\mathbb{P})$ holds if $-\ln\omega(0,e) \in L^p(\mathbb{P})$ for all $e \sim 0$. Moreover for any $\gamma \geq 0$ by Jensen's inequality

$$\begin{aligned} &\mathbb{E}\left[\left(\lambda - \gamma + \frac{1}{n}\sum_{m=1}^{n} -\ln\omega(x_{m-1},x_m)\right)_+\right] \\ &\leq \frac{1}{n}\sum_{m=1}^{n} \mathbb{E}\left[(\lambda - \gamma - \ln\omega(x_{m-1},x_m))_+\right] \;\leq\; \max_{e\sim 0} \mathbb{E}\left[(\lambda - \gamma - \ln\omega(0,e))_+\right] \end{aligned}$$

which tends to zero as $\gamma \to \infty$ if $-\ln\omega(0,e)$ is integrable for all $e \sim 0$. This proves the statement about uniform integrability. With $\gamma = 0$ and (9) we get the right inequality of (8). For the lower bound on $\mathbb{E}[a_\lambda(x,y,\omega)]$ observe that $H(y) \geq |y-x|$ under $P_{x,\omega}$ and hence $e_\lambda(x,y,\omega) \leq \exp(-\lambda|y-x|)$ for all $\omega \in \Omega$ which implies the left inequality in (8). □

We now introduce the Lyapounov exponents $\alpha_\lambda(x)$.

PROPOSITION 3. (Lyapounov exponents) *Let $\lambda \geq 0$ and suppose that $-\ln\omega(0,e)$ has finite expectation for all $e \sim 0$. Then there exists a nonrandom function $\alpha_\lambda : \mathbb{R}^d \to [0,\infty)$ such that on a set $\Omega_4(\lambda)$ of full $\mathbb{P}$-measure and in $L^1(\mathbb{P})$ for all $x \in \mathbb{Z}^d$*

$$\begin{aligned} \alpha_\lambda(x) &= \lim_{n\to\infty} \frac{1}{n} a_\lambda(0,nx,\omega) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}[a_\lambda(0,nx,\omega)] \\ &= \inf_{n\in\mathbb{N}} \frac{1}{n}\mathbb{E}[a_\lambda(0,nx,\omega)]. \end{aligned} \tag{10}$$

*$\alpha_\lambda(x)$ is concave increasing in $\lambda$ and homogeneous and convex in $x$. Furthermore it is jointly continuous in $\lambda$ and $x$ and satisfies for all $x \in \mathbb{R}^d$*

$$\lambda|x| \le \alpha_\lambda(x) \le \left(\lambda + \max_{e\sim 0} \mathbb{E}[-\ln \omega(0,e)]\right)|x|. \tag{11}$$

PROOF. Let $\lambda \ge 0, x \in \mathbb{Z}^d$ and consider the doubly indexed sequence $a_\lambda(nx, mx),\ 0 \le n \le m,\ n, m \in \mathbb{N}_0$ of integrable (see (8)) random variables. This process satisfies the conditions of the subadditive ergodic theorem (see [11, p. 277]) thanks to ergodicity properties and translation invariance of $\mathbb{P}$. Consequently there is some finite constant $\alpha_\lambda(x) \ge 0$ such that (10) is fulfilled $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$. This and (8) imply (11) for all $x \in \mathbb{Z}^d$. Furthermore it is easy to conclude from (10) and (6) that

$$\alpha_\lambda(qx) = q\alpha_\lambda(x) \quad \text{and} \quad \alpha_\lambda(x+y) \le \alpha_\lambda(x) + \alpha_\lambda(y) \tag{12}$$

holds for any $0 < q \in \mathbb{Q}$ and $x, y \in \mathbb{Z}^d$. By setting $\alpha_\lambda(x/q) := \alpha_\lambda(x)/q$ we extend $\alpha_\lambda$ first to a function on $\mathbb{Q}^d$ and then by continuity to a function on $\mathbb{R}^d$ which satisfies (11) and (12) for any $q > 0$ and $x, y \in \mathbb{R}^d$. Hence $\alpha_\lambda(x)$ is homogeneous and convex in $x$. It increases with $\lambda$ since $a_\lambda(0, x, \omega)$ increases and is concave in $\lambda$ due to Hölder's inequality which implies

$$e_{(\lambda_1+\lambda_2)/2}(0,x,\omega) \le e_{\lambda_1}(0,x,\omega)^{1/2} \cdot e_{\lambda_2}(0,x,\omega)^{1/2}.$$

As a consequence the map $\lambda \mapsto \alpha_\lambda(x)$ is lower semicontinuous. On the other hand by dominated convergence $\mathbb{E}[a_\lambda(0,x,\omega)]$ depends continuously on $\lambda$. Therefore for $x \in \mathbb{Z}^d$ due to (10), $\lambda \mapsto \alpha_\lambda(x)$ is upper semicontinuous as infimum of continuous functions. This implies continuity for arbitrary $x \in \mathbb{R}^d$. Moreover it now follows from a Dini type argument that $\alpha_\lambda(x)$ is jointly continuous in $\lambda$ and $x$. □

REMARKS. 1. Due to the triangle inequality the two-point function $a_\lambda$ induces a random distance function

$$d_\lambda(x,y,\omega) := \max\{a_\lambda(x,y,\omega), a_\lambda(y,x,\omega)\} \tag{13}$$

for $\lambda > 0$. If $\lambda = 0$, $d_\lambda$ might be only a semimetric or even identically vanishing as for example in the case of the usual simple random walk with $d \le 2$.
Unlike the Lyapounov exponents of Brownian motion in Poissonian potential [20, Chapter 5, Theorem 2.5] or random walk in random potential [21, Theorem 8], $\alpha_\lambda(x)$ is in general not invariant under reflection at the origin since we did not assume any invariance of the distribution of $(\omega(0,e))_e$ (for an example see Figure 2). Thus the exponential decay rate $\max\{\alpha_\lambda(x), \alpha_\lambda(-x)\}$ of $d_\lambda(nx)$ as $n \to \infty$ may differ from $\alpha_\lambda(x)$.

2. In the one-dimensional case the proof of Lemma 1 shows that equality in (6) holds also for negative $\lambda$. Note that in this case both sides of (6) may equal $-\infty$ if $\lambda$ is less than some critical value. Thus for $d = 1$ the ergodic theorem gives us $\alpha_\lambda$ also for negative $\lambda$. However we shall not make use of these exponents in the present work.

**2. Fluctuations around the mean value.** The main result of this section is

THEOREM 4. (Upper bound on fluctuations) *Suppose that* $-\ln\omega(0,e)$ *is square integrable for all* $e \sim 0$ *and let* $\lambda > 0$. *Then for some finite constant* $c_1(\lambda)$

$$\mathrm{Var}(a_\lambda(x,y,\omega)) \le c_1(\lambda)\cdot|x-y| \qquad \text{for all } x,y \in \mathbb{Z}^d. \tag{14}$$

The proof is similar to that of [9, (1.13)], [19, Theorem 2.1] and [21, Th. 11]. We use a lemma that gives us upper bounds on how much $a_\lambda(0,y,\omega)$ can change when $\omega$ is changed at a single site. One bound is formulated in terms of the transformed path measure

$$\hat{P}^y_{x,\lambda,\omega} := e_\lambda(x,y,\omega)^{-1} e^{-\lambda H(y)} 1_{\{H(y)<\infty\}} \cdot P_{x,\omega} \tag{15}$$

under which the process $(X_n)_{n\ge 0}$ is roughly speaking a "mortal" RWIRE that starts at $x$, is conditioned to reach $y$ and survives each step with probability $e^{-\lambda}$.

LEMMA 5. (Rank-one perturbation formula) *Let* $\lambda > 0$, $z \in \mathbb{Z}^d$ *and* $\omega_1, \omega_2 \in \Omega$ *such that* $\omega_1(x,u) = \omega_2(x,u)$ *for all* $x \ne z$ *and* $u \sim x$. *Then for all* $y \in \mathbb{Z}^d$

$$\frac{e_\lambda(0,y,\omega_2)}{e_\lambda(0,y,\omega_1)} \ge \max\left\{\hat{P}^y_{0,\lambda,\omega_1}[H(y)\le H(z)],\ (1-e^{-\lambda})\min_{u\sim z}\omega_2(z,u)\right\} \tag{16}$$

PROOF OF LEMMA 5. We only need to consider the case where the quotient on the left side of (16) is less than 1 since the right member of (16) is at most 1. In particular we may assume that $y \ne z$ and $y \ne 0$ and that even in the case $d = 1$ there is some path from the origin to $z$ that does not touch $y$ because otherwise $e_\lambda(0,y,\omega_1)$ and $e_\lambda(0,y,\omega_2)$ do not depend on $\omega_i(z,u)$ and thus coincide. By the strong Markov property

$$\begin{aligned} e_\lambda(0,y,\omega_2) &= E_{0,\omega_2}[e^{-\lambda H(y)}, H(y) < H(z)] \\ &\quad + E_{0,\omega_2}[e^{-\lambda H(z)}, H(z) < H(y)]\cdot E_{z,\omega_2}[e^{-\lambda H(y)}, H(y)<\infty] \\ &= E_{0,\omega_1}[e^{-\lambda H(y)}, H(y) < H(z)] \\ &\quad + E_{0,\omega_1}[e^{-\lambda H(y)}, H(z) < H(y)]\cdot \frac{e_\lambda(z,y,\omega_2)}{e_\lambda(z,y,\omega_1)} \end{aligned}$$

and consequently

$$\begin{aligned} 1 &\ge \frac{e_\lambda(0,y,\omega_2)}{e_\lambda(0,y,\omega_1)} \\ &= \hat{P}^y_{0,\lambda,\omega_1}[H(y)\le H(z)] + \frac{e_\lambda(z,y,\omega_2)}{e_\lambda(z,y,\omega_1)}\hat{P}^y_{0,\lambda,\omega_1}[H(z) < H(y)] \end{aligned} \tag{17}$$

Since both summands on the right-hand side of (17) are nonnegative this implies the first part of inequality (16). For the second part observe that (17) also implies $e_\lambda(z,y,\omega_2)/e_\lambda(z,y,\omega_1) \le 1$ because otherwise the right-hand side of (17) would be strictly greater than 1 due to $\hat{P}^y_{0,\lambda,\omega_1}[H(z) < H(y)] > 0$. Hence

$$\begin{aligned}\frac{e_\lambda(0,y,\omega_2)}{e_\lambda(0,y,\omega_1)} &\ge \frac{e_\lambda(z,y,\omega_2)}{e_\lambda(z,y,\omega_1)}\Big(\hat{P}^y_{0,\lambda,\omega_1}[H(y)\le H(z)] + \hat{P}^y_{0,\lambda,\omega_1}[H(z)<H(y)]\Big)\\ &= \frac{e_\lambda(z,y,\omega_2)}{e_\lambda(z,y,\omega_1)} \qquad (18)\end{aligned}$$

