VELOCITY AND LYAPOUNOV EXPONENTS OF SOME RANDOM WALKS IN RANDOM ENVIRONMENT

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Abstract. – We express the asymptotic velocity of random walks in random environment satisfying Kalikow's condition in terms of the Lyapounov exponents which have previously been used in the context of large deviations.

Résumé. – Nous exprimons la vitesse asymptotique des marches aléatoires en environnement aléatoire qui vérifient la condition de Kalikow en function des exposants de Lyapounov qui ont déjà été utilisés dans le contexte des grandes déviations.

0. Introduction and result

There was recently some progress in understanding the asymptotic behavior of multidimensional random walks in random environment (RWRE), see [9], [10], [11] and [13]. On the one hand, in [11] certain renewal times are constructed, which enable one to prove a law of large numbers (see [11]) and a central limit theorem for the walk (see [10]). This approach leads to an expression for the asymptotic velocity of the random walk in terms of these renewal times. On the other hand, one uses in [13] certain Lyapounov exponents to derive the rate function for some large deviation principle for the walk. The purpose of the present note is to link these two different concepts by giving an alternative expression for the asymptotic velocity of the walk in terms of the Lyapounov exponents.

Let us present the precise model and recall some of the results mentioned above. We assign to the lattice sites $z \in \mathbb{Z}^d$ $(d \geq 1)$ i.i.d. $2d$-dimensional vectors $(\omega(z, z+e))_{|e|=1, e \in \mathbb{Z}^d}$ with a common distribution μ and with positive components which add up to one and are uniformly bounded away from 0. That is, we assume that there is some fixed $\kappa > 0$ such that μ is supported on the set $\mathcal{P}_\kappa$ of $2d$-vectors $(p(e))_{|e|=1, e\in\mathbb{Z}^d}$ with $p(e) \geq \kappa$ and $\sum_e p(e) = 1$. The random variables

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$\omega(z, z+e)$ can then be realized as the canonical projections on the product space $\Omega := \mathcal{P}_\kappa^{\mathbb{Z}^d}$ endowed with the canonical product σ-algebra and the product measure $\mathbb{P} := \otimes_{\mathbb{Z}^d}\mu$. Given such an environment ω, the values $\omega(z, z+e)$ serve as transition probabilities for the Markov chain $(X_n)_{n\geq 0}$, called random walk in random environment. This walk moves on $\mathbb{Z}^d$ and is for fixed starting point $x \in \mathbb{Z}^d$ defined on the sample space $(\mathbb{Z}^d)^{\mathbb{N}}$ endowed with the so-called quenched measure $P_{x,\omega}$ which satisfies

$$\begin{aligned} P_{x,\omega}[X_0 = x] &= 1 \quad \text{and} \\ P_{x,\omega}[X_{n+1} = X_n + e \mid X_0, X_1, \dots, X_n] &= \omega(X_n, X_n + e) \quad P_{x,\omega} - a.s. \end{aligned}$$

for all $e \in \mathbb{Z}^d$ with $|e| = 1$ and all $n \geq 0$. The so-called annealed measures P_x, $x \in \mathbb{Z}^d$, are then defined as the semi-direct products $P_x := \mathbb{P} \times P_{x,\omega}$ on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$. The corresponding expectations are denoted by $E_{x,\omega}$ and E_x, respectively.

While the one-dimensional case is very well understood (see e.g. [7], [4], [3] and the references therein) there are still many open questions for higher dimensions. For $d \geq 2$ this model has been introduced by Kalikow [5] and has subsequently been studied in [6] and [1] and, as mentioned above, recently in [9], [10], [11] and [13].

For the study of large deviations of X_n/n under typical measures $P_{0,\omega}$ in general dimension, one introduces in [13] for $\lambda \geq 0$, $y \in \mathbb{R}^d$ and $\omega \in \Omega$, the transform

$$e_\lambda(0, y, \omega) := E_{0,\omega}[\exp(-\lambda H(y)) \, 1\{H(y) < \infty\}]$$

of the first passage time $H(y) := \inf\{n \geq 0 : \; X_n = [y]\}$ of the walk through the lattice point $[y] \in \mathbb{Z}^d$, which is closest to y. It turns out that $e_\lambda(0, ny, \omega)$ decays exponentially as $n \to \infty$ with a deterministic rate $\alpha_\lambda(y)$, the so-called Lyapounov exponent, which depends on the direction. The next theorem shows that even more holds.

Theorem A. (Point-to-point Lyapounov exponents and shape theorem, [13, Th. A and Prop. 3]). *There exists a continuous function $\alpha : [0, \infty) \times \mathbb{R}^d \to [0, \infty)$, $(\lambda, x) \mapsto \alpha_\lambda(x)$, which is concave increasing in λ and homogeneous and convex in x, and a set $\Omega_1 \subseteq \Omega$ of full $\mathbb{P}$-measure such that the following holds: For all $\lambda \geq 0$ and all sequences $(y_m)_m$ with $y_m \in \mathbb{R}^d$ and $|y_m| \to \infty$,*

$$\lim_{m\to\infty} \frac{\ln e_\lambda(0, y_m, \omega) + \alpha_\lambda(y_m)}{|y_m|} = 0 \quad \textit{for all } \omega \in \Omega_1 \textit{ and in } L^1(\mathbb{P}). \tag{1}$$

Furthermore,

$$\lambda|x| \leq \alpha_\lambda(x) \leq (\lambda - \ln\kappa)|x| \quad \textit{for all } x \in \mathbb{R}^d \textit{ and all } \lambda \geq 0. \tag{2}$$

Here $|x| := |x_1| + \ldots + |x_d|$ is the ℓ_1-norm of x.

The law of large numbers derived in [11] does not make any use of these Lyapounov exponents but of a condition which has already been introduced by Kalikow in [5]. For the precise definition of this condition consider strict subsets U of $\mathbb{Z}^d$ and denote the exterior boundary of U by

$$\partial U := \Big\{x \in \mathbb{Z}^d \backslash U : \text{ There is some } x' \in U \text{ with } |x - x'| = 1\Big\}. \tag{3}$$

On $U \cup \partial U$ one defines an auxiliary Markov chain with transition probability

$$\begin{aligned} \hat{P}_U(x, x+e) &:= \frac{E_0\left[\sum_{n=0}^{H(U^c)} 1\{X_n = x\}\ \omega(x, x+e)\right]}{E_0\left[\sum_{n=0}^{H(U^c)} 1\{X_n = x\}\right]} \\ &\quad \text{if } x \in U \text{ and } |e| = 1, \\ \hat{P}_U(x, x) &:= 1 \quad \text{if } x \in \partial U, \end{aligned}$$

where $H(U^c)$ is the exit time from U. For $\ell \in \mathbb{R}^d \setminus \{0\}$, Kalikow's condition relative to ℓ is the requirement that

$$\varepsilon_\ell := \inf_{U, x \in U} \sum_{|e|=1} \ell \cdot e\ \hat{P}_U(x, x+e) > 0,$$

where U runs over all connected strict subsets of $\mathbb{Z}^d$ containing 0. For sufficient conditions which imply Kalikow's condition and are easier to check see [5, p. 759-760] and [11, Prop. 2.4]. The limit theorems of [11] and [10] assume Kalikow's condition and rely on the following renewal structure. For fixed $\ell \in \mathbb{R}^d \setminus \{0\}$, one constructs successive times $\tau_k,\ k \geq 1$, which have the property that P_0-a.s.

$$X_m \cdot \ell < X_{\tau_k} \cdot \ell \leq X_n \cdot \ell \qquad \text{for all } 0 \leq m < \tau_k < n. \tag{4}$$

For details we again refer to [11]. Although these times are not stopping times with respect to the canonical filtration of $(X_n)_{n \geq 0}$, they are very useful as is shown by the following theorem. Here $D := \inf\{n \geq 0 :\ X_n \cdot \ell < X_0 \cdot \ell\}$ is the first time the walk reaches a level strictly below its starting level.