By counting the returns to $z$ before the first visit of $y$ and using the strong Markov property again we get for $i = 1, 2$

$$e_\lambda(z,y,\omega_i) = \frac{E_{z,\omega_i}[\exp(-\lambda H(y)), H(y) < H_2(z)]}{1 - E_{z,\omega_i}[\exp(-\lambda H_2(z)), H_2(z) < H(y)]} \qquad (19)$$

where $H_2(z) := \inf\{n > H(z) :\ X_n = z\}$ is the time of the second visit of $z$. We estimate the denominator in (19) for $i = 2$ from above by 1 and for $i = 1$ from below by $c_2(\lambda) := 1 - e^{-\lambda}$ and thus get by partition over the first step

$$\begin{aligned}\frac{e_\lambda(z,y,\omega_2)}{e_\lambda(z,y,\omega_1)} &\ge c_2(\lambda)\frac{E_{z,\omega_2}[\exp(-\lambda H(y)), H(y) < H_2(z)]}{E_{z,\omega_1}[\exp(-\lambda H(y)), H(y) < H_2(z)]}\\ &= c_2(\lambda)\frac{\sum_{u\sim z}\omega_2(z,u)e^{-\lambda}E_{u,\omega_2}[\exp(-\lambda H(y)), H(y) < H(z)]}{\sum_{u\sim z}\omega_1(z,u)e^{-\lambda}E_{u,\omega_1}[\exp(-\lambda H(y)), H(y) < H(z)]}\\ &\ge c_2(\lambda)\frac{\max_{u\sim z}\omega_2(z,u)E_{u,\omega_2}[\exp(-\lambda H(y)), H(y) < H(z)]}{\sum_{u\sim z}\omega_1(z,u)\max_{v\sim z}E_{v,\omega_1}[\exp(-\lambda H(y)), H(y) < H(z)]}\\ &\ge c_2(\lambda)\min_{w\sim z}\omega_2(z,w)\frac{\max_{u\sim z}E_{u,\omega_2}[\exp(-\lambda H(y)), H(y) < H(z)]}{\max_{v\sim z}E_{v,\omega_1}[\exp(-\lambda H(y)), H(y) < H(z)]}\\ &= c_2(\lambda)\min_{w\sim z}\omega_2(z,w).\end{aligned}$$

This and (18) yield the second part of (16). □

Proof of Theorem 4. Fix $x, y \in \mathbb{Z}^d$ and $\lambda > 0$. We use a martingale method by representing $a_\lambda(x,y,\omega) - \mathbb{E}[a_\lambda(x,y,\omega)]$ as a sum of martingale differences. To this end we introduce an arbitrary but fixed enumeration $x_k,\ k \in \mathbb{N}$, of $\mathbb{Z}^d$ and let $\mathcal{F}_k,\ k \in \mathbb{N}_0$, be the $\sigma$-field generated by the $2dk$ random variables $\omega(x_i,z_i),\ 1 \le i \le k,\ z_i \sim x_i$. $\mathcal{F}_0$ denotes the trivial $\sigma$-field. Then the martingale $M_k := \mathbb{E}[a_\lambda(x,y,\omega) \mid \mathcal{F}_k],\ k \ge 0$, converges $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$ to $a_\lambda(x,y,\omega)$ as $k \to \infty$. Moreover $a_\lambda(x,y,\omega)$ is square integrable due to Lemma 2 with $p = 2$. Hence the convergence of the martingale takes also place in $L^2(\mathbb{P})$. Note that $M_k(\omega)$ can be represented as $\mathbb{E}_\sigma[a_\lambda(x,y,[\omega,\sigma]_k)]$ where $\mathbb{E}_\sigma$ denotes the expectation with respect to the variable $\sigma$ and the environment $\tilde{\omega} := [\omega,\sigma]_k \in \Omega$ is a mixture of $\omega$ and $\sigma$ for which $\tilde{\omega}(x_i,\cdot)$ agrees with $\omega(x_i,\cdot)$ if $i \le k$ and with

$\sigma(x_i,\cdot)$ otherwise. Thus

$$\operatorname{Var}(a_\lambda(x,y,\omega)) = \mathbb{E}\left[\left(\sum_{k\geq 1} M_k - M_{k-1}\right)^2\right] \;=\; \sum_{k\geq 1}\mathbb{E}\left[(M_k - M_{k-1})^2\right]$$

$$\leq \sum_{k\geq 1}\mathbb{E}_\omega\otimes\mathbb{E}_\sigma\left[\left(\ln\frac{e_\lambda(x,y,[\omega,\sigma]_{k-1})}{e_\lambda(x,y,[\omega,\sigma]_k)}\right)^2\right]. \tag{20}$$

By symmetry in the $k$th variable in (20), those configurations for which the quotient in (20) is not greater than one provide at least half of the value of (20) such that (20) is thanks to Lemma 5 less than

$$2\sum_{k\geq 1}\left(\mathbb{E}_\omega\otimes\mathbb{E}_\sigma\left[\left(\ln(1-e^{-\lambda})\min_{u\sim x_k}\sigma(x_k,u)\right)^2, \hat{P}^y_{x,\lambda,[\omega,\sigma]_k}\left[H(y)\leq H(x_k)\right] < \frac{1}{2}\right]\right.$$

$$\left. + \,\mathbb{E}_\omega\otimes\mathbb{E}_\sigma\left[\left(\ln\hat{P}^y_{x,\lambda,[\omega,\sigma]_k}\left[H(y)\leq H(x_k)\right]\right)^2, \hat{P}^y_{x,\lambda,[\omega,\sigma]_k}\left[H(y)\leq H(x_k)\right]\geq\frac{1}{2}\right]\right).$$

Since $(\ln x)^2 \leq 1-x$ for $1/2\leq x\leq 1$, this is smaller than

$$2\sum_{k\geq 1}\left(\mathbb{E}_\sigma\left[\left(\ln(1-e^{-\lambda})\min_{u\sim x_k}\sigma(x_k,u)\right)^2\right]\cdot\mathbb{P}_\omega\left[\hat{P}^y_{x,\lambda,\omega}\left[H(y)>H(x_k)\right]\geq\frac{1}{2}\right]\right.$$

$$\left. + \,\mathbb{E}_\omega\left[\hat{P}^y_{x,\lambda,\omega}\left[H(y)>H(x_k)\right]\right]\right)$$

$$\leq 2\left(2\mathbb{E}_\sigma\left[\left(\ln(1-e^{-\lambda})\min_{u\sim 0}\sigma(0,u)\right)^2\right]+1\right)\sum_{k\geq 1}\mathbb{E}_\omega\left[\hat{P}^y_{x,\lambda,\omega}\left[H(y)>H(x_k)\right]\right]$$

$$= c_3(\lambda)\mathbb{E}_\omega\left[\hat{E}^y_{x,\lambda,\omega}\left[\sharp\{k\geq 1:\; H(x_k)<H(y)\}\right]\right]$$

$$\leq \frac{c_3(\lambda)}{\lambda}\mathbb{E}_\omega\left[\ln\hat{E}^y_{x,\lambda,\omega}\left[\exp(\lambda\sharp\{k\geq 1:\; H(x_k)<H(y)\})\right]\right]$$

$$= c_4(\lambda)\mathbb{E}_\omega\left[a_\lambda(x,y,\omega)+\ln E_{x,\omega}\left[\exp(\lambda(\sharp\{k\geq 1:\; H(x_k)<H(y)\}-H(y)))\right]\right].$$

Since the rightmost expectation is smaller than one this is less than

$$c_4(\lambda)\mathbb{E}_\omega\left[a_\lambda(x,y,\omega)\right] \;\leq\; c_1(\lambda)|y-x|$$

due to (8). □

**3. Maximal lemma and shape theorems.** The next lemma plays the role of the maximal lemmas of first-passage percolation (e.g. [8, Lemma (3.5), (3.6)]), Brownian motion in a Poissonian potential [18, Lemma 1.3], [20, Chapter 5, Lemma 2.6] and random walk in a nonnegative random potential [21, Lemma 7]. For the definition of the random metric $d_\lambda$ recall (13).

LEMMA 6. (Maximal lemma) *Suppose that* $-\ln\omega(0,e)$ *has finite* $d$*th moment for all* $e\sim 0$ *and let* $\lambda\geq 0$. *Then there are a constant* $c_5(\lambda)$ *and a set* $\Omega_5(\lambda)$ *of full* $\mathbb{P}$*-measure such that for all* $\omega\in\Omega_5(\lambda)$ *and for all* $0<\varepsilon\in\mathbb{Q}$ *there is some finite number* $R(\varepsilon,\omega)$ *such that*

$$\sup\left\{d_\lambda(x,y,\omega):\ y\in\mathbb{R}^d, |y-x|\leq\varepsilon|x|\right\}<c_5(\lambda)\varepsilon|x| \tag{21}$$

*for all* $x\in\mathbb{R}^d$ *with* $|x|\geq R(\varepsilon,\omega)$.

PROOF. We split the proof into two parts, $d\geq 2$ and $d=1$. In the multidimensional case $d\geq 2$ for each pair $x,y\in\mathbb{Z}^d$, $x\neq y$ there are $2d$ self-avoiding nearest-neighbor paths $x^{(i)}=(x_0^{(i)}=x,x_1^{(i)},x_2^{(i)},\ldots,x_{m_i}^{(i)}=y)$ from $x$ to $y$, each containing $m_i\leq|y-x|+8$ edges and being pairwise site disjoint except for the starting and the end point $x$ and $y$. For each $i\in\{1,\ldots,2d\}$

$$\begin{aligned} e_\lambda(x,y,\omega) &\geq E_{x,\omega}[\exp(-\lambda H(y)),\ X_n=x_n^{(i)}\quad \text{for all } n=0,\ldots,m_i]\\ &= e^{-\lambda m_i}\prod_{n=0}^{m_i-1}\omega(x_n^{(i)},x_{n+1}^{(i)})\ \geq\ e^{-\lambda(|y-x|+8)}\prod_{n=0}^{m_i-1}\min_{z\sim x_n^{(i)}}\omega(x_n^{(i)},z)\end{aligned}$$

Hence by independence and identical distribution of the transition vectors and pairwise site disjointness of the paths for any $t>0$

$$\mathbb{P}[a_\lambda(x,y,\omega)+\ln\min_{z\sim x}\omega(x,z)\geq t] \tag{22}$$

$$\leq\mathbb{P}\left[\sum_{n=1}^{|y-x|+8}-\ln\min_{z\sim ne_1}\omega(ne_1,z)\geq t-\lambda(|y-x|+8)\right]^{2d}$$

$$\leq\mathbb{P}\Bigg[\left|\sum_{n=1}^{|y-x|+8}-c_6-\ln\min_{z\sim ne_1}\omega(ne_1,z)\right| \tag{23}$$

$$\geq(t-(\lambda+c_6)(|y-x|+8))_+\Bigg]^{2d}$$

with $c_6:=\mathbb{E}[-\ln\min_{e\sim 0}\omega(0,e)]$. Since the above calculation still holds after interchanging $x$ and $y$ we get by Chebyshev's inequality and square integrability of $-\ln\omega(0,e)$

$$\mathbb{P}[\hat{a}_\lambda(x,y,\omega)\geq t]\leq 2\left(\frac{(|y-x|+8)\mathrm{Var}\,(-\ln\min_{e\sim 0}\omega(0,e))}{(t-(\lambda+c_6)(|y-x|+8))_+^2}\right)^{2d}$$

$$\leq\frac{c_7|y-x|^{2d}}{(t-c_8(\lambda)|y-x|)_+^{4d}}\qquad(\leq\infty) \tag{24}$$

where

$$\hat{a}_\lambda(x,y,\omega):=\max\{a_\lambda(u,v,\omega)+\ln\min_{z\sim u}\omega(u,z):\ (u,v)\in\{(x,y),(y,x)\}\}.$$