Theorem B. (Renewal structure, [11, Prop. 1.2, Cor. 1.5, Th. 2.3], [10, Lemma 1.2]).
Assume Kalikow's condition relative to some $\ell \in \mathbb{R}^d \setminus \{0\}$. Then P_0-a.s., $0 =: \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots$ and under P_0, $(X_{\tau_1}, \tau_1), (X_{\tau_2} - X_{\tau_1}, \tau_2 - \tau_1), \ldots, (X_{\tau_{k+1}} - X_{\tau_k}, \tau_{k+1} - \tau_k), \ldots$ are independent variables. Furthermore, $P_0[D = \infty] > 0$ and $(X_{\tau_2} - X_{\tau_1}, \tau_2 - \tau_1), \ldots, (X_{\tau_{k+1}} - X_{\tau_k}, \tau_{k+1} - \tau_k)$ are distributed under P_0 as (X_{τ_1}, τ_1) under $P_0[\ \cdot \mid D = \infty]$.
Moreover, under $P_0[\ \cdot \mid D = \infty]$, τ_1 has finite expectation and $X_{\tau_1} \cdot \ell$ has some finite exponential moment.

It follows from this result that under Kalikow's condition relative to ℓ the walk has a non-vanishing velocity as stated in the next theorem.

Theorem C. (Law of large numbers, [11, Th. 2.3], [10, Prop. 1.6]).
Assume Kalikow's condition relative to some $\ell \in \mathbb{R}^d \setminus \{0\}$. Then P_0-a.s.,

$$\lim_{n \to \infty} \frac{X_n}{n} = v := \frac{E_0[X_{\tau_1} \mid D = \infty]}{E_0[\tau_1 \mid D = \infty]}, \qquad \text{where } v \cdot \ell > 0.$$

Moreover, $\alpha_0(x) = 0$ if and only if $x = tv$ for some $t \geq 0$.

In the following we shall only need the if-part of the last statement of Theorem C, which is the easier part. It follows from the fact that in the transient case P_0-a.s.

$\alpha_0(X_n/|X_n|) \to 0$ as $n \to \infty$ which is a consequence of (1) and the Borel-Cantelli lemma.

However, if one uses the full equivalence given in the last statement of Theorem C then under Kalikow's condition the 0-th Lyapounov exponent α_0 characterizes at least the direction $v/|v|$ of the velocity v. If one additionally assumes the so-called nestling property, which is a centering condition for the environment, see [13], then it follows from [13, Section 5, Remark 3] and [10, Proposition 5.10] that v is characterized by

$$\alpha_0(v) = 0 \qquad \text{and} \qquad \alpha'_{0+}(v) = 1,$$

where $\alpha'_{0+}(v)$ is the right-hand side derivative of $\alpha_\lambda(v)$ with respect to λ at $\lambda = 0$.

In the present note we combine the two approaches of [11] and [13] without assuming the nestling property. We assume Kalikow's condition only and derive a different formula for the asymptotic velocity v in terms of the Lyapounov exponents. To this end, we first use the point-to-point exponents α_λ to define some new exponents γ_λ, which could be called point-to-hyperplane exponents. This name will be justified in Lemma 2.

Definition. For $\lambda \geq 0$ and $\ell \in \mathbb{R}^d \backslash \{0\}$ set

$$\gamma_\lambda(\ell) := \inf\{\alpha_\lambda(x) : \; x \in \mathbb{R}^d, x \cdot \ell \geq 1\}.$$

Our main result is the following.

Theorem 1. *Assume that the set of $\ell \in \mathbb{R}^d \backslash \{0\}$ relative to which Kalikow's condition holds is not empty. Then on this set the right hand side derivative*

$$\gamma'_{0+}(\ell) := \frac{d}{d\lambda}\gamma_\lambda(\ell)\Big|_{\lambda \searrow 0}$$

exists, is smooth in ℓ, and satisfies the identity

$$\gamma'_{0+}(\ell) = \frac{1}{v \cdot \ell} \in (0, \infty) \qquad \textit{and hence} \tag{5}$$

$$v = -\frac{\nabla \gamma'_{0+}(\ell)}{\left(\gamma'_{0+}(\ell)\right)^2}, \tag{6}$$

where v is the asymptotic velocity defined in Theorem C.