Now (21) follows in exactly the same way from (24) as in the proof of [21, Lemma 7] (21) follows from (20). The idea is to take for fixed $0 < \varepsilon \in \mathbb{Q}$ some finite subset $Z$ of the $|\cdot|$-unit sphere $S^{d-1}$ such that the open balls $B(z,\varepsilon)$ with center $z \in Z$ and radius $\varepsilon$ cover $S^{d-1}$. Then for any large $x \in \mathbb{R}^d$ the closed ball $\bar{B}(x,\varepsilon|x|)$ is completely contained in the open ball $B(nz, 3\varepsilon n)$ with $n = \lfloor |x| \rfloor$ for some $z \in Z$. Hence the supremum in (21) is thanks to the triangle inequality (6) less than

$$2\sup\{d_\lambda(nz,y,\omega):\ z \in Z, y \in B(nz,3\varepsilon n)\}$$

$$\leq 2\sup\{\hat{a}_\lambda(nz,y,\omega):\ z \in Z, y \in B(nz,3\varepsilon n)\} \tag{25}$$

$$+\ 2\sup\{-\ln \min_{u\sim y} \omega(y,u):\ z \in Z, y \in B(nz,3\varepsilon n)\}. \tag{26}$$

Since the $d$th moment of $-\ln\omega(0,e)$ is finite for all $e \sim 0$ the supremum in (26) is $\mathbb{P}$-a.s. of order $o(n)$ thanks to the Borel-Cantelli lemma. The supremum in (25) is under control too because it is $\mathbb{P}$-a.s. less than $5c_8(\lambda)\varepsilon n$ for $n$ large. Indeed, the probability that it is bigger than $5c_8(\lambda)\varepsilon n$ is due to (24) for $n \geq d/\varepsilon$ less than

$$\sum_{z\in Z}\sum_{y\in[B(nz,3\varepsilon n)]} \frac{c_7|y-[nz]|^{2d}}{(5c_8(\lambda)\varepsilon n - c_8(\lambda)|y-[nz]|)_+^{4d}} \leq \sum_{z\in Z} \sharp[B(nz,3\varepsilon n)] \frac{c_7(4\varepsilon n)^{2d}}{(c_8(\lambda)\varepsilon n)^{4d}}$$
$$\leq c_9(\lambda,\varepsilon) n^{-d}$$

which is summable since $d \geq 2$. The Borel-Cantelli lemma finishes the proof of (21) for $d \geq 2$.

In the case $d = 1$, $a_\lambda$ is additive in the sense of the last statement of Lemma 1. Thus the supremum in (21) is dominated by $\max\{d_\lambda(x, x+\varepsilon|x|, \omega), d_\lambda(x, x-\varepsilon|x|,\omega)\}$. Hence it suffices to construct a constant $c_5(\lambda)$ such that

$$d_\lambda(x, x \pm \varepsilon|x|, \omega) \leq c_5(\lambda)\varepsilon|x| \tag{27}$$

$\mathbb{P}$-a.s. for large $x$. We only consider the case with $+\varepsilon$ in (27) and $x > 0$. To do this we define recursively $x_0 := 0$ and $x_{n+1} = [(1+\varepsilon)x_n] + 1$ and observe that

$$a_\lambda(x_n, x_{n+1}, \omega) \leq \sum_{k=1}^{x_{n+1}-x_n} \lambda - \ln\omega(x_n + k - 1, x_n + k)$$

which is $\mathbb{P}$-a.s. less than $c_{10}(\lambda)(x_{n+1} - x_n)$ for $n$ large since $x_n$ grows geometrically fast (see e.g. [13, subsection 6.8.5]). An analogous statement holds for $a_\lambda(x_{n+1}, x_n, \omega)$. Hence $\mathbb{P}$-a.s. for $x$ large enough and $n$ with $x_n \leq x \leq x_{n+1}$

$$d_\lambda(x, x+\varepsilon x, \omega) \leq d_\lambda(x_n, x_{n+2}, \omega) \leq c_{10}(\lambda)(x_{n+2} - x_n) \leq c_5(\lambda)\varepsilon x.$$

□

It has been shown in Proposition 3 that the subadditive ergodic theorem gives us the existence of the limit of $a_\lambda(0, ny, \omega)/n$ as $n \to \infty$ where the starting point is fixed. This assertion would suffice for the proof of the usual shape theorem (2).

However, for the uniform shape theorem (3) we need a partially stronger version of (10) that gives us convergence along parallel shifts.

Lemma 7. *Let $\lambda > 0$ and suppose that the dth moment of $-\ln \omega(0,e)$ is finite for all $e \sim 0$. Then on a set $\Omega_6(\lambda)$ of full $\mathbb{P}$-measure for all $x, y \in \mathbb{Z}^d$*

$$\lim_{n\to\infty} \frac{a_\lambda(nx, ny, \omega)}{n} = \alpha_\lambda(y-x). \tag{28}$$

The proof is exactly the same as that of [21, Lemma 14]. The idea for $d \geq 2$ is to use Chebyshev's inequality, the bounds on the variance of Theorem 4, the Borel-Cantelli lemma, and (10) to prove (28) for the subsequence $n^2$. The statement for the full sequence then follows from the triangle inequality (6) and the maximal lemma. For $d = 1$ the claim follows from (10) and the additivity of $a_\lambda$ as stated in Lemma 1.

We now come to the

Proof of Theorem A. The proof goes along the same line as the proofs of [21, Th. 8, Th. 13, Cor. 18]. For completeness we shall give here the proof of the uniform shape theorem (3), which is slightly more involved than that of the simple shape theorem (2) and includes (2) as a special case if $\lambda > 0$. The reason why (2) is valid also for $\lambda = 0$ whereas (3) is only proven for strictly positive $\lambda$ is that our proof of (3) uses Lemma 7 which assumes $\lambda > 0$ whereas the proof of (2) only makes use of (10) and thus holds for all $\lambda \geq 0$. Observe that $L^1(\mathbb{P})$ convergence follows from $\mathbb{P}$-a.s. convergence by uniform integrability provided by Lemma 2. Let us prove $\mathbb{P}$-a.s. convergence in (3) first for some fixed $\lambda > 0$. To this end we fix some $\omega \in \Omega_7(\lambda) := \Omega_5(\lambda) \cap \Omega_6(\lambda)$ (see Lemma 6, Lemma 7) and some $c > 0$ and let $x_n, y_n \in \mathbb{R}^d$ be arbitrary sequences such that $c(|x_n| \vee |y_n|) \leq |y_n - x_n|$ holds for all $n$ and such that $|y_n - x_n|$ tends to infinity as $n$ goes to infinity. We may assume without loss of generality that either $|x_n| \leq |y_n|$ for all $n$ or $|y_n| \leq |x_n|$ for all $n$. We only treat the case $|x_n| \leq |y_n|$ since the other one is similar. Due to compactness of $S^{d-1}$ and of the unit interval we may assume furthermore $y_n/|y_n| \to e_y \in S^{d-1}$ and $|x_n|/|y_n| \to r \in [0,1]$. If there are infinitely many $x_n \neq 0$ let us assume $x_n/|x_n| \to e_x \in S^{d-1}$ for those $n$ with $x_n \neq 0$, otherwise let $e_x \in S^{d-1}$ be arbitrary.

Now let $0 < \varepsilon \in \mathbb{Q}$ and choose $v_x, v_y \in S^{d-1} \cap \mathbb{Q}^d$, $M \in \mathbb{N}$ and $q = q_1/q_2 \in (0,1]$ with $q_1, q_2 \in \mathbb{N}$ such that $Mv_x, Mv_y \in \mathbb{Z}^d$, $|v_x - e_x|, |v_y - e_y| < \varepsilon$, and $|q - r| < \varepsilon$. We approximate $x_n$ and $y_n$ by the lattice vertices

$$x'_n := \left\lfloor \frac{|y_n|}{q_2 M} \right\rfloor q_1 M v_x \in \mathbb{Z}^d \qquad \text{and} \qquad y'_n := \left\lfloor \frac{|y_n|}{q_2 M} \right\rfloor q_2 M v_y \ \in \mathbb{Z}^d$$

where $\lfloor z \rfloor$ denotes the largest integer $\leq z$ (see Figure 3).

Then for $n$ large enough we have

$$|y_n - y'_n| \leq \big|y_n - |y_n| v_y\big| + \big||y_n| v_y - y'_n\big| < \varepsilon |y_n| + q_2 M < 2\varepsilon |y_n| \tag{29}$$

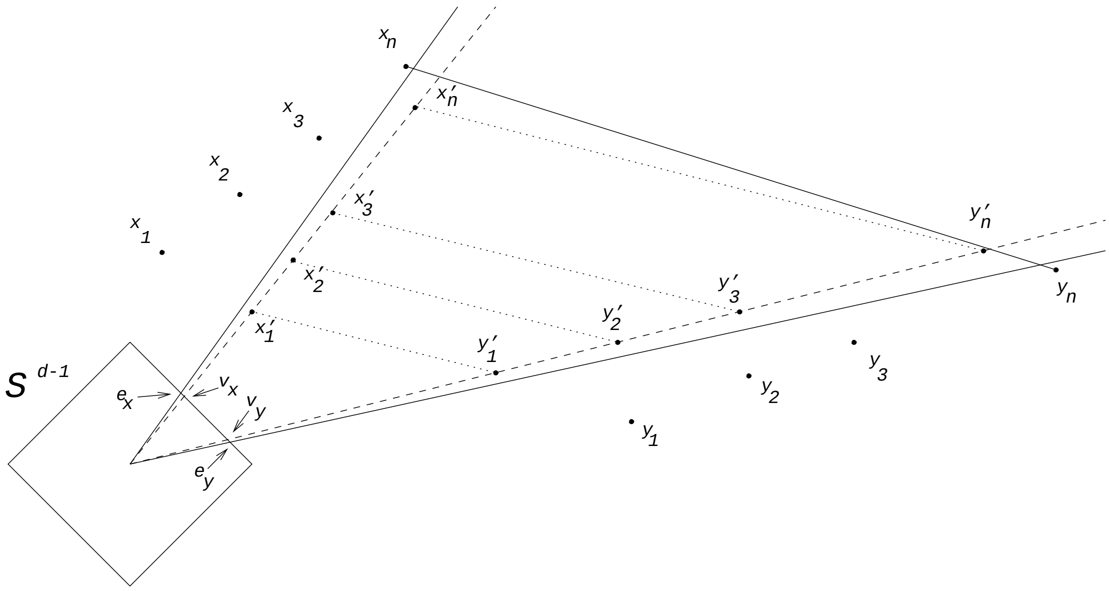


FIG. 3. *Sketch for the proof of the uniform shape theorem*

and

$$
\begin{aligned}
|x_n - x_n'| &\le \big|x_n - |x_n|v_x\big| + \big||x_n|v_x - |y_n|qv_x\big| + \big||y_n|qv_x - x_n'\big| \\
&< \varepsilon|x_n| + \varepsilon|y_n| + q_1 M < 3\varepsilon|y_n| \le \frac{4\varepsilon}{q}|x_n'| \le 4\varepsilon|y_n| \qquad (30)
\end{aligned}
$$

and consequently by $c|y_n| \le |y_n - x_n|$

$$|y_n' - x_n'| \le |y_n' - y_n| + |y_n - x_n| + |x_n - x_n'| \le (5\varepsilon/c + 1)|y_n - x_n|. \qquad (31)$$

Using the triangle inequality (6) and suppressing for brevity the $\omega$ dependence in the notation we estimate $a_\lambda(x_n, y_n)$ from above by

$$
\begin{aligned}
a_\lambda(x_n, y_n) &\le a_\lambda(x_n, x_n') + a_\lambda(x_n', y_n') + a_\lambda(y_n', y_n) \\
&\le d_\lambda(x_n, x_n') + a_\lambda(x_n', y_n') + d_\lambda(y_n', y_n)
\end{aligned}
$$

and from below by

$$
\begin{aligned}
a_\lambda(x_n, y_n) &\ge a_\lambda(x_n', y_n') - a_\lambda(x_n', x_n) - a_\lambda(y_n, y_n') \\
&\ge a_\lambda(x_n', y_n') - d_\lambda(x_n', x_n) - d_\lambda(y_n, y_n').
\end{aligned}
$$

Thus we get

$$|a_\lambda(x_n, y_n) - \alpha_\lambda(y_n - x_n)| \le d_\lambda(x_n, x_n') + d_\lambda(y_n, y_n')$$

$$+\ |a_\lambda(x'_n, y'_n) - \alpha_\lambda(y'_n - x'_n)| \qquad (32)$$
$$+\ |\alpha_\lambda(y'_n - x'_n) - \alpha_\lambda(y_n - x_n)|.$$

We use the maximal lemma and the estimates (30) and (29) to bound the first two summands on the right-hand side of (32) from above, getting

$$d_\lambda(x_n, x'_n) \le 4c_5(\lambda)\varepsilon|y_n| \quad \text{and} \quad d_\lambda(y_n, y'_n) \le 2c_5(\lambda)\varepsilon|y_n|$$

for $n$ large. For the third term on the right side of (32) we use Lemma 7 and estimation (31) which yield

$$|a_\lambda(x'_n, y'_n) - \alpha_\lambda(y'_n - x'_n)| = o(|y'_n - x'_n|) = o(|y_n - x_n|).$$

Finally by the triangle inequality for $\alpha_\lambda$ the fourth term is for large $n$ less than

$$\max\{\alpha_\lambda(x'_n - x_n), \alpha_\lambda(x_n - x'_n)\} + \max\{\alpha_\lambda(y'_n - y_n), \alpha_\lambda(y_n - y'_n)\} \le c_{11}(\lambda)\varepsilon|y_n|$$

where we used the bounds of (11) for $\alpha_\lambda$ and once more the estimates (30) and (29). Consequently and by $c|y_n| \le |y_n - x_n|$ we get that the left-hand side of (32) is less than $c_{12}(\lambda, c)\varepsilon|y_n - x_n|$ for $n$ large. Letting $\varepsilon$ tend to zero yields convergence in (3) for our fixed but arbitrary $\omega \in \Omega_7(\lambda)$.

This implies convergence in (3) for all $\lambda > 0$ and all $\omega \in \Omega_1 := \bigcap_{0<\lambda\in\mathbb{Q}} \Omega_7(\lambda)$ which has full $\mathbb{P}$-measure. Indeed, let $\omega \in \Omega_1$, $\lambda > 0$ and $\varepsilon > 0$ be arbitrary where $\lambda$ need not be rational. Due to the joint continuity of $\alpha_\lambda(x)$ there are $\lambda_1, \lambda_2 \in \mathbb{Q}$ with $0 < \lambda_1 < \lambda < \lambda_2$ such that $\alpha_{\lambda_2}(x) - \alpha_{\lambda_1}(x) < \varepsilon$ for all $x \in S^{d-1}$. Using the monotonicity of $a_\mu := a_\mu(x_n, y_n, \omega)$ and $\alpha_\mu := \alpha_\mu(y_n - x_n)$ in $\mu$ we get

$$\begin{aligned}
|a_\lambda - \alpha_\lambda| &\le a_\lambda - a_{\lambda_1} + |a_{\lambda_1} - \alpha_{\lambda_1}| + (\alpha_\lambda - \alpha_{\lambda_1}) \\
&\le (a_{\lambda_2} - \alpha_{\lambda_2}) + (\alpha_{\lambda_1} - a_{\lambda_1}) + |a_{\lambda_1} - \alpha_{\lambda_1}| + 2(\alpha_{\lambda_2} - \alpha_{\lambda_1}) \\
&\le o(|y_n - x_n|) + 2\varepsilon|y_n - x_n|
\end{aligned}$$

due to $\omega \in \Omega_7(\lambda_1) \cap \Omega_7(\lambda_2)$ The claim now follows by letting $\varepsilon$ tend to zero. □

**4. Nestling walks.** In this section we give three further conditions, each of which is equivalent to the nestling property (NP).

PROPOSITION 8. *The nestling property (NP) is equivalent to each of the following conditions.*

*(I) For all $\varepsilon > 0$ there is some $2 \le n \in \mathbb{N}$ such that*

$$\mathbb{P}\left[P_{0,\omega}[X_n = 0] > e^{-\varepsilon n}\right] > 0.$$

*(II) On a set of full $\mathbb{P}$-measure $e_\lambda(x, y, \omega) = \infty$ for all $\lambda < 0$ and all $x, y \in \mathbb{Z}^d$ with $x \ne y$.*

*(III) On a set of full $\mathbb{P}$-measure*

$$\limsup_{n\to\infty} \frac{\ln P_{0,\omega}[X_n = 0]}{n} = 0.$$

Of course, (NP) can be checked much more easier than (I) – (III) as soon as the distribution of the local drifts $\sum_{e\sim 0}\omega(0,e)e$ is known. However, (I) especially will help us to proof the lower large deviation estimate (5) in the next section. The name "nestling property" is supposed to describe figuratively the behavior of a random walker who in the sense of (III) sticks to the starting point roughly comparable to a young bird that often returns to its nest before it finally leaves it.

Examples. 1. Let $\xi_e > 0$ $(e\sim 0)$ be independent random variables with probability distribution $\nu_e$ and define $\omega(0,e) := \xi_e/\sum_{e'\sim 0}\xi_{e'}$. If $\inf\operatorname{supp}\nu_e \le \sup\operatorname{supp}\nu_{-e}$ for all $e\sim 0$ then the distribution of $(\omega(0,e))_{e\sim 0}$ has the nestling property (NP). The environments underlying Figure 1 and 2 are of this type.

2. Kalikow [7, Lemma 1] showed that any RWIRE is either $\mathbb{P}$-a.s. recurrent or $\mathbb{P}$-a.s. transient, i.e. $P_{0,\omega}[X_n = 0$ infinitely often$]$ equals either $\mathbb{P}$-a.s. one or $\mathbb{P}$-a.s. zero. In the recurrent case the nestling property is fulfilled. Indeed, otherwise $P_{0,\omega}[X_n = 0]$ would be less than $\exp(-\varepsilon n)$ for some $\varepsilon > 0$ and all $n$ greater than some $N_1 \in \mathbb{N}$ on a set of positive $\mathbb{P}$-probability due to (III). Therefore on this set $P_{0,\omega}[X_n = 0$ for some $n \ge N_2]$ would be strictly less than one for some $N_2 \in \mathbb{N}$ in contradiction to the recurrence.
However, even in the transient case the nestling property may hold. See [7, Example 2] for an example in two dimensions and [17], [6], [16] for the classification of recurrence and transience in one dimension.

Remark. If (NP) does not hold then $P_{0,\omega}[X_n = 0]$ decays with strictly positive exponential decay rate $\lambda_c$. In general it is impossible to give a formula for $\lambda_c$ just in terms of the distribution of the local drifts $\sum_{e\sim 0}\omega(0,e)e$. For example, let $d = 2$ and consider the nonrandom environment $\omega$ with

$$\omega(0,e_1) = \omega(0,-e_1) = \frac{1}{5} + \varepsilon, \quad \omega(0,e_2) = \frac{2}{5} - \varepsilon, \quad \omega(0,-e_2) = \frac{1}{5} - \varepsilon$$

where $\varepsilon \in (0,1/5)$ is a parameter. Then $\sum_{e\sim 0}\omega(0,e)e = e_2/5$ does not depend on $\varepsilon$. However it follows from Cramér's theorem (see e.g. [2, Theorem 2.2.30]) that in the nonrandom constant case

$$\lambda_c = -\ln 2\sum_{i=1}^{d}\sqrt{\omega(0,e_i)\omega(0,-e_i)}$$

which does depend on $\varepsilon$ for our particular choice of $\omega$.

For the proof of Proposition 8 we need the following simple geometric observation.

Lemma 9. *There is a constant $0 < c_{13} < 1$ just depending on the dimension $d$ such that for all $x \in \mathbb{Z}^d\backslash\{0\}$ there is some $y_x \sim x$ with $\|y_x\|_2 \le \|x\|_2 - c_{13}$.*

Proof of Lemma 9. Assume $x \neq 0$ and choose $i \in \{1, \ldots, d\}$ which maximizes $|x_i|$. Now set $y_x := x - (\text{sign } x_i)e_i$. Then

$$\|x\|_2 - \|y_x\|_2 = \frac{\|x\|_2^2 - \|y_x\|_2^2}{\|x\|_2 + \|y_x\|_2} \;=\; \frac{|x_i|^2 - (|x_i|-1)^2}{\|x\|_2 + \|y_x\|_2} \;\geq\; \frac{2|x_i| - 1}{\sqrt{d}|x_i| + \sqrt{d}|x_i|}$$
$$= \frac{1}{\sqrt{d}} - \frac{1}{2\sqrt{d}|x_i|} \;\geq\; \frac{1}{2\sqrt{d}} \;=:\; c_{13}.$$

□

Proof of Proposition 8. (NP) $\Rightarrow$ (I): The idea is to construct a finite environment of the origin in which the local drifts $\sum_{e \sim 0} \omega(x, x+e)e$ point from $x$ as best as possible into the direction of the origin. This environment tends to push the random walker back to the origin and thus prevents the walker from leaving the origin's neighborhood too early.
To be precise let $\varepsilon > 0$. Without restriction we may assume $\varepsilon < 1$. The proof consists of two parts. In the first one we prove the following statement:

(33) There exists a continuous function $F : \mathbb{R}^d \to [0,1]$ with bounded support, $F(0) = 1$ and $\mathbb{P}[A] > 0$, where
$$A := \left\{ \omega \in \Omega : \sum_{y \sim x} F(y)\omega(x,y) \geq e^{-\varepsilon/3} F(x) \text{ for all } x \in \mathbb{Z}^d \right\}.$$

In the second part we use a submartingale argument to complete the proof of (I).