1. Point-to-hyperplane exponents

In this section we list some basic properties of $\gamma_\lambda(\ell)$ and justify its name point-to-hyperplane exponent by showing that $\gamma_\lambda(\ell)$ is essentially the Laplace transform of the hyperplane-hitting time

$$T_m(\ell) := \inf\{n \geq 0 : \; X_n \cdot \ell \geq m\} \qquad (m \geq 0, \ell \in \mathbb{R}^d \backslash \{0\}).$$

Lemma 2. (Point-to-hyperplane exponents). *For all* $\ell \in \mathbb{R}^d\backslash\{0\}$ *the function* $\gamma.(\ell) : [0,\infty) \to [0,\infty)$, $\lambda \mapsto \gamma_\lambda(\ell)$ *is continuous and concave increasing. Furthermore,*

$$(7)\quad \gamma_\lambda(\ell) \geq \frac{\lambda}{\|\ell\|_2} \quad \text{and} \quad \gamma_\lambda(\sigma\ell) = \frac{\gamma_\lambda(\ell)}{\sigma} \quad \text{for all } \lambda \geq 0,\ \ell \in \mathbb{R}^d\backslash\{0\} \text{ and } \sigma > 0,$$

where $\|\cdot\|_2$ *is the Euclidean norm. Moreover, on a set* $\Omega_2 \subseteq \Omega$ *of full* $\mathbb{P}$*-measure for all* $\lambda > 0$ *and all* $\ell \in \mathbb{R}^d\backslash\{0\}$,

$$\gamma_\lambda(\ell) = \lim_{m\to\infty} \frac{-\ln E_{0,\omega}\left[\exp\left(-\lambda T_m(\ell)\right)\right]}{m}. \tag{8}$$

Proof. The map $\lambda \mapsto \gamma_\lambda(\ell)$ is concave increasing since it is the infimum of the concave increasing functions $\lambda \mapsto \alpha_\lambda(x)$, see Theorem A. Hence the only possible location of discontinuity of $\gamma_\lambda(\ell)$ could be $\lambda = 0$. However, since $\alpha_\lambda(x)$ is also continuous in λ, $\gamma.(\ell)$ is upper semicontinuous and thus continuous also in $\lambda = 0$. The first part of (7) follows from (2) since for all $\lambda > 0$ and x with $x \cdot \ell \geq 1$,

$$1 \leq x \cdot \ell \leq \|x\|_2\|\ell\|_2 \leq |x|\|\ell\|_2 \leq \frac{\alpha_\lambda(x)}{\lambda}\|\ell\|_2.$$

The second part is a consequence of the homogeneity of α_λ. The proof of (8) is similar to the proof of [8, Corollary 1.9] and of [12, Corollary 16]. Indeed, observe that for any m and $x \in \mathbb{R}^d$ with $x \cdot \ell \geq 1$,

$$[mx]\cdot\ell \;=\; mx\cdot\ell + ([mx]-mx)\cdot\ell \;\geq\; m - \|[mx]-mx\|_2\|\ell\|_2 \;\geq\; m - d|\ell|$$

and therefore $H(mx) \geq T_{m-d|\ell|}(\ell)$ for $m \geq d|\ell|$. Consequently, for fixed $\omega \in \Omega_2 := \Omega_1$ (see Theorem A), for any $\lambda > 0$ and any $\ell \in \mathbb{R}^d\backslash\{0\}$ due to (1),

$$\begin{aligned}
\gamma_\lambda(\ell) &= \inf_{x\cdot\ell\geq 1}\lim_{m\to\infty}\frac{-\ln E_{0,\omega}[\exp(-\lambda H(mx))]}{m} \\
&\geq \limsup_{m\to\infty}\frac{-\ln E_{0,\omega}[\exp(-\lambda T_{m-d|\ell|}(\ell))]}{m-d|\ell|}\cdot\frac{m-d|\ell|}{m},
\end{aligned}$$

which proves one inequality of (8). For the reverse inequality set

$$D_m := \left\{x \in \mathbb{Z}^d:\ x\cdot\ell \geq m \quad \text{or} \quad \frac{\lambda|x|}{m} \geq \frac{(\lambda - \ln\kappa)|\ell|}{\ell\cdot\ell}\right\} \qquad (m \geq 1).$$