1. Choose $\delta \in (0,1)$ such that $\mathbb{P}[\omega(0,e) \geq \delta] > 0$ for all $e \sim 0$. We use the constant $c_{13}$ of Lemma 9 to define

$$\eta := \frac{\varepsilon c_{13} \delta}{16} > 0, \quad G(x) := 1 - \eta^2 \|x\|_2^2 \quad \text{for } x \in \mathbb{R}^d \quad \text{and} \quad F := \max\{G, 0\}.$$

Obviously $F$ is a continuous function from $\mathbb{R}^d$ to [0,1] with bounded support and $F(0) = 1$. For the proof of $\mathbb{P}[A] > 0$ consider the event

$$B := \Bigg\{ \omega \in \Omega : \; x \cdot \sum_{e \sim 0} \omega(x, x+e)e \leq 1 \quad \text{for all} \quad \|x\|_2 \leq \frac{1}{\eta} - \frac{c_{13}\delta}{2},$$
$$\omega(x,y) \geq \delta \quad \text{for all} \quad \frac{1}{\eta} - \frac{c_{13}\delta}{2} < \|x\|_2 \leq \frac{1}{\eta},\; y \sim x \quad (x \in \mathbb{Z}^d) \Bigg\}.$$

Note that $1/\eta - c_{13}\delta/2$ is positive due to $\varepsilon, c_{13}, \delta < 1$. Thanks to (NP) we have

$$\mathbb{P}\left[ x \cdot \sum_{e \sim 0} \omega(x, x+e)e \leq 1 \right] > 0$$

for all $x \in \mathbb{Z}^d$. Therefore and due to the choice of $\delta$ we get by independence that $\mathbb{P}[B] > 0$. Hence it suffices to show $B \subseteq A$. To this end let $\omega \in B$ and $x \in \mathbb{Z}^d$.

We want to show

$$\sum_{y\sim x} F(y)\omega(x,y) \geq e^{-\varepsilon/3}F(x). \tag{34}$$

If $\|x\|_2 \geq 1/\eta$ this is obvious since in this case $F(x)=0$.

Now assume $1/\eta - c_{13}\delta/2 < \|x\|_2 < 1/\eta$. Then $\|y_x\|_2 \leq \|x\|_2 - c_{13} < 1/\eta - c_{13}$ with the notations of Lemma 9. This implies on the one hand

$$F(x) \leq 1-\eta^2\left(\frac{1}{\eta}-\frac{c_{13}\delta}{2}\right)^2 = \eta c_{13}\delta - \left(\frac{\eta c_{13}\delta}{2}\right)^2 \leq \eta c_{13}\delta$$

and on the other hand

$$F(y_x) \geq 1-\eta^2\left(\frac{1}{\eta}-c_{13}\right)^2 = 2\eta c_{13} - (\eta c_{13})^2 \geq \eta c_{13}.$$

Hence

$$\sum_{y\sim x} F(y)\omega(x,y) \geq F(y_x)\omega(x,y_x) \geq \eta c_{13}\delta \geq F(x) \geq e^{-\varepsilon/3}F(x)$$

because of $\omega\in B$, which proves (34).

Finally we consider the case $\|x\|_2 \leq 1/\eta - c_{13}\delta/2$ in which

$$F(x) \geq \eta c_{13}\delta - \left(\frac{\eta c_{13}\delta}{2}\right)^2 \geq \frac{\eta c_{13}\delta}{2}. \tag{35}$$

By Taylor expansion there are $\theta_e \in [0,1]$ such that

$$\begin{aligned}
\sum_{y\sim x} F(y)\omega(x,y) &\geq \sum_{e\sim 0} G(x+e)\omega(x,x+e)\\
&= \sum_{e\sim 0}\left(G(x)+\nabla G(x+\theta_e e)\cdot e\right)\omega(x,x+e)\\
&= G(x) - \sum_{e\sim 0} 2\eta^2(x+\theta_e e)\cdot e\ \omega(x,x+e)\\
&= F(x) - 2\eta^2 x\cdot\sum_{e\sim 0}\omega(x,x+e)e - 2\eta^2\cdot\sum_{e\sim 0}\theta_e(e\cdot e)\omega(x,x+e)
\end{aligned}$$

which is due to $\omega\in B$ and $\theta_e \leq 1$

$$\begin{aligned}
&\geq F(x) - 2\eta^2 - 2\eta^2 \;=\; \left(1-\frac{\varepsilon}{2}\right)F(x) + \frac{\varepsilon}{2}F(x) - 4\eta^2\\
&\geq e^{-\varepsilon}F(x) + \frac{\varepsilon}{2}\cdot\frac{\eta c_{13}\delta}{2} - 4\eta\frac{\varepsilon c_{13}\delta}{16} \;=\; e^{-\varepsilon}F(x)
\end{aligned}$$

where we used (35) and $1-\varepsilon/2 \geq e^{-\varepsilon}$ for $\varepsilon\in(0,1)$. This completes the proof of (34) and (33).

2. In this second part we use (33) to prove (I). For any $\omega\in A$ the sequence

$$e^{\varepsilon n/3}F(X_n) \quad (n\geq 0)$$

is a submartingale under $P_{0,\omega}$ with respect to the filtration generated by $X_n, n \geq 0$. Indeed

$$E_{0,\omega}[e^{\varepsilon(n+1)/3}F(X_{n+1}) \mid X_n] = e^{\varepsilon(n+1)/3}\sum_{y\sim X_n} F(y)\omega(X_n,y) \geq e^{\varepsilon n/3}F(X_n)$$

because of $\omega \in A$. Consequently for all $n \geq 0$

$$\begin{aligned} 1 = E_{0,\omega}[F(X_0)] \ &\leq \ E_{0,\omega}\left[e^{\varepsilon n/3}F(X_n)\right] \ \leq \ e^{\varepsilon n/3}P_{0,\omega}[X_n \in \text{supp } F] \\ = e^{\varepsilon n/3} &\sum_{y\in\mathbb{Z}^d\cap\text{supp } F} P_{0,\omega}[X_n = y] \ \leq \ e^{\varepsilon n/3}\sharp(\mathbb{Z}^d \cap \text{supp } F)P_{0,\omega}[X_n = y_n(\omega)] \end{aligned}$$

where $y_n(\omega) \in \mathbb{Z}^d \cap \text{supp } F$ is choosen to maximize $P_{0,\omega}[X_n = y]$. Hence

$$P_{0,\omega}[X_n = y_n(\omega)] \geq \frac{e^{-\varepsilon n/3}}{\sharp\mathbb{Z}^d \cap \text{supp } F} \geq e^{-\varepsilon n/2}$$

for $n \geq c_{14}(\varepsilon)$. Since $\mathbb{Z}^d \cap \text{supp } F$ is finite and independent of $n$ there is some $\bar{y} \in \mathbb{Z}^d \cap \text{supp } F$ and some $\gamma > 0$ such that

$$\mathbb{P}[P_{0,\omega}[X_n = y_n(\omega)] \geq e^{-\varepsilon n/2}, y_n(\omega) = \bar{y}, A] > \gamma \tag{36}$$

for infinitely many $n$. For this fixed $\bar{y}$ we get by the simple Markov property

$$\begin{aligned} &\mathbb{P}\left[P_{0,\omega}[X_{n+|\bar{y}|} = 0] > e^{-\varepsilon(n+|\bar{y}|)}\right] \\ &\geq \mathbb{P}\left[P_{0,\omega}[X_n = \bar{y}]P_{\bar{y},\omega}[X_{|\bar{y}|} = 0] > e^{-\varepsilon n}, y_n(\omega) = \bar{y}, A\right] \\ &\geq \mathbb{P}\left[P_{0,\omega}[X_n = \bar{y}] > e^{-\varepsilon n/2}, P_{\bar{y},\omega}[X_{|\bar{y}|} = 0] > e^{-\varepsilon n/2}, y_n(\omega) = \bar{y}, A\right] \\ &\geq \mathbb{P}\left[P_{0,\omega}[X_n = \bar{y}] > e^{-\varepsilon n/2}, y_n(\omega) = \bar{y}, A\right] \\ &\quad -\mathbb{P}\left[P_{\bar{y},\omega}[X_{|\bar{y}|} = 0] \leq e^{-\varepsilon n/2}\right] \end{aligned} \tag{37}$$

which is bigger than $\gamma$ for some large $n$ due to (36) and since the term in (37) tends to zero as $n \to \infty$.

(I) $\Rightarrow$ (II): It suffices to show that for fixed $x, y \in \mathbb{Z}^d$ with $x \neq y$ and fixed rational $\lambda < 0$ on a set of full $\mathbb{P}$-measure $e_\lambda(x, y, \omega) = \infty$ holds. Let $0 < \varepsilon < -\lambda$ and pick $2 \leq n \in \mathbb{N}$ according to (I). Observe that the occurence of the event $\{\omega : P_{0,\omega}[X_n = 0] > e^{-\varepsilon n}\}$ depends only on the environment inside a bounded neighborhood of the origin. Thus by the Borel-Cantelli lemma and (I) there is $\mathbb{P}$-a.s. some $z(\omega) \in \mathbb{Z}^d$ with $|y - z| \geq n$ such that

$$P_{z,\omega}[X_n = z] > e^{-\varepsilon n}. \tag{38}$$

In the one-dimensional case $z$ we may assume furthermore that $x$ lies between $z$ and $y$. Now by the strong Markov property

$$\begin{aligned} e_\lambda(x,y,\omega) &\geq E_{x,\omega}[e^{-\lambda H(y)}, H(z) < H(y) < \infty] \\ &= E_{x,\omega}[e^{-\lambda H(z)}, H(z) < H(y)] \cdot e_\lambda(z,y,\omega). \end{aligned} \tag{39}$$

Since the first factor on the right side of (39) is $\mathbb{P}$-a.s. positive, it suffices to prove $e_\lambda(z,y,\omega) = \infty$. This holds since for all $k \in \mathbb{N}$

$$\begin{aligned} e_\lambda(z,y,\omega) &\geq E_{z,\omega}\left[e^{-\lambda H(y)}, X_{\ell n} = z \text{ for all } \ell \in \{1,\ldots,k\}, H(y) < \infty\right] \\ &\geq e^{-\lambda nk} P_{z,\omega}[X_n = z]^k P_{z,\omega}[H(y) < \infty] \\ &\geq e^{(-\lambda-\varepsilon)nk} P_{z,\omega}[H(y) < \infty] \end{aligned}$$

due to (38) which tends to infinity as $k \to \infty$.

(II) $\Rightarrow$ (III): Consider the power series

$$e_\lambda(0,e_1,\omega) = \sum_{n=0}^{\infty} P_{0,\omega}[H(e_1) = n](e^{-\lambda})^n.$$

(II) implies for the radius $r(\omega)$ of convergence of this series

$$1 \geq r(\omega) = \liminf_{n\to\infty} \left(P_{0,\omega}[H(e_1) = n]\right)^{-1/n} \qquad \mathbb{P}\text{-a.s.} \tag{40}$$

Hence

$$\begin{aligned} \limsup_{n\to\infty} \frac{\ln P_{0,\omega}[X_n = 0]}{n} &\geq \limsup_{n\to\infty} \frac{\ln P_{0,\omega}[H(e_1) = n-1, X_n = 0]}{n} \\ &= \limsup_{n\to\infty} \frac{\ln P_{0,\omega}[H(e_1) = n-1]}{n-1} \cdot \frac{n-1}{n} + \frac{\ln P_{e_1,\omega}[X_1 = 0]}{n} \end{aligned}$$

which is nonnegative by (40).