The complement D_m^c of D_m contains the origin and is finite since $\lambda > 0$. Denote by H_m the first time the walk visits the interior boundary $\partial(D_m^c)$ of D_m [see (3)]. Since the walk with start at the origin must pass the boundary $\partial(D_m^c)$ before it can hit the halfspace $\{x:\ x\cdot\ell \geq m\}$ we find that

$$\begin{aligned}
E_{0,\omega}[\exp(-\lambda T_m(\ell))] &\leq E_{0,\omega}[\exp(-\lambda H_m)] \leq \sum_{x\in\partial(D_m^c)} e_\lambda(0,x,\omega) \\
&\leq \sharp\partial(D_m^c)e_\lambda(0,x_m(\omega),\omega)
\end{aligned}$$

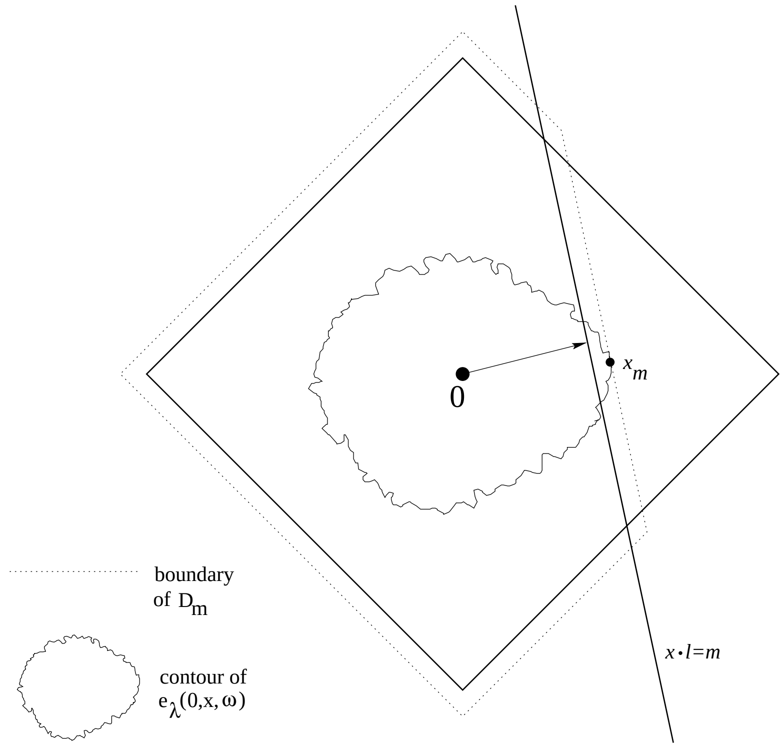

FIGURE 1. Sketch for the proof of Lemma 2. The box is chosen large enough to include the random contour of $e_\lambda(0,x,\omega)$ that touches the hyperplane $x\cdot\ell=m$.

for some maximizing $x_m(\omega)\in\partial(D_m^c)$ (see Figure 1). Because $\sharp\partial(D_m^c)$ grows only polynomially in m we get

$$\liminf_{m\to\infty}\frac{-\ln E_{0,\omega}\left[\exp\left(-\lambda T_m(\ell)\right)\right]}{m} \;\geq\; \liminf_{m\to\infty}\frac{-\ln e_\lambda(0,x_m(\omega),\omega)}{m}$$

$$\geq\; \liminf_{m\to\infty}\frac{-\ln e_\lambda(0,x_m(\omega),\omega)-\alpha_\lambda(x_m(\omega))}{|x_m(\omega)|}\cdot\frac{|x_m(\omega)|}{m} \tag{9}$$

$$+\liminf_{m\to\infty}\alpha_\lambda\left(\frac{x_m(\omega)}{m}\right). \tag{10}$$

The term in (9) vanishes due to (1) and the fact that $|x_m(\omega)|/m$ is bounded. The term in (10) is larger than $\gamma_\lambda(\ell)$. Indeed, since $x_m\in D_m$ we are in one of the following cases:

$$\frac{x_m(\omega)}{m}\cdot\ell\geq\frac{m}{m}=1\quad\text{and therefore}\quad\alpha_\lambda\left(\frac{x_m(\omega)}{m}\right)\geq\inf_{x\cdot\ell\geq1}\alpha_\lambda(x)=\gamma_\lambda(\ell)$$

or due to (2)

$$\alpha_\lambda\left(\frac{x_m(\omega)}{m}\right) \ \geq\ \lambda\frac{|x_m(\omega)|}{m} \ \geq\ \frac{(\lambda-\ln\kappa)|\ell|}{\ell\cdot\ell} \ \geq\ \alpha_\lambda\left(\frac{\ell}{\ell\cdot\ell}\right) \ \geq\ \inf_{x\cdot\ell\geq 1}\alpha_\lambda(x)=\gamma_\lambda(\ell).$$

This proves (8). □

2. Proof of Theorem 1

The following lemma holds regardless of Kalikow's condition.

Lemma 3. *For all $\ell\in\mathbb{R}^d\backslash\{0\}$ P_0-a.s.,*

$$\limsup_{n\to\infty}\frac{X_n\cdot\ell}{n}\leq\frac{1}{\gamma'_{0+}(\ell)}\qquad\in[0,\infty). \tag{11}$$

Proof. Since $\gamma_\cdot(\ell)$ is concave in λ by Lemma 2 it follows from the first part of (7) that $\gamma'_{0+}(\ell)>0$. Now fix some $\omega\in\Omega_2$ (see Lemma 2). We first show that

$$\liminf_{m\to\infty}\frac{T_m(\ell)}{m}\geq\gamma'_{0+}(\ell)\qquad P_{0,\omega}\text{-a.s.} \tag{12}$$

Pick some $0<\gamma<\gamma'_{0+}(\ell)$. Then there are some $\lambda_0>0$ and $\varepsilon>0$ with

$$\gamma<\frac{\gamma_{\lambda_0}(\ell)-\gamma_0(\ell)}{\lambda_0}-\varepsilon\leq\frac{\gamma_{\lambda_0}(\ell)}{\lambda_0}-\varepsilon, \tag{13}$$

where we used $\gamma_0(\ell)\geq 0$. Observe that for any positive integer m

$$\begin{aligned} P_{0,\omega}\left[T_m(\ell)\leq m\gamma\right] &= E_{0,\omega}\left[\exp(-\lambda_0 T_m(\ell))\exp(\lambda_0 T_m(\ell)),T_m(\ell)\leq m\gamma\right]\\ &\leq \exp(\lambda_0 m\gamma)E_{0,\omega}\left[\exp(-\lambda_0 T_m(\ell))\right]. \end{aligned}$$

Due to (8) this is for large m less than

$$\exp(m(\lambda_0\gamma-\gamma_{\lambda_0}(\ell)+\lambda_0\varepsilon)).$$

Since this is summable in m due to (13) it follows from the Borel Cantelli lemma that the left hand side of (12) is $P_{0,\omega}$-a.s. at least γ. Letting $\gamma\nearrow\gamma'_{0+}(\ell)$ proves (12). For the proof of (11) we distinguish two cases. If the sequence $X_n\cdot\ell, n\in\mathbb{N}$, is bounded from above, (11) is obvious. If the sequence is unbounded from above then the sequence

$$u_n:=\max_{m\leq n}X_m\cdot\ell$$

tends to infinity as $n\to\infty$. Observe that $T_{u_n}(\ell)\leq n$ for all n. Hence the left hand side of (11) is less than

$$\limsup_{n\to\infty}\frac{u_n}{T_{u_n}(\ell)} \ \leq\ \limsup_{n\to\infty}\frac{\lfloor u_n\rfloor+1}{T_{\lfloor u_n\rfloor}(\ell)} \ \leq\ \limsup_{m\to\infty}\frac{m}{T_m(\ell)}\leq\frac{1}{\gamma'_{0+}(\ell)}$$

due to (12). Here $\lfloor u\rfloor$ denotes the largest integer less than or equal to u. □