(III) $\Rightarrow$ (NP): Assume that (NP) does not hold. Then the origin and the closed convex hull in (NP) can be separated by some hyperplane, i.e. there is some $\varepsilon \in (0,1)$ and some $h \in \mathbb{R}^d$ with $\|h\|_2 = 1$ such that $\mathbb{P}$-a.s. for all $x \in \mathbb{Z}^d$

$$h \cdot \sum_{e\sim 0} \omega(x,x+e)e \leq -2\varepsilon.$$

Hence for all $n \in \mathbb{N}$, $\mathbb{P}$-a.s.

$$\begin{aligned} E_{0,\omega}[\exp(\varepsilon X_{n+1}\cdot h)] &= E_{0,\omega}[E_{0,\omega}[\exp(\varepsilon X_{n+1}\cdot h) \mid X_n]] \\ &= E_{0,\omega}\left[\sum_{e\sim 0} \omega(X_n, X_n+e)\exp(\varepsilon(X_n+e)\cdot h)\right] \\ &\leq E_{0,\omega}\left[\exp(\varepsilon X_n \cdot h)\sum_{e\sim 0} \omega(X_n,X_n+e)(1+\varepsilon e\cdot h + (\varepsilon e\cdot h)^2)\right] \end{aligned}$$

since $e^x \leq 1 + x + x^2$ for $x = \varepsilon e \cdot h < 1$. By our choice of $h$ and $\varepsilon$ the last expression is less than

$$E_{0,\omega}\left[\exp(\varepsilon X_n \cdot h)\right](1 - 2\varepsilon^2 + \varepsilon^2).$$

Thus we get by induction over $n$ that $\mathbb{P}$-a.s.

$$(1-\varepsilon^2)^n \geq E_{0,\omega}[\exp(\varepsilon X_n \cdot h)] \geq P_{0,\omega}[X_n = 0],$$

a contradiction to (III). □

## 5. Large deviation estimates.

PROOF OF THEOREM B. The proof uses the technique developed in [18], see also [20, Chapter 5.4], and modified in [21]. Let us first investigate the properties of $I$. It follows from (11) and the definition of $I$ that

$$0 \leq I(z) \leq |z| \max_{e \sim 0} \mathbb{E}[-\ln \omega(0,e)] \qquad \text{if } |z| \leq 1$$

and $I(z) = \infty$ otherwise. Moreover $I$ is convex thanks to convexity of $\alpha_\lambda(z)$ in $z$ and lower semicontinuous as a supremum of continuous functions. Thus $I$ is continuous on the closed $|\cdot|$-unit ball. $I(0) = 0$ follows from $\alpha_\lambda(0) = 0$.

We now come to the proof of the upper bound (4). We let $\omega \in \Omega_2 := \Omega_1$ (see Theorem A) and assume $x \notin A$ since otherwise (4) is immediate from $I(0) = 0$. Moreover since the law of $X_n/n$ under $P_{nx,\omega}$ is supported on a compact set independent of $n$ and since $I$ is infinite outside a compact set, we may assume that $A$ is compact, too. Furthermore we only need to consider the case where $A$ intersects $\bar{B}(x,1)$ because in the other case both sides of (4) equal $-\infty$. For all $\lambda \geq 0$ with a self-explanatory extension of the definition (1) of $H$

$$\begin{aligned} P_{nx,\omega}[X_n \in [nA]] &= e^{\lambda n} E_{nx,\omega}\left[e^{-\lambda n}, X_n \in [nA]\right] \\ &\leq e^{\lambda n} E_{nx,\omega}\left[e^{-\lambda H([nA])}, H([nA]) < \infty\right] \\ &= e^{\lambda n} \sum_{y \in [nA]} E_{nx,\omega}\left[e^{-\lambda H([nA])}, H([nA]) = H(y) < \infty\right] \\ &\leq e^{\lambda n}\ \sharp[nA]\ e_\lambda(nx, y_{n,\lambda}, \omega) \end{aligned}$$

for some maximizing $y_{n,\lambda}(\omega) \in nA$. Observe that $\sharp[nA]$ grows just polynomially with $n$. Therefore by letting $\lambda > 0$ vary, we see that the left-hand side of (4) is less than

$$\begin{aligned} &\inf_{\lambda>0} \limsup_{n\to\infty} \left(\lambda - \frac{1}{n} a_\lambda(nx, y_{n,\lambda}, \omega)\right) \\ &\leq \inf_{\lambda>0} \left(\lambda - \liminf_{n\to\infty} \frac{1}{n}\left(a_\lambda(nx, y_{n,\lambda}, \omega) - \alpha_\lambda(y_{n,\lambda} - nx) + \inf_{y \in A} \alpha_\lambda(ny - nx)\right)\right) \end{aligned}$$

$$= \inf_{\lambda>0}\left(\lambda - \inf_{y\in A}\alpha_\lambda(y-x)\right)$$

where we used dist$\,(x,A)>0$ and the uniform shape theorem. Thus

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\ln P_{nx,\omega}[X_n\in nA] &\le -\sup_{\lambda>0}\inf_{y\in A}\left(\alpha_\lambda(y-x)-\lambda\right)\\ &= -\sup_{\lambda\ge 0}\inf_{y\in A}\left(\alpha_\lambda(y-x)-\lambda\right)\end{aligned}\tag{41}$$

by joint continuity of $\alpha$. For the proof of (4) we therefore need to exchange infimum and supremum in (41). This is done by a classical argument (see e.g. [20, Section 5.4]) as follows. For any $\varepsilon>0$

$$A=\bigcup_{\lambda\ge 0}A_\lambda \quad \text{where} \quad A_\lambda := \left\{z\in A:\ \alpha_\lambda(z-x)-\lambda > \inf_{y\in A} I(y-x)-\varepsilon\right\}.$$

From the compactness of $A$ we can choose $\lambda_1,\ldots,\lambda_m$ such that $A$ is covered by the finite collection $A_{\lambda_i},\ i=1,\ldots,m$. Applying (41) to $\overline{A_{\lambda_i}}$ we see that the left-hand side of (41) equals

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n}\ln P_{nx,\omega}\left[X_n\in[n\overline{A_{\lambda_1}}]\cup\ldots\cup[n\overline{A_{\lambda_m}}]\right]\\ &= \sup_{i=1,\ldots,m}\ \limsup_{n\to\infty}\frac{1}{n}\ln P_{nx,\omega}\left[X_n\in[n\overline{A_{\lambda_i}}]\right]\\ &\le -\inf_{i=1,\ldots,m}\ \sup_{\lambda\ge 0}\ \inf_{z\in\overline{A_{\lambda_i}}}(\alpha_\lambda(z-x)-\lambda) \ \le\ -\inf_{i=1,\ldots,m}\ \inf_{z\in\overline{A_{\lambda_i}}}(\alpha_{\lambda_i}(z-x)-\lambda_i)\\ &\le -\inf_{i=1,\ldots,m}\ \inf_{y\in\overline{A_{\lambda_i}}}(I(y-x)-\varepsilon) \ =\ \varepsilon-\inf_{y\in A}I(y-x).\end{aligned}$$

Since $\varepsilon$ is arbitrary this proves (4).

We now establish the equivalence of the lower estimate in (5) with the nestling property. First we show that the nestling property as formulated in (I) follows from the lower large deviation principle with rate function $I$. To this end fix $\omega\in\Omega_3\cap\Omega_5(0)$ (see Theorem B and Lemma 6) and let $\delta>0$. By assumption $\mathbb{P}$-a.s.

$$\begin{aligned}0=I(0) \ &\le\ -\inf_{|y|<\delta}I(y) \ \le\ \liminf_{n\to\infty}\frac{1}{n}\ln P_{0,\omega}[|X_n|<n\delta]\\ &\le \liminf_{n\to\infty}\frac{1}{n}\ln\left(\sharp\{y\in\mathbb{Z}^d:\ |y|<n\delta\}\cdot\max_{|y|<n\delta}P_{0,\omega}[X_n=y]\right)\\ &= \liminf_{n\to\infty}\frac{1}{n}\ln P_{0,\omega}[X_n=y_n]\end{aligned}\tag{42}$$

for some maximizing $y_n(\omega)\in\mathbb{Z}^d$ with norm less than $n\delta$. Then

$$P_{0,\omega}[X_m=0 \text{ for some } m\ge n] \ge P_{0,\omega}[X_n=y_n]P_{y_n,\omega}[H(0)<\infty].\tag{43}$$

The first factor on the right side of (43) decays subexponentially due to (42). For the second factor observe that by the maximal lemma with $\varepsilon = 1$ we have $a_0(y_n, 0, \omega) < c_5(0)|y_n|$ provided $|y_n|$ is large enough. Thus $a_0(y_n, 0, \omega) < c_5(0)n\delta$ due to $|y_n| < n\delta$ for large $n$. Hence by (43)

$$\liminf_{n\to\infty} \frac{1}{n} \ln P_{0,\omega}[X_m = 0 \text{ for some } m \geq n] \geq -c_5(0)\delta. \tag{44}$$

Since $\delta > 0$ was arbitrary, the left member of (44) is nonnegative. This implies that $P_{0,\omega}[X_m = 0]$ decays subexponentially, too. Indeed, otherwise $P_{0,\omega}[X_m = 0]$ would be less than $e^{-\varepsilon m}$ for some $\varepsilon > 0$ and large $m$ and thus for large $n$

$$P_{0,\omega}[X_m = 0 \text{ for some } m \geq n] \leq \sum_{m\geq n} P_{0,\omega}[X_m = 0] \leq \sum_{m\geq n} e^{-\varepsilon m} = \frac{e^{-\varepsilon n}}{1 - e^{-\varepsilon}}.$$

We now come to the proof of the opposite direction of the equivalence in the lower large deviation principle . Let us first describe informally the rough idea of our strategy.

One way for the walker to be at time $n$ in the set $nB$ is to hit the set $nB$ for the first time approximately at time $n$, i.e. $H(nB) \approx n$. Thus we want to derive a lower bound on the probability of the event $H(nB) \approx n$. Hence there are two different cases we shall treat separately. In the first case the walk if it ever reaches $nB$ does this usually before time $n$, that is typically $H(nB) << n$. In the other case the walks needs typically much more time to reach $nB$ than just $n$ steps, i.e. $H(nB) >> n$.

In the second case (which will be called the case $\lambda_1 > 0$ in the following) we introduce in order to force the walk to hurry up some appropriate killing rate $\lambda > 0$ and thus get rid of those trajectories which need too much time to reach $nB$. In the first case (the case $\lambda_1 = 0$) it is pointless to use a positive killing rate since this would only accelerate the walker even more. Perhaps it would help to choose some negative $\lambda$ in order to slow down the walk. However we are not able to handle the case of negative $\lambda$ for reasons explained before. At this point the nestling property comes into play. It guarantees the existence of a 'waiting room' inside $nB$ in which the walker after entering $nB$ can wait up to time $n$ without paying an exponential cost.