Proof of Theorem 1. Fix some $\ell \in \mathbb{R}^d \setminus \{0\}$ relative to which Kalikow's condition holds. For the proof of (5), thanks to Lemma 3 we only need to derive the inequality opposite to the one in (11). To this end we first estimate γ_0, which is defined in terms of quenched exponents, from below by some annealed exponent as follows. We proceed as in the first part of the proof of (8), but this time we use $L^1(\mathbb{P})$ convergence in (1) and Jensen's inequality to get for all $\lambda > 0$,

$$\begin{aligned} \gamma_\lambda(\ell) &\geq \limsup_{m\to\infty} \frac{\mathbb{E}[-\ln E_{0,\omega}[\exp(-\lambda T_m(\ell)]]}{m} \\ &\geq \limsup_{m\to\infty} \frac{-\ln E_0[\exp(-\lambda T_m(\ell))]}{m}. \end{aligned} \tag{14}$$

Now fix some $\varepsilon > 0$ and set

$$c_\varepsilon := \left(E_0\left[X_{\tau_1} \cdot \ell \mid D = \infty\right] + \varepsilon\right)^{-1}.$$

Then for $\lambda > 0$ and $m \geq 0$,

$$\begin{aligned} E_0\left[\exp(-\lambda T_m(\ell))\right] &\leq A_m + B_{m,\lambda}, \qquad \text{where} \\ A_m &:= P_0\left[T_m(\ell) < \tau_{\lfloor mc_\varepsilon \rfloor}\right] \qquad \text{and} \\ B_{m,\lambda} &:= E_0\left[\exp(-\lambda \tau_{\lfloor mc_\varepsilon \rfloor})\right]. \end{aligned} \tag{15}$$

Since $X_n \cdot \ell \leq X_{\tau_i} \cdot \ell$ for $n \leq \tau_i$ [see (4)] we deduce

$$A_m \leq P_0\left[X_{\tau_{\lfloor mc_\varepsilon \rfloor}} \cdot \ell \geq m\right] \leq P_0\left[\frac{X_{\tau_{\lfloor mc_\varepsilon \rfloor}} \cdot \ell}{\lfloor mc_\varepsilon \rfloor} \geq E_0\left[X_{\tau_1} \cdot \ell \mid D = \infty\right] + \varepsilon\right].$$

Recall from Theorem B that $E_0[\exp(cX_{\tau_1} \cdot \ell)] < \infty$ for some $c > 0$. Using the part of Theorem B concerning the increments of X_{τ_i} it follows from a Cramér type argument (see e.g. [2, Chapter 2.2.1]) that A_m decays exponentially as $m \to \infty$ with a rate depending only on ε but not on λ. On the other hand, we can use the part of Theorem B concerning the increments of τ_i to obtain for $\lfloor mc_\varepsilon \rfloor \geq 1$,

$$B_{m,\lambda} = E_0[\exp(-\lambda\tau_1)]\ E_0[\exp(-\lambda\tau_1) \mid D = \infty]^{\lfloor mc_\varepsilon \rfloor - 1}.$$

Thus $B_{m,\lambda}$ decays exponentially as $m \to \infty$ like A_m but with a rate that tends to 0 if $\lambda \searrow 0$. Hence there are some $\lambda_0(\varepsilon) > 0$ and some m_0 such that $A_m \leq B_{m,\lambda}$ for all $0 < \lambda \leq \lambda_0(\varepsilon)$ and $m \geq m_0$. Consequently, we get by (14) and (15) for all $0 < \lambda \leq \lambda_0(\varepsilon)$,

$$\gamma_\lambda(\ell) \geq \limsup_{m\to\infty} \frac{-\ln B_{m,\lambda}}{m} = -c_\varepsilon \ln E_0[\exp(-\lambda\tau_1) \mid D = \infty]. \tag{16}$$