For the precise proof we need the following lemma that shows that under $\hat{P}^y_{x,\lambda,\omega}$ (recall (15)) the first-passage time $H(y)$ is roughly speaking centered around the derivative $\alpha'_\lambda(y - x)$ of $\alpha_\lambda(y - x)$ with respect to $\lambda$. However, we do not know, whether this derivative always exists, but since $\alpha_\lambda$ is concave increasing in $\lambda$ the left-hand derivatives $\alpha'_{\lambda-}$ and the right-hand derivatives $\alpha'_{\lambda+}$ exist for all $\lambda > 0$ and $\lambda \geq 0$, respectively.

LEMMA 10. (Crossing velocities) *Suppose that* $-\ln \omega(0, e)$ *has finite dth moment for all* $e \sim 0$*. Then there is a set* $\Omega_1$ *of full* $\mathbb{P}$*-measure such that the following holds: Let* $\omega \in \Omega_1$*,* $x, y \in \mathbb{R}^d, x \neq y$*, and* $x_n, y_n \in \mathbb{R}^d$ $(n \in \mathbb{N})$ *with* $x_n/n \to x$ *and* $y_n/n \to y$*. Furthermore let* $\lambda > 0$ *and* $0 \leq \gamma_1 < \gamma_2 < \infty$ *such that*

$$\gamma_1 < \alpha'_{\lambda+}(y - x) \leq \alpha'_{\lambda-}(y - x) < \gamma_2.$$

*Then*

$$\lim_{n\to\infty} \hat{P}^{y_n}_{x_n,\lambda,\omega}\left[H(y_n)/n \in (\gamma_1,\gamma_2)\right] = 1. \tag{45}$$

Proof of Lemma 10. Let $\omega \in \Omega_1$ (see Theorem A). Then by the uniform shape theorem for any $0<\mu<\lambda$

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \ln \hat{P}^{y_n}_{x_n,\lambda,\omega}\left[H(y_n) \geq \gamma_2 n\right] \\
&= \alpha_\lambda(y-x) + \limsup_{n\to\infty} \frac{1}{n} \ln E_{x_n,\omega}\left[e^{(-\lambda+\mu)H(y_n)} \cdot e^{-\mu H(y_n)}, \gamma_2 n \leq H(y_n) < \infty\right] \\
&\leq \alpha_\lambda(y-x) - \mu\gamma_2 - \alpha_{\lambda-\mu}(y-x) \;=\; \mu\left(\frac{\alpha_\lambda(y-x) - \alpha_{\lambda-\mu}(y-x)}{\mu} - \gamma_2\right)
\end{aligned}$$

which is negative for some small $\mu$. Since a corresponding statement holds for the event $\{H(y_n) \leq \gamma_1 n\}$ this implies (45). $\square$

We now assume that the nestling property holds. It suffices to construct for all $z \in \mathbb{Q}^d$ a set $\Omega_3(z)$ of full $\mathbb{P}$-measure such that

$$\liminf_{n\to\infty} \frac{1}{n} \ln P_{nx,\omega}[X_n \in nB(z,r)] \geq -I(z-x) \tag{46}$$

for all $\omega \in \Omega_3(z)$, all $x \in \mathbb{R}^d$ with $0<|z-x|<1$ and all $r>0$. $\Omega_3 := \bigcap_z \Omega_3(z)$ will then have the desired properties. Indeed, assume that $\Omega_3(z)$ has been constructed for all $z \in \mathbb{Q}^d$ and let $x \in \mathbb{R}^d$, $B \subseteq \mathbb{R}^d$ open and $\omega \in \Omega_3$. If $\operatorname{dist}(x,B) \geq 1$ then nothing has to be shown because in this case $\inf_{y\in B} I(y-x) = \infty$ as $B$ is open. If $\operatorname{dist}(x,B)<1$ then there is for any $\varepsilon>0$ some $z \in \mathbb{Q}^d$ and some $r>0$ such that $B(z,r) \subseteq B$, $0<|z-x|<1$ and $\inf_{y\in B} I(y-x) \geq I(z-x)-\varepsilon$ since $I$ is continuous on $\bar{B}(0,1)$ and infinite outside $\bar{B}(0,1)$. Hence the left member of (5) is greater than

$$\liminf_{n\to\infty} \frac{1}{n} \ln P_{nx,\omega}[X_n \in nB(z,r)] \geq -I(z-x) \geq -\inf_{y\in B} I(y-x) - \varepsilon$$

since $\omega \in \Omega_2(z)$. $\varepsilon \searrow 0$ then proves (5). Now let us construct $\Omega_3(z)$. To do this, observe that thanks to the nestling property (I) there is for all $0<\varepsilon\in\mathbb{Q}$ some $2 \leq R(\varepsilon) \in \mathbb{N}$ such that the independent events

$$A(y,\varepsilon) := \left\{\omega:\ P_{y,\omega}\left[X_{R(\varepsilon)} = y\right] > e^{-\varepsilon R(\varepsilon)}\right\} \qquad (y \in (R(\varepsilon)+1)\mathbb{Z}^d)$$

have positive $\mathbb{P}$-probability $\delta>0$. Let $y(n,z,\varepsilon,\omega) \in (R(\varepsilon)+1)\mathbb{Z}^d$ be some vertex with minimal distance from $nz$ such that $\omega \in A(y(n,z,\varepsilon,\omega),\varepsilon)$. It follows from the Borel-Cantelli lemma that on a set $\Omega_3(z,\varepsilon)$ of full $\mathbb{P}$-measure these $y(n,z,\varepsilon,\omega)$ exist for all $n\in\mathbb{N}$ and satisfy

$$|y(n,z,\varepsilon,\omega) - nz| \leq (\ln n)^2 \qquad \text{for } n \text{ large enough} \tag{47}$$

which is only a rough upper bound. Now define $\Omega_3(z) := \Omega_1 \cap \bigcap_{0<\lambda\in\mathbb{Q}} \Omega_5(\lambda) \cap \bigcap_{0<\varepsilon\in\mathbb{Q}} \Omega_3(z,\varepsilon)$ (see Theorem A and Lemma 6) and let $\omega \in \Omega_3(z), x \in \mathbb{R}^d$ with $0 < |z-x| < 1$ and $r > 0$. We must prove (46). To this end set $u := z - x$. Then $\lambda_1 := \sup\{\lambda > 0 : \ \alpha'_{\lambda-}(u) \geq 1\}$, which is defined to be 0 if the supremum is taken over the empty set, is a finite number due to (11) and $|u| < 1$. It maximizes $\alpha_\lambda(u) - \lambda$, i.e. $\alpha_{\lambda_1}(u) - \lambda_1 = I(u)$.

We distinguish two cases, $\lambda_1 = 0$ and $\lambda_1 > 0$ and treat first the case $\lambda_1 = 0$. Due to $\lambda_1 = 0$ and Lemma 10 most walkers that travel from $nx$ to $nz$ under the law $\hat{P}^{nz}_{nx,0,\omega}$ arrive before time $n$ in the ball $nB(z,r)$. One strategy to stay in this ball up to time $n$ is to go to a secure place inside this ball, a 'nest', that can only be left at an arbitrary small exponential rate. The existence of such a nest is guaranteed by the nestling property. So let $0 < \varepsilon \in \mathbb{Q}$. Then due to (47) the ball with center $y_n := y(n,z,\varepsilon,\omega)$ and radius $R(\varepsilon)$ is finally contained in the ball $nB(z,r)$. Thus the left-hand side of (46) is greater than

$$\liminf_{n\to\infty} \frac{1}{n} \ln P_{nx,\omega}\left[H(y_n) \leq n, \left|X_{m+H(y_n)} - y_n\right| \leq R(\varepsilon) \quad \text{for all } m = 0,\ldots,n\right].$$

By the strong Markov property this is greater than

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n} \ln P_{nx,\omega}\left[H(y_n) \leq n\right] \\
&\quad + \liminf_{n\to\infty} \frac{1}{n} \ln P_{y_n,\omega}\left[|X_m - y_n| \leq R(\varepsilon) \quad \text{for all } m = 0,\ldots,n\right] \\
&\geq \sup_{0<\lambda} \liminf_{n\to\infty} \frac{1}{n} \ln E_{nx,\omega}\left[e^{-\lambda H(y_n)}, H(y_n) \leq n\right] \\
&\quad + \liminf_{n\to\infty} \frac{1}{n} \ln P_{y_n,\omega}\left[X_{R(\varepsilon)} = y_n\right]^{n/R(\varepsilon)}.
\end{aligned}$$

By the uniform shape theorem and (47) this equals

$$\begin{aligned}
&= \sup_{0<\lambda}\left(\liminf_{n\to\infty} \frac{1}{n} \ln \hat{P}^{y_n}_{nx,\lambda,\omega}[H(y_n) \leq n] - \alpha_\lambda(z-x)\right) \\
&\quad + \frac{1}{R(\varepsilon)} \liminf_{n\to\infty} \ln P_{y_n,\omega}\left[X_{R(\varepsilon)} = y_n\right] \\
&\geq \sup_{0<\lambda} -\alpha_\lambda(u) + \frac{1}{R(\varepsilon)}(-\varepsilon R(\varepsilon)) \;=\; -\alpha_0(u) - \varepsilon \;=\; -I(u) - \varepsilon
\end{aligned}$$

where we used $\omega \in A(y_n, \varepsilon)$ and Lemma 10 with $\gamma_1 = 0$ and $\gamma_2 = 1$. Since $0 < \varepsilon \in \mathbb{Q}$ was arbitrary, this proves (46).

Now we treat the case $\lambda_1 > 0$. The idea is that under $\hat{P}^{nz}_{nx,\lambda_1,\omega}$ the walk reaches $nz$ with high probability approximately at time $n$ such that $X_n \in nB(z,r)$. This follows from Lemma 10 if $\alpha_\lambda(u)$ is differentiable at $\lambda = \lambda_1$. However, we do not know whether $\alpha'_{\lambda_1}(u)$ exists, since $\lambda_1$ might be in the at most countable set of locations $\lambda$ where the left-hand and the right-hand derivatives do not coincide. So at first glance we loose control of $H(nz)$ under $\hat{P}^{nz}_{nx,\lambda_1,\omega}$. We cope with this problem by putting in a stopover in some intermediate point $n\xi$ between $nx$ and $nz$. For the first part of the journey from $nx$ to $n\xi$ we use some $\lambda_0 \leq \lambda_1$ for which $\alpha'_{\lambda_0}(u)$ exists. For the second part from $n\xi$ to $nz$ we use some $\lambda_2 \geq \lambda_1$ with the same property. Since we are able to control the travel times for both stretches due to Lemma 10 we are also able to control the time needed for the whole journey from $nx$ to $nz$ after all. So let $0 < \varepsilon < 1$. Then there are $\rho \in (0,1), \eta > 0$ and $\lambda_0, \lambda_2$ such that $\alpha'_{\lambda_0}(u)$ and $\alpha'_{\lambda_2}(u)$ exist and

$$\max\{0, \lambda_1 - \varepsilon\} \;<\; \lambda_0 \;\leq\; \lambda_1 \leq \lambda_2, \tag{48}$$

$$\rho\alpha'_{\lambda_0}(u) + (1-\rho)\alpha'_{\lambda_2}(u) + [-\eta, \eta] \subseteq (1 - \varepsilon r, 1 + \varepsilon r), \qquad \text{and} \tag{49}$$

$$\alpha_{\lambda_2}(u) < \alpha_{\lambda_1}(u) + \varepsilon. \tag{50}$$

Hence

$$\begin{aligned}
\{X_n \in nB(z,r)\} \supseteq A_1 &:= \{H(nz) \in n(1 + \varepsilon r(-1,1))\} \\
&\supseteq \left\{H(nz) \in n\left(\rho\alpha'_{\lambda_0}(u) + (1-\rho)\alpha'_{\lambda_2}(u) + [-\eta,\eta]\right)\right\} \\
&\supseteq A_2 := \left\{H(n\xi) \in n\rho\left(\alpha'_{\lambda_0}(u) + [-\eta,\eta]\right), \right. \\
&\qquad \left. \inf\{m \geq H(n\xi) : X_m = [nz]\} \in n(1-\rho)\left(\alpha'_{\lambda_2}(u) + [-\eta,\eta]\right) \right\}
\end{aligned}$$

where $\xi := (1-\rho)x + \rho z$ lies between $x$ and $z$. Since $n \leq H(nz) + n\varepsilon r$ on the event $A_1$, the left hand side of (46) is bigger than

$$\begin{aligned}
&\lambda_1 + \liminf_{n\to\infty} \frac{1}{n} \ln E_{nx,\omega}\left[e^{-\lambda_1(H(nz)+n\varepsilon r)}, A_2\right] \\
&\geq \lambda_1(1-\varepsilon r) + \liminf_{n\to\infty} \frac{1}{n} \ln E_{nx,\omega}\left[e^{-\lambda_1 H(n\xi)}, H(n\xi) \in n\rho\left(\alpha'_{\lambda_0}(u) + [-\eta,\eta]\right)\right] \\
&\quad + \liminf_{n\to\infty} \frac{1}{n} \ln E_{n\xi,\omega}\left[e^{-\lambda_1 H(nz)}, H(nz) \in n(1-\rho)\left(\alpha'_{\lambda_2}(u) + [-\eta,\eta]\right)\right]
\end{aligned}$$

where we used the strong Markov property. We use (48) to get

$$\begin{aligned}
&-\lambda_1 H(n\xi) \geq -\lambda_0 H(n\xi) + (\lambda_0 - \lambda_1)n\rho\left(\alpha'_{\lambda_0}(u) + \eta\right) \qquad \text{and} \\
&-\lambda_1 H(nz) \geq -\lambda_2 H(nz)
\end{aligned}$$

and thus see by the uniform shape theorem that the expression from above is greater than

$$\lambda_1(1-\varepsilon r) + (\lambda_0 - \lambda_1)\rho\left(\alpha'_{\lambda_0}(u) + \eta\right) - \alpha_{\lambda_0}(\rho u) - \alpha_{\lambda_2}((1-\rho)u) \tag{51}$$

$$+ \liminf_{n\to\infty} \frac{1}{n} \ln \hat{P}^{n\xi}_{nx,\lambda_0,\omega}\left[H(n\xi)/n \in \rho(\alpha'_{\lambda_0}(u) + [-\eta,\eta])\right] \tag{52}$$

$$+ \liminf_{n\to\infty} \frac{1}{n} \ln \hat{P}^{nz}_{n\xi,\lambda_2,\omega}\left[H(nz)/n \in (1-\rho)(\alpha'_{\lambda_2}(u) + [-\eta,\eta])\right]. \tag{53}$$

The terms in (52) and (53) vanish because of Lemma 10. Due to (48) and (49) we have $\lambda_0 - \lambda_1 > -\varepsilon$ and $\rho(\alpha'_{\lambda_0}(u) + \eta) \le 1 + \varepsilon r$, respectively. Thus (51) is greater than

$$\begin{aligned} &\lambda_1(1-\varepsilon r) - \varepsilon(1+\varepsilon r) - \alpha_{\lambda_1}(u) - (1-\rho)\left(\alpha_{\lambda_2}(u) - \alpha_{\lambda_1}(u)\right) \\ \ge\ &\lambda_1(1-\varepsilon r) - \varepsilon(1+\varepsilon r) - \alpha_{\lambda_1}(u) - \varepsilon \end{aligned}$$

by (50). Again letting $\varepsilon \searrow 0$ proves (46). □

REMARKS. 1. If one is interested only in large deviations for fixed starting point $x = 0$ then one does not need the uniform shape theorem but only the simple shape theorem. This is obvious for the proof of the upper bound (4). However, even in the proof of the lower bound (5) where it is natural to apply the uniform shape theorem since we use intermediate points one can avoid the uniform shape theorem by using the technique developed by Sznitman in the context of Brownian motion in Poissonian potentials, see [18, Theorem 2.1] or [20, Theorem 5.4.2].

2. It is an open problem to prove or disprove differentiability or analyticity of $\alpha_\lambda$ with respect to $\lambda$ for $d \ge 2$. This problem is unsolved also for the Lyapounov exponents belonging to Brownian motion in a Poissonian potential [20] or random walks in random nonnegative potentials [21], see also [10, Section 5 (iii)].

3. The set $Z = \{x \in \mathbb{R}^d : \ I(x) = 0\}$ of zeros of the rate function is of particular interest. Since $I$ is nonnegative, convex and continuous on the closed $|\cdot|$-unit ball $\bar{B}(0,1)$, and infinite outside this ball, $Z$ is a closed convex subset of $\bar{B}(0,1)$. Moreover it follows from the definition of $I$ that

$$Z = \{x \in \mathbb{R}^d : \ \alpha_0(x) = 0,\ \alpha'_{0+}(x) \le 1\}$$

since $\alpha_\lambda(x)$ is nonnegative and concave in $\lambda$.

**6. The one-dimensional case.** In this section we assume $d = 1$. Since in one dimension the process $a_\lambda(x,y,\omega)$ is additive in the sense of Lemma 1 we can apply the ergodic theorem instead of the subadditive ergodic theorem and obtain

$$\alpha_\lambda(\pm 1) = \mathbb{E}[a_\lambda(0,\pm 1,\omega)]. \tag{54}$$

Consequently

$$\alpha_\lambda(x) = |x|\ \mathbb{E}[a_\lambda(0,\operatorname{sign} x,\omega)] \tag{55}$$

for all $x \in \mathbb{R}^d$ and $\lambda \ge 0$. This is the counterpart to [18, eq. (1.30)] and [21, eq. (39)] and appears in a similar context in [3, eq. (5.3) and (5.4), p. 502]. Moreover by partition over the first step and the strong Markov property we get

$$\begin{aligned} e_\lambda(1,0,\omega) &= e^{-\lambda}(\omega(1,0) + \omega(1,2)e_\lambda(2,0,\omega)) \\ &= e^{-\lambda}(\omega(1,0) + \omega(1,2)e_\lambda(2,1,\omega)e_\lambda(1,0,\omega)) \end{aligned}$$

and thus

$$e_\lambda(1,0,\omega) = \frac{\omega(1,0)}{e^\lambda - \omega(1,2)e_\lambda(2,1,\omega)} = \frac{\omega(1,0)/\omega(1,2)}{\dfrac{e^\lambda}{\omega(1,2)} - e_\lambda(2,1,\omega)}. \tag{56}$$

Note that $e_\lambda(2,1,\omega)$ has the same distribution as $e_\lambda(1,0,\omega)$ and is independent of $\omega(1,0)$ and $\omega(1,2)$. Thus (56) can be read as an implicit equation for the distribution of $e_\lambda(1,0,\omega)$ in terms of $\lambda$ and the distribution of $\omega(0,1)$.
Iterating (56) we get the following continued fraction representation.

$$e_\lambda(1,0,\omega) = \cfrac{\omega(1,0)/\omega(1,2)}{\cfrac{e^\lambda}{\omega(1,2)} - \cfrac{\omega(2,1)/\omega(2,3)}{\cfrac{e^\lambda}{\omega(2,3)} - \cfrac{\omega(3,2)/\omega(3,4)}{\cfrac{e^\lambda}{\omega(3,4)} - \dots}}}.$$

This continued fraction converges due to the Theorem of Śleszyński-Pringsheim (e.g. [12, Theorem I.4.1]) that states that

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \dots}}}$$

converges if $|b_n| \geq |a_n| + 1$ for all $n \in \mathbb{N}$. This criterion is fulfilled here because

$$\frac{e^\lambda}{\omega(n,n+1)} \geq \frac{\omega(n,n-1)}{\omega(n,n+1)} + 1 \quad \text{since} \quad e^\lambda \geq 1 = \omega(n,n-1) + \omega(n,n+1)$$

for all $n \geq 1$. Thus the random continued fraction $f(r,\omega)$ introduced by Greven and den Hollander in [5, (0.20), (3.12), (4.2)] is in our notation $e_r(0,-1,\bar\omega)$ where $\bar\omega(x,x+1) := \omega(x,x-1)$ and $\bar\omega(x,x-1) := \omega(x,x+1)$. Hence $f(r,\omega)$ has the same distribution as $e_r(0,1,\omega)$. The further correspondence of our notation with Greven's and den Hollander's notation is as follows. Due to (54) their $\log\lambda(r)$ [5, (0.21)] equals $-\alpha_r(1)$. Their $\theta_c^{-1}$ [5, (0.22)] is $\alpha'_{0+}(1)$ and their $r(\theta)$ [5, (0.23)] is our $\lambda_1$, which maximizes the supremum in the definition of the rate function, see the proof of Theorem B. The nestling property is equivalent to the assumption $r_c = 0$ [5, (0.19)]. Under this assumption we recover the rate function $I$ obtained in [5] as follows. Due to Theorem B and (55) the rate function can be written as

$$I(x) = \sup_{\lambda \geq 0} \left(|x|\ \mathbb{E}[a_\lambda(0, \operatorname{sign} x, \omega)] - \lambda\right). \tag{57}$$

It follows from (55) that $\alpha_\lambda(x)$ is differentiable in $\lambda \in (0,\infty)$. Indeed, one could even show that it depends analytically on $\lambda$ (compare [18, Theorem 2.6]). We have

$$\alpha'_\lambda(x) = |x|\ \mathbb{E}\left[\frac{-e'_\lambda(0,\operatorname{sign} x,\omega)}{e_\lambda(0,\operatorname{sign} x,\omega)}\right] = |x|\ \mathbb{E}\left[\hat{E}^{\operatorname{sign} x}_{0,\lambda,\omega}[H(\operatorname{sign} x)]\right].$$

So if $\alpha_0'(x) \leq 1$ then the supremum in (57) is attained at $\lambda_1 = 0$ (compare [5, first line of (0.23)]). Hence in this case $I(x) = \alpha_0(x) = |x|\mathbb{E}[-\ln P_{0,\omega}[H(\operatorname{sign} x) < \infty]]$. If $\alpha_0'(x) > 1$ and $|x| < 1$ then the number $\lambda_1$ which maximizes $\alpha_\lambda(x) - \lambda$ is characterized by $1 = \alpha_{\lambda_1}'(x)$ (compare [5, second line of (0.23)]). In this case $I(x) = \alpha_{\lambda_1}(x) - \lambda_1$. Thus we arrive at the same expression as in [5, (0.24)].

REMARK. In one dimension it could be possible to drop the nestling property in the derivation of the lower bound (5) by taking into account also exponents $\alpha_\lambda$ for negative $\lambda$ (see Remark 2 at the end of Section 1). However we are not carrying this out here.

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