Since $\gamma_\lambda(\ell)$ is concave increasing and continuous in λ by Lemma 2 we can use the fact that $\alpha_0(v) = 0$, see Theorem C, which implies $\gamma_0(\ell) = 0$, and (16) to get

$$\gamma'_{0+}(\ell) = \sup_{0<\lambda<\lambda_0(\varepsilon)} \frac{\gamma_\lambda(\ell)}{\lambda} \geq c_\varepsilon \sup_{0<\lambda<\lambda_0(\varepsilon)} \frac{-\ln E_0[\exp(-\lambda\tau_1) \mid D = \infty]}{\lambda}.$$

Note that by the Cauchy-Schwarz inequality the numerator in the last expression is concave in λ, too, and vanishes for $\lambda = 0$. Therefore the last expression equals

$$c_\varepsilon \left(\frac{d}{d\lambda} - \ln E_0[\exp(-\lambda\tau_1) \mid D = \infty]\right)\Big|_{\lambda=0} = \frac{E_0[\tau_1 \mid D = \infty]}{E_0[X_{\tau_1} \cdot \ell \mid D = \infty] + \varepsilon}.$$

Now we let $\varepsilon \searrow 0$ and use Theorem C to arrive at the conclusion that $\gamma'_{0+}(\ell) \geq 1/(v \cdot \ell)$ as desired, which proves (5).

For the proof of (6), that the set of directions relative to which Kalikow's condition holds is open, cf. [10, after equation (1.7)]. Indeed, if Kalikow's condition holds relative to ℓ with $\varepsilon_\ell > 0$, then it is also fulfilled for $\ell + h$ with $h \in \mathbb{R}^d$ if $|h| < \varepsilon_\ell$ because of

$$\begin{aligned}
&\inf_{U, x \in U} \sum_{|e|=1} (\ell + h) \cdot e \hat{P}_U(x, x+e) \\
\geq\ & \inf_{U, x \in U} \sum_{|e|=1} \ell \cdot e \hat{P}_U(x, x+e) + \inf_{U, x \in U} \sum_{|e|=1} h \cdot e \hat{P}_U(x, x+e) \\
\geq\ & \varepsilon_\ell + \inf_{U, x \in U} - \sum_{|e|=1} |h \cdot e| \hat{P}_U(x, x+e) \ \geq\ \varepsilon_\ell - |h| \ >\ 0.
\end{aligned}$$

Thus (5) is also valid in a neighborhood of ℓ. Therefore $\gamma'_{0+}(\ell) = 1/(v \cdot \ell)$ is smooth in ℓ with

$$\frac{\partial}{\partial \ell_i} \gamma'_{0+}(\ell) \ =\ \frac{-v_i}{(v \cdot \ell)^2} \ =\ -v_i (\gamma'_{0+}(\ell))^2$$

for all $i = 1, \dots, d$ which proves (6).

Remark. In one dimension Kalikow's condition relative to $\ell > 0$ is equivalent to the existence of a non-vanishing velocity of the walk into positive direction, cf. [11, Remark 2.5]. In order to evaluate formula (6) for the velocity in this situation observe that it follows from the ergodic theorem that for $\ell > 0$ and $\lambda \geq 0$,

$$\alpha_\lambda(\ell) = \ell \mathbb{E}[-\ln e_\lambda(0, 1, \omega)]$$

(cf. [12, (39)]). Recall that we denote by $H(1)$ the first-passage time through 1. Since $\gamma_\lambda(\ell) = \alpha_\lambda(1/\ell)$ we get

$$\begin{aligned}
\gamma'_{0+}(\ell) &= E_0[H(1)]/\ell \quad \text{and therefore} \\
\frac{d}{d\ell} \gamma'_{0+}(\ell) &= -E_0[H(1)]/\ell^2 \quad \text{and thus by (6)} \\
v &= 1/E_0[H(1)],
\end{aligned}$$

which is not very explicit but still the correct expression for the velocity v since $E_{0,\omega}[H(n)]/n$ converges $\mathbb{P}$-a.s. due to ergodicity to $E_0[H(1)]$.

